A refinement of Betti numbers and homology in the presence of a continuous function II (the case of an angle valued map).

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Abstract

This paper is a sequel to [2]. We propose refinements of the Novikov-Betti numbers of the Novikov homology (w.r. to afield κ) of a pair (X, ξ) consisting of a compact ANR X and a degree one integral cohomology class ξ, in the presence of a continuous angle valued map representing the cohomology class ξ.

The first refinement consists of finite configurations of points with multiplicity located in $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}$ which can be identified to the punctured complex plane, of total cardinality the Novikov-Betti numbers; equivalently of monic polynomials with nonzero free coefficient.

The refinement of Novikov homology consists in configurations of free modules over the ring of Laurent polynomials with coefficients in the field κ, indexed by the points of the previous configurations, each of rank equal to the multiplicity of the point.

In case κ is the field of complex numbers these configurations of free modules can be canonically converted into configurations of Hilbert modules over the von Neumann algebra of the group $\mathbb{Z}$, all closed submodules of the L_2-homology of the infinite cyclic cover defined by the angle valued map.

A number of properties of these configurations are discussed.

Contents

1 Introduction

This paper is a sequel of [2] and we suppose the reader familiar with the notations and the basic concepts considered there. Here we treat the case of a circle valued map $f : X \to \mathbb{S}^1$. In this paper without any

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additional specifications we assume that for such map the space X is a compact ANR and f is a continuous map. In this case the map f determines a degree one integral cohomology class $\xi_f \in H^1(X;\mathbb{Z})$.

As in part I we fix a field κ and an integer r, $r = 0, 1, 2, \dim X$, and provide first a configurations δ^f_r of finitely many points with specified multiplicity located in the space $\mathbb{T} := \mathbb{R}^2/\mathbb{Z}$ [1] which can be identified to the punctured plane $\mathbb{C} \setminus 0$. It will be shown that the set of points (counted with multiplicity) of this configuration has cardinality the Novikov–Betti number $\beta^N_r(X;\xi_f)$. In view of the identification of $\mathbb{T}$ with $\mathbb{C} \setminus 0$ the configuration δ^f_r can be interpreted as a monic polynomial with complex coefficients and nonzero free term, $P^f_r(z)$, of degree the Novikov–Betti number whose roots are the points of the configuration δ^f_r; however this will be used in this paper only for notational simplifications.

As in part I (reference [2]) we refine the configuration δ^f_r to the configuration $\hat\delta^f_r$ of $\kappa[t^{-1},t]-$free modules (each one a quotient of split free submodules of the $r-$th Novikov homology of $(X;\xi_f)$) and in case $\kappa = \mathbb{C}$ to the configuration $\hat{\hat\delta}^f_r$ of closed Hilbert submodules of L_2-homology of $\tilde X$, the infinite cyclic cover of X defined by ξ. Each $\hat{\hat\delta}^f_r(z), z \in \mathbb{C} \setminus 0,$ [2] is a Hilbert module over the finite von Neumann algebra $L^\infty(\mathbb{S}^1)$ of von Neumann dimension equal to $\delta^f_r(z)$.

The results about these configurations formulated in Theorems 1.1, 1.2 and 1.3 are formally similar to Theorems 4.1, 4.2, and 4.3 in part I, but conceptually more complex and technically more difficult to conclude. There are however a number of differences and new features which deserve to be pointed out.

- The location of the points in the support of the configurations $\delta^f_r, \hat\delta^f_r, \hat{\hat\delta}^f_r$ is the space $\mathbb{T} := \mathbb{R}^2/\mathbb{Z}$ identified to the punctured complex plane $\mathbb{C} \setminus 0$ by the map $\mathbb{T} \ni \langle a, b\rangle \to z = e^{ia+(b-a)} \in \mathbb{C} \setminus 0$ and not $\mathbb{R}^2 = \mathbb{C}$ as in [2].

- The Betti numbers $\beta_r(X)$ are replaced by the Novikov-Betti numbers $\beta^N_r(X;\xi)$ and the homology $H_r(X)$ by the Novikov homology. In this paper the Novikov homology $H^N_r(X;\xi)$ for the field κ is a free $\kappa[t^{-1},t]-$ module (see definition in section 2 below), whose rank is equal to $\beta^N_r(X;\xi)$.

- For $z = \langle a, b\rangle \in supp\ \delta^f_r$ the configuration $\hat\delta^f_r$ has as value a free $\kappa[t^{-1},t]-$module, $\hat\delta^f_r(\langle a, b\rangle) = \hat\delta^f_r(z)$, which is a quotient $\hat{\mathbb{F}}_r(z)/\hat{\mathbb{F}}'_r(z)$ of split free submodules $\hat{\mathbb{F}}'_r(z) \subseteq \hat{\mathbb{F}}_r(z) \subseteq H^N_r(X;\xi)$. Moreover this configuration is derived from a *special configuration* of subquotients of $H^N_r(X,\xi)$, in the sense explained in section 2.

- In case $\kappa = \mathbb{C}$, the ring of Laurent polynomials $C[t^{-1},t]$ has a natural completion to the finite von Neumann algebra $L^\infty(S^1)$ and $H^N_r(X;\xi)$ to a $L^\infty(\mathbb{S}^1)-$Hilbert module. The Hilbert module structure, although unique up to isomorphism, depends on a chosen $C[t^{-1},t]-$inner product on $H^N_r(X;\xi)$, cf. section 2, which always exists. With respect to a given $\mathbb{C}[t^{-1},t]-$inner product the free module $H^N_r(X;\xi)$ can be canonically converted into the $L^\infty(\mathbb{S}^1)-$Hilbert module $H^{L_2}_r(\tilde X)$, and the configuration $\hat\delta^f_r(z)$ into a configuration of mutually orthogonal closed Hilbert submodules $\hat{\hat\delta}^f_r(z) \subseteq H^{L_2}_r(X;\xi)$ with $\sum_{z \in supp\delta^f_r} \hat{\hat\delta}^f_r(z) = H^{L_2}_r(\tilde X)$. This conversion is referred below as the *von Neumann completion* and is described in section 2.

- The Poincaré Duality refinement stated in Theorem 1.3 is derived from the Poincaré Duality between Borel–Moore homology and cohomology of the open manifold $\tilde M$.

The configurations $\delta^f_r, \hat\delta^f_r, \hat{\hat\delta}^f_r$ are defined in section 3 and all have the same support located in $\mathbb{T}$ or $\mathbb{C} \setminus 0$. A point when in $\mathbb{T}$ will be specified as $\langle a, b\rangle$ and when in $\mathbb{C} \setminus 0$ as z. To formulate the results, for the reader's convenience we recall some notions.

[1] $\mathbb{R}^2$ is equipped with the action $\mu(n,(a,b)) \to (a + 2\pi n, b + 2\pi n)$

[2] the configurations $\delta^f_r, \hat\delta^f_r, \hat{\hat\delta}^f_r$ are viewed as a functions with finite support on $\mathbb{C} \setminus 0$

For $\xi \in H^1(X;\mathbb{Z})$ and κ a fixed field one denotes by:

- $C_\xi(X,\mathbb{S}^1)$, the space of continuous maps in the homotopy class defined by ξ equipped with the compact open topology,
- $\tilde{X}$, an infinite cyclic cover defined by ξ_f,
- $H^N_r(X;\xi)$, the Novikov homology in dimension r with coefficients in κ, and $\beta^N_r(X;\xi)$, the $r-$th Novikov–Betti number,
- in case $\kappa = \mathbb{C}$ the L_2-homology in dimension r will be denoted by $H^{L_2}_r(\tilde{X})$. In this case the von-Neumann dimension equals the Novikov–Betti number.

For $f : X \to \mathbb{S}^1$ *a map* one denotes by :

- $\xi_f \in H^1(X;\mathbb{Z})$, the integral cohomology class represented by f,
- $\tilde{f} : \tilde{X} \to \mathbb{R}$, an infinite cyclic cover of the map f,
- $\tilde{\epsilon}(f) := \tilde{\epsilon}(\tilde{f})$, the smallest distance between homological critical values of $\tilde{f}$.

In section (2) one defines the space of configurations of mutually orthogonal closed Hilbert submodules of V, a $\mathcal{N}-$ Hilbert module of finite type over $\mathcal{N}$ a finite von Neumann algebra $\mathcal{N}$, and denotes this space by $\mathcal{C}^O_V(X)$ and describe the relevant topologies on this space.

Theorem 1.1 *(Topological results)*

1. *If $\delta^f_r(z) \neq 0$, $z = e^{ia+(b-a)}$ then both e^{ia} and e^{ib}, $(e^{ia}, e^{ib} \in \mathbb{S}^1)$ are homological critical values of f, equivalently a and b are homological critical values of $\tilde{f}$ hence also critical values*[3].

2. *(a)* $\sum_{z\in\mathbb{C}\setminus 0} \delta^f_r(z) = \beta^N_r(X;\xi_f)$,

   (b) $\bigoplus_{z\in\mathbb{C}\setminus 0} \hat{\delta}^f_r(z) \simeq H^N_r(X;\xi_f)$,

   (c) If $\kappa = \mathbb{C}$ a $\mathbb{C}[t^{-1},t]-$inner product on $H^N_r(X;\xi_f)$ (cf section 2 for definition) canonically converts $\hat{\delta}^f_r$ into a configuration $\hat{\hat{\delta}}^f_r$ of closed Hilbert submodules of $H^{L_2}_r(\tilde{X})$ which satisfy $\sum_{z\in\mathbb{C}\setminus 0} \hat{\hat{\delta}}^f_r(z) = H^{L_2}_r(\tilde{X})$, and $\hat{\hat{\delta}}^f_r(z) \perp \hat{\hat{\delta}}^f_r(z')$ for $z \neq z'$.

3. *If X is homeomorphic to a finite simplicial complex or to a compact Hilbert cube manifold then for an open and dense set of maps $f \in C_\xi(X,\mathbb{S}^1)$ one has $\delta^f(z) = 0$ or 1.*

Items 1.and 2. a) were first established in [4] for tame maps.

Item 2. a) indicates that $\delta^f_r \in \mathcal{C}_{\beta^N_r(X;\xi)}(\mathbb{C}\setminus 0)$, where $\mathcal{C}_N(\mathbb{C}\setminus 0)$ denotes the space of configuration of points with multiplicity located in $\mathbb{C}\setminus 0 = \mathbb{T}$ of cardinality N. This space equipped with the collision topology (described in [2]) identifies to the N-fold symmetric product of $\mathbb{T}$ equipped with the induced topology and to the space of degree $N-$monic polynomials with nonzero free coefficient hence to $\mathbb{C}^{N-1} \times (\mathbb{C}\setminus 0)$.

Item 2. b) implies that any family of splittings as defined in section 3, makes from $\hat{\delta}^f_r$ an element in $\mathcal{C}_V(\mathbb{T})$, $V = H^N_r(X;\xi)$, the space of configurations of split submodules of V, described in section 2.

Item 2 c) indicates that $\hat{\hat{\delta}}^f_r \in \mathcal{C}^O_{H^{L_2}_r(\tilde{X})}(\mathbb{T})$, the space configurations of mutually orthogonal closed Hilbert submodules of the $L^\infty(\mathbb{S}^1)-$Hilbert module $H^{L_2}_r(\tilde{X})$. This space can be equipped with two collision topologies, the *fine* and the *natural* collision topology, both described in section 2.

Theorem 1.2 *(Stability) Suppose X is a compact ANR, $\xi \in H^1(X;\mathbb{Z})$.*

1. The assignment

$$C(X,\mathbb{S}^1)_\xi \ni f \rightsquigarrow \delta^f_r = P^f_r(z) \in \mathbb{C}^{\beta^N_r(X;\xi)} \times (\mathbb{C}\setminus 0)$$

[3] the homotopy type of the fibers in any neighborhood of the critical values changes

is a continuous map.

Moreover with respect to the canonical metric $\underline{D}$ provided by the identification of the space of configurations with the $\beta^N_r(X;\xi)-$fold symmetric product of $\mathbb{T}$, one has the estimate

$$\underline{D}(\delta^f,\delta^g)<2D(f,g),$$

where $D(f,g):=||f-g||_\infty = sup_{x\in X}|f(x)-g(x)|$.

2. If $\kappa=\mathbb{C}$ and the spaces of configurations $\mathcal{C}^O_{H^{L_2}_r(\tilde X)}(\mathbb{C}\setminus 0)$ is equipped with either the fine or the natural collision topology then the assignment $f\rightsquigarrow \hat{\hat{\delta}}^f_r$ is continuous.

Item 1. was first established in [4] for X homeomorphic to a simplicial complex .

Theorem 1.3 *(Poincaré Duality) Suppose M is a closed topological manifold of dimension n which is $\kappa-$orientable and $f: M\to\mathbb{S}^1$ a continuous map with $\xi_f\neq 0$.Then one has*

1. *$\delta^f_r(\langle a,b\rangle)=\delta^f_{n-r}(\langle b,a\rangle)$, equivalently $\delta^f_r(z)=\delta^f_{n-r}(\tau z)$ with $\tau(z)=z^{-1}e^{i\ln|z|}$.*

2. *The Poincaré Duality (between Borel Moore homology of $\tilde M$ and the cohomology of $\tilde M$) induces the isomorphisms*
$$PD_r(\langle a,b\rangle):\hat\delta^f_r(\langle a,b\rangle)\to \hom_{\kappa[t^{-1},t]}(\hat\delta^f_{n-r}(\langle b,a\rangle),\kappa[t^{-1},t])$$
and establishes the isomorphism of $H^N_r(M;\xi)$ and $\hom_{\kappa[t^{-1},t]}(H^N_{n-r}(M;\xi),\kappa[t^{-1},t])$ as $\kappa[t^{-1},t]-$modules (both free modules).

3. *If $\kappa=\mathbb{C}$ and M is a closed Riemannian manifold then the canonical isomorphism of $H^{L_2}_r(\tilde M)$ to $H^{L_2}_{n-r}(\tilde M)$ induced by the Riemannian metric (via L_2 harmonic forms and the Hodge star operator) intertwines the configuration $\hat{\hat{\delta}}^f_r(\langle a,b\rangle)$ with $\hat{\hat{\delta}}^f_{n-r}(\langle b,a\rangle)$.*

Item 1. was first established in [4]. The isomorphism claimed in Item 2. is not canonical; it depends on the choices of splittings.
Item 3. will be discuss in this paper only informally. It requires a number of additions too long to be reviewed in full details and is actually part of a more general discussion on Poincaré Duality and refinements planned for a future paper.

Last section of the paper uses the Poincaré Duality for Novikov homology discussed in relation with the proof of Theorem 1.3 to conclude a few result about the Novikov–Betti numbers, the Betti numbers and the Jordan cells for some compact manifolds , cf Corollary 6.2, of possibly of relevance for the topology of complements of complex hypersurfaces. None of these results are new, at least in case of complement of hyper surfaces, but most likely derived by a different approach. At this point we thank L.Maxim for challenging questions and informations about some of his work.

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2 Additional preparatory material

The reader can skip most of the material of this section, which contains mostly definitions and notations, and return selectively when some notations or concepts need clarifications.

Angles and angle valued maps

An *angle* is a complex number $\theta = e^{it} \in \mathbb{C}, t \in \mathbb{R}$ and the set of all angles is denoted by $\mathbb{S}^1 = \{\theta = e^{it} \mid t \in \mathbb{R}\}$. The space of angles, $\mathbb{S}^1$, is equipped with the distance

$$d(\theta_2, \theta_1) = \inf\{|t_2 - t_1| \mid e^{it_1} = \theta_1, e^{it_2} = \theta_2\}.$$

One has $d(\theta_1, \theta_2) \leq \pi$. With this description $\mathbb{S}^1$ is an oriented one dimensional manifold with the orientation provided by a specified generator u of $H_1(\mathbb{S}^1; \mathbb{Z})$, the infinite cyclic group.

A closed interval in $I \subset \mathbb{S}^1$ with ends the angles $\theta_1 = e^{it_1}$ and $\theta_2 = e^{it_2}$ is the set $I := \{e^{it} \mid t_1 \leq t \leq t_2, t_2 - t_1 < 2\pi\}$.

More general configurations

If A is a unital ring without any additional specification a *configuration of free $A-$modules indexed by points in* X is a map with finite support $\hat\omega : X \rightsquigarrow$free $A-$modules. This set is denoted by $\mathcal{C}^A(X)$.

If V is a finite generated free $A-$module and $\mathcal{P}(V)$ denotes the collection of split submodules [4] of V a more interesting set denoted by $\mathcal{C}_V(X)$ can be considered and referred to as the set of configurations of split submodules of V.

$$\mathcal{C}_V(X) : \{\hat\omega : X \to \mathcal{P}(V) \mid \begin{cases} \hat\omega \in \mathcal{C}^A(X) \\ \text{for any x, } \{\hat\omega(\text{x})\} \text{ linearly independent} \\ \sum_{x \in X} \hat\omega(x) = V \end{cases} \}$$

It comes equipped with a topology defined by specifying for each element a system of *fundamental neighborhoods* (as described in [2] in case A is a field). A fundamental neighborhood of a configuration $\hat\omega \in \mathcal{C}_V(X)$ with support $\{x_1, x_2, \cdots x_k\}$ and values $\hat\omega(x_i) = V_i$ is specified by a collection of disjoint open sets of X, $(U_1, U_2, \cdots U_k)$, each U_i neighborhood of x_i, and consists of

$$\{\hat\delta \in \mathcal{C}_V(X) \mid \sum_{x \in U_i \cap supp \hat\delta} \hat\delta(x) = V_i\}^5.$$

This topology is referred to as the *fine collision topology*.

If $\mathcal{N}$ is a finite von -Neumann algebra and V is a finite type $\mathcal{N}-$Hilbert module, then $\mathcal{P}(V)$ will denote the collection of closed Hilbert submodules of V and $\mathcal{C}_V^O(X)$ the set

$$\mathcal{C}_V^O(X) := \{\hat\omega : X \to \mathcal{P}(V) \mid \begin{cases} \sharp(supp\, \hat\omega) < \infty \\ \hat\omega(x) \perp \hat\omega(y), x \neq y \\ \sum_{x \in X} \hat\omega(x) = V \end{cases} \}.$$

This set is referred to as the *configurations of mutually orthogonal closed submodules of* V. In this case one can consider in addition to the fine collision topology a second and coarser topology, the *natural collision topology*. A fundamental neighborhood of $\hat\delta$ is specified by $(U_1, U_2, \cdots U_k;\ O_1, O_2, \cdots O_k)$, U_i as before and O_i open neighborhood of V_i in $G_{k_i}(V)$. Here $G_k(V)$ denotes the subset of closed Hilbert submodules of V of von Neumann dimension k equipped with the obvious topology [6] . The neighborhood consists of configurations $\hat\delta'$ with the property that $\sum_{x \in U_i \cap supp\ \hat\delta'} \hat\delta'(x) \in O_i$.

[4] a submodule V' of V is *split* if the inclusion $V' \subseteq V$ has a a left inverse

[5] when referred to vector subspaces Σ will always "sum" inside of the bigger space

[6] the topology induced by the distance between bounded operators in Hilbert space; any such submodule corresponds to a selfadjoint projector

Special configurations

Let V be a free f.g. $A-$module, with A a unital (commutative) ring. A subquotient of V is a pair $\omega = (W, W')$, with $W' \subset W$ split submodules of V. For each such subquotient ω one considers the free module $\hat{\omega} = W/W'$. Call *splitting* any linear map $i_\omega : W/W' \to W$ which is a right inverse of the canonical projection $\pi_\omega : W \to W/W'$, i.e. $\pi_\omega \cdot i_\omega = id$. A splitting of ω realizes the quotient W/W' as a split submodule of W and then of V.

Given two different subquotients $\omega_1 = (W_1, W_1')$ and $\omega_2 = (W_2, W_2')$ one writes $\omega_1 \leq \omega_2$ when$W_1 \subseteq W_2$. Denote by $\tilde{\mathcal{P}}(V)$ the collection of subquotients of V to which we add the symbol 0 as a symbol for the "trivial" subquotient.

A *special configuration of subquotients of V indexed by points in the space X* is given by a map $\tilde{\omega} : X \to \tilde{\mathcal{P}}(V)$ with finite support and different values for different points in the support, which satisfies properties P1., P2., P3. below.

Let $\omega_\alpha = (W_\alpha, W_\alpha'), \alpha \in \mathcal{A}$ be the set of nontrivial subquotients which appear as values of $\tilde{\omega}$. The set $\mathcal{A}$ is finite. For any $\alpha \in \mathcal{A}$ denote by $\mathcal{A}_\alpha := \{\beta \in \mathcal{A} \mid \omega_\beta \leq \omega_\alpha\}$.

P1. For any subset $\mathcal{B} \subseteq \mathcal{A}_\alpha \setminus \alpha,\ \sum_{\beta \in \mathcal{B}} W_\beta \subset W_\alpha \Rightarrow \sum_{\beta \in \mathcal{B}} W_\beta \subset W_\alpha'$.

P2. For any subset $\mathcal{B} \subseteq \mathcal{A}_\alpha \setminus \alpha,\ \sum_{\beta \in \mathcal{B}} W_\beta \cap W_\alpha \subseteq \sum_{\beta \in \mathcal{B}} W_\beta \cap W_\alpha'$.

P3. For any $\alpha \in \mathcal{A},\ \sum_{\beta \in \mathcal{A}_\alpha} \mathrm{rank}(\hat{\omega}_\beta) = \mathrm{rank} W_\alpha$ and $\sum_{\alpha \in \mathcal{A}} \mathrm{rank}(\hat{\omega}_\alpha) = \mathrm{rank} V$.

Any special configuration $\tilde{\omega}$ provides a configuration of free modules $\hat{\tilde{\omega}}(x) = W(x)/W'(x)$ with

$$\oplus_{x \in supp\ \tilde{\omega}} W(x)/W'(x) \simeq V,$$

and once a collection of splittings $i_\alpha := i_{\omega_\alpha} : W_\alpha/W_\alpha' \to W_\alpha$ is given, one can realize these free modules inside V and convert the special configuration into a configuration split free submodules of V. Of course the realization is not unique and different splittings lead to different configurations. To check this one shows first that for any α the submodules $i_\beta(\hat{\omega}_\beta) \subset W_\alpha$ are linearly independent using P1. and P2. , and that $\sum_{\beta \in \mathcal{A}_\alpha} i_\beta(\hat{\omega}_\beta) = W_\alpha$ and $\sum_{\beta \in \mathcal{A}} i_\beta(\hat{\omega}_\beta) = V$ using P 3..

In case A is the field $\mathbb{C}$ or $\mathbb{R}$ and V is a Hilbert space each subquotient $(W(x), W'(x))$ has a *canonical splitting*, defined by the orthogonal complement of $W'(x)$ in $W(x)$. The canonical splittings defined using orthogonality permit to convert this configuration into an element $\hat{\tilde{\omega}} \in \mathcal{C}_V^O(X)$.

Infinite cyclic cover

For an angle valued map $f : X \to \mathbb{S}^1$ let $f^* : H^1(\mathbb{S}^1; \mathbb{Z}) \to H^1(X; \mathbb{Z})$ be the homomorphism induced by f in integral cohomology let and $\xi_f = f^*(u) \in H^1(X; \mathbb{Z})$. The assignment $f \rightsquigarrow \xi_f$ establishes a bijective correspondence between the set of homotopy classes of continuous maps from X to $\mathbb{S}^1$ and $H^1(X; \mathbb{Z})$.

Recall the following:

- An infinite cyclic cover of X is a map $\pi : \tilde{X} \to X$ together with a free action $\mu : \mathbb{Z} \times \tilde{X} \to \tilde{X}$ such that $\pi(\mu(n, x)) = \pi(x)$ and the map induced by π from $\tilde{X}/\mathbb{Z}$ to X is a homeomorphism. An infinite cyclic cover is said to be *associated to* ξ if any continuous proper map $\tilde{f} : \tilde{X} \to \mathbb{R}$ which satisfies $\tilde{f}(\mu(n, x)) = \tilde{f}(x) + 2\pi n$ induces a map $f : X \to \mathbb{R}/2\pi\mathbb{Z} = \mathbb{S}^1$ with ξ_f equal to ξ.

- For two infinite cyclic covers $\pi_i : \tilde{X}_i \to X$ associated to ξ there exists homeomorphisms $\omega : \tilde{X}_1 \to \tilde{X}_2$ which intertwine the free actions μ_1 and μ_2 and satisfy $\pi_2 \cdot \omega = \pi_1$.

- Given $\pi : \tilde{X} \to X, \mu : \mathbb{Z} \times \tilde{X} \to \tilde{X}$ an infinite cyclic cover and $f : X \to \mathbb{S}^1$ an angle valued map the map $\tilde{f} : \tilde{X} \to \mathbb{R}$ is a called a *lift of* f if $\tilde{f}(\mu(n, x)) = \tilde{f}(x) + 2\pi n$ and by passing to the quotients

$X = \tilde{X}/\mathbb{Z}$ and $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ the map f induces exactly f. A lift $\tilde{f}$ provides the following pull back diagram

$$\begin{tikzcd} \mathbb{R} \arrow[r, "p"] & \mathbb{S}^1 \\ \tilde{X} \arrow[u, "\tilde{f}"] \arrow[r, "\pi"] & X. \arrow[u, "f"'] \end{tikzcd} \tag{1}$$

where $p(t)$ is given by $p(t) = e^{it} \in \mathbb{S}^1$.

- Given $f : X \to \mathbb{S}^1$ there exists a *canonical infinite cyclic cover* associated to ξ_f, the pullback of $p : \mathbb{R} \to \mathbb{S}^1$ by f, precisely $\tilde{X} = \{x, r) \mid f(x) = p(t)\}$ and $\tilde{f}(x,t) = t$.

Observation 2.1

1. *Any map $f : X \to \mathbb{S}^1$ has lifts. Two lifts of f , $\tilde{f}_1$ and $\tilde{f}_2$ differ by a deck transformation, i.e. there exists $k \in \mathbb{Z}$ with $f_2 = f_1 \cdot \mu(k, \cdots)$.*

2. *If $f, g : X \to \mathbb{S}^1$ are two maps with $D(f,g) < \pi$ then they are homotopic and if $\tilde{f}$ is a lift of f there exists a lift $\tilde{g}$ of g such that $D(\tilde{f}, \tilde{g}) = D(f,g)$.*

3. *If $D(f_1, f_2) < \pi, f_1, f_2 : X \to \mathbb{S}^1$, then there exists a canonical homotopy $f_t : X \to \mathbb{S}^1$. This homotopy has lifts $\tilde{f}_t$, which satisfy $\tilde{f}_t = t\tilde{f}_1 + (1-t)\tilde{f}_2$ for any pair $\tilde{f}_1, \tilde{f}_2$, of lifts which satisfy $D(\tilde{f}_1, \tilde{f}_2) = D(f_1, f_2)$. Moreover for the canonical homotopy and any sequence $1 = t_0 < t_1 < t_2 < \cdots t_k < t_{k+1} = 1$ one has $D(f_1, f_2) = \sum_{0 \leq i \leq k} D(f_{t_i}, f_{i+1})$.*

Tame angle valued maps, critical values

For an angle valued map $f : X \to \mathbb{S}^1$ consider $\tilde{f} : \tilde{X} \to \mathbb{R}$ an infinite cyclic cover. The map f is weakly tame, resp. tame, resp. homologically tame if so is $\tilde{f}$. If X is a finite simplicial complex then a map $f : X \to \mathbb{S}^1$ is called p.l (piecewise linear) if for some and then any lift $\tilde{f} : \tilde{X} \to \mathbb{R}$ is a p.l. An angle $\theta \in \mathbb{S}^1$ is a regular value, resp. critical value, resp. homologically critical value if $\theta = e^{it}$ with t a regular value, resp. critical value resp. homologically value for $\tilde{f}$. The above definitions are independent on the lift. Since an infinite cyclic cover of a map $f : X \to \mathbb{S}^1$ is proper the set of homological critical values is discrete and $2\pi-$periodic and therefore is homologically tame w.r. to any field.

Novikov homology:

Let κ be a field and let $\kappa[t^{-1}, t]$ be the $\kappa-$algebra of Laurent polynomials with coefficients in κ. This is a commutative algebra which is an integral domain and a principal ideal domain. For a pair (X, ξ) $\xi \in H^1(X; \mathbb{Z})$, X a compact ANR, let $\tilde{X}$ be the associated infinite cyclic cover and let $\tau : \tilde{X} \to \tilde{X}$ be the positive generator of the group of deck transformation $\mathbb{Z}$ viewed as a homeomorphism of $\tilde{X}$. Since X is compact, the $\kappa-$vector space $H_k(\tilde{X})$ is actually a finitely generated $\kappa[t^{-1}, t]-$module whose multiplication by t is given by the linear isomorphism induced by the homeomorphism τ.

Since $\kappa[t^{-1}, t]$ is a PID the collection of torsion elements form a $\kappa[t^{-1}, t]-$submodule $V_r(X; \xi) := \mathrm{Torsion}\ (\mathrm{H_r}(\tilde{\mathrm{X}})) = \mathrm{TH_r}(\tilde{\mathrm{M}})$ (usually referred to as *monodromy*) which as a $\kappa-$ vector space is of finite dimension. The quotient module $H_r(\tilde{X})/T(H_r(\tilde{X}))$ is a finitely generated free $\kappa[t^{-1}, t]-$module. In this paper this free $\kappa[t^{-1}, t]-$module and its rank are called the *Novikov homology* and the *Novikov–Betti number* and are denoted by $H_r^N(X; \xi)$ and $\beta_r^N(X; \xi)$.

Since $\kappa[t^{-1}, t]$ is a principal ideal domain one has $H_r(\tilde{X}) \simeq V_r(X; \xi) \oplus H_r^N(X; \xi)$. [7]

[7] classically, the Novikov homology is the $\kappa[t^{-1}, t]]-$ vector space $H_r(\tilde{X}) \otimes_{\kappa[t^{-1}, t]} \kappa[t^{-1}, t]]$ with $\kappa[t^{-1}, t]]$ the field of Laurent power series; clearly $\beta_r^N = \dim(H_r^N(\tilde{X}) \otimes_{\kappa[t^{-1}, t]} \kappa[t^{-1}, t]]) = \mathrm{rank}(H_r^N(\tilde{X}))$.

The von-Neumann completion:

When $\kappa = \mathbb{C}$, the ring of Laurent polynomials $C[t^{-1}, t]$, equivalently the group ring $\mathbb{C}[\mathbb{Z}]$ of the infinite cyclic group $\mathbb{Z}$, is an algebra with involution $*$ and trace tr.

If $a = \sum_{n\in\mathbb{Z}} a_n t^n$ then:

$$*(a) := a^* = \sum_{n\in\mathbb{Z}} \overline{a}_n t^{-n}$$
$$tr(a) = a_0.$$

with $\overline{a}$ denoting the complex conjugate of the complex number a.

The algebra $\mathbb{C}[\mathbb{Z}]$ can be considered as a sub algebra of the algebra of bounded linear operators on the separable Hilbert space

$$l_2(\mathbb{Z}) = \{a_n, n \in \mathbb{Z} \mid \sum_{n\in\mathbb{Z}} |a_n|^2 < \infty\}.$$

The linear operator defined by a Laurent polynomial is given by the multiplication of the Laurent polynomial (regarded as a sequence with all but finitely many components equal to zero) with a sequences in $l_2(\mathbb{Z})$.

One denotes by $\mathcal{N}$ the *weak closure* of $\mathbb{C}[\mathbb{Z}]$[8] which is a finite von Neumann algebra, with involution and trace extending the ones defined above, cf [10].

This algebra $\mathcal{N}$ is referred below as the von–Neumann completion of the group ring $\mathbb{C}(\mathbb{Z})$ and is isomorphic to the familiar $L^\infty(\mathbb{S}^1)$ via Fourier series transform (which assigns to a complex valued function defined on $\mathbb{S}^1$ its Fourier series).

Given a free $\mathbb{C}[t^{-1}, t]$–module M a $\mathbb{C}[t^{-1}, t]$-*valued inner product* in M is a map $\mu : M \times M \to \mathbb{C}[t^{-1}, t]$ which satisfies:

1. $\mathbb{C}[t^{-1}, t]-$linear in the first variable,
2. symmetric in the sense that $\mu(x, y) = \mu(y, x)^*$, $x, y \in M$,
3. positive definite in the sense that satisfies
   (a) $\mu(x, x) \in \mathbb{C}[t^{-1}, t]_+$ with $\mathbb{C}[t^{-1}, t]_+$ the set of elements of the form aa^* and
   (b) $\mu(x, x) = 0$ iff $x = 0$,

   and
4. the map $M \to Hom_{\mathbb{C}[t^{-1},t]}(M, \mathbb{C}[t^{-1}, t])$ defined by $\mu(y)(x) = \mu(x, y)$ is one to one.

Clearly $\mathbb{C}[t^{-1}, t]$-valued inner products exist. Indeed, if $e^1, e^2, \cdots e^k$ is a base of M then

$$\mu(\sum a_i e^i, \sum b_j e^j) := \sum a_i (b_i)^*$$

provides such inner product.

By completing the $\mathbb{C}-$vector space M w.r. to the Hermitian inner product $< x, y >:= tr(\mu(x, y))$ one obtains a Hilbert space $\overline{M}$ which is an $\mathcal{N}$-Hilbert module, cf [10], isomeric to $l_2(\mathbb{Z})^{\oplus k}$, k the rank of M. Two different $\mathbb{C}[t^{-1}, t]$-valued inner products, μ_1 and μ_2, lead to the isomorphic (and then also isometric) Hilbert modules $\overline{M}_{\mu_1}$ and $\overline{M}_{\mu_2}$. This justifies dropping μ from notation. If one identifies $\mathcal{N}$ to $L^\infty(\mathbb{S}^1)$ and $l_2(\mathbb{Z})^{\oplus k}$ to $L^2(\mathbb{S}^1)^{\oplus k}$ (by interpreting the sequence $\sum_{n\in\mathbb{Z}} a_n t^n$ as the complex valued function $\sum_{n\in\mathbb{Z}} a_n e^{in\theta}$) the $\mathcal{N}-$ module structure on $l_2(\mathbb{Z})^{\oplus k}$ becomes the $L^\infty(\mathbb{S}^1)$-module structure on $(L^2(\mathbb{S}^1)^{\oplus k}$ and is given by the

[8] in the space of bounded operators of the Hilbert space $l_2(\mathbb{Z})$ when each element of $\mathbb{C}[\mathbb{Z}]$ is regarded as such operator

component-wise multiplication of the L^∞-function in $L^\infty(\mathbb{S}^1)$ with the $k-$tuple of L^2-functions, an element in $(L^2(\mathbb{S}^1)^{\oplus k}$.

If one has $N \subset M$ a free split submodule of the f.g free $\mathbb{C}[t^{-1},t]$-module M and μ is an $\mathbb{C}[t^{-1},t]$-valued inner product on M, then $\overline{N}_\mu$ is a closed Hilbert submodule of $\overline{M}_\mu$. Moreover if $N'_i \subseteq N_i \subseteq M$, $i = 1, 2, \cdots$ is a collection of split submodules and N_i/N'_i is a collection of free modules, quotient of submodules of M, then one can canonically convert N_i/N'_i into closed Hilbert submodules of $\overline{M}$ simply by taking the closure of the kernel of the projection $N_i \to N_i/N'_i$ inside $\overline{M}$. The process of passing from $(\mathbb{C}[t^{-1},t], M)$ to $(\mathcal{N}, \overline{M})$ is referred to as *von Neumann completion* and was pioneered in [11] for any group ring $\mathbb{C}[\Gamma]$ and f.g. projective $\mathbb{C}[\Gamma]$-module.

3 The configurations δ^f_r, $\hat\delta^f_r$ and $\hat{\hat\delta}^f_r$.

3.1 Recollections from Part 1, ([2])

In this subsection we recall a number of observations already made in [2] but we reconsider them in the context of proper maps defined on locally compact ANR's in order to be applied to the lifts of an angle valued maps in the next subsection.

Consider $h : Y \to \mathbb{R}$ with Y a locally compact ANR Y and h a proper map. Suppose that $\epsilon(h) > 0$. Recall that $\epsilon(h)$ is the shortest distance between homological critical values. Recall that for $a, b \in \mathbb{R}$

$$\mathbb{F}^h_r(a,b) = \mathbb{I}^h_a(r) \cap \mathbb{I}^b_h(r) \subseteq H_r(Y).$$

Proposition 3.1

1. *$\mathbb{F}^h_r(a,b)$ is a finite dimensional vector space with the property that:*
   - *(a)* $\mathbb{F}^h_r(a',b') \subseteq \mathbb{F}^h_r(a,b)$ *for* $a' \leq a, b \leq b'$,
   - *(b)* $\mathbb{F}^h_r(a',b') = \mathbb{F}^h_r(a',b) \cap \mathbb{F}^h_r(a,b')$.
2. *For a box* $B = (a',a] \times [b,b')$, $a' < a, b < b'$

$$\mathbb{F}^h_r(B) := \mathbb{F}^h_r(a,b)/\mathbb{F}^h_r(a',b) + \mathbb{F}^h_r(a,b'),$$

and for $a'' < a' < a;\ b < b' < b''$ *with*

$$\begin{aligned} B'_1 :=&(a',a] \times [b,b'),\\ B_1 :=&(a'',a] \times [b,b')\\ B''_1 :=&(a'',a'] \times [b,b'),\end{aligned}$$

$$\begin{aligned} B'_2 :=&(a',a] \times [b',b''),\\ B_2 :=&(a'',a] \times [b',b''),\\ B''_2 :=&(a'',a'] \times [b',b'')\end{aligned}$$

and

$$B := (a'',a] \times [b,b'') \tag{2}$$

the inclusions $B''_1 \subseteq B_1 \supseteq B'_1$ *and* $B''_2 \subseteq B_2 \supseteq B'_2$ *and* $B_1 \subseteq B \supseteq B_2$ *induce the short exact sequences*

$$0 \longrightarrow \mathbb{F}^h_r(B''_1) \xrightarrow{i^{B_1}_{B''_1,r}} \mathbb{F}^h_r(B_1) \xrightarrow{\pi^{B'_1}_{B_1,r}} \mathbb{F}^h_r(B'_1) \longrightarrow 0\ ,$$

$$0 \longrightarrow \mathbb{F}_r^h(B_2'') \xrightarrow{i^{B_2}_{B_2'',r}} \mathbb{F}_r^h(B_2) \xrightarrow{\pi^{B_2'}_{B_2,r}} \mathbb{F}_r^h(B_2') \longrightarrow 0\ ,$$

$$0 \longrightarrow \mathbb{F}_r^h(B_2) \xrightarrow{i^{B}_{B_2,r}} \mathbb{F}_r^h(B) \xrightarrow{\pi^{B_1}_{B,r}} \mathbb{F}_r^h(B_1) \longrightarrow 0\ .$$

For proof see Propositions 3.6 and 3.7 in [2].

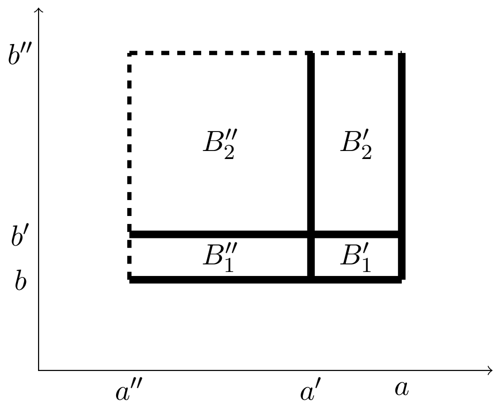

Figure 1

$$\begin{aligned} B_1 =& B_1' \sqcup B_1'' \\ B_2 =& B_2' \sqcup B_2'' \\ B\ =& B_1 \sqcup B_2 \end{aligned}$$

As in part I, for $\epsilon > 0$ one denotes by $B(a,b;\epsilon) := (a-\epsilon, a] \times [b, b+\epsilon)$. In view of Proposition (3.1) item 1. the inclusion $B(a,b;\epsilon) \subseteq B(a,b;\epsilon')$ for $\epsilon' > \epsilon$ induces the surjective linear map $\mathbb{F}_r^h(B(a,b;\epsilon')) \to \mathbb{F}_r^h(B(a,b;\epsilon))$. Then the limit

$$\hat{\delta}^h_r(a,b) := \varinjlim_{\epsilon \to 0} \mathbb{F}_r^h((B(a,b;\epsilon))$$

is a finite dimensional vector space. Define

$$\boxed{\delta^h_r(a,b) := \dim \hat{\delta}^h_r(a,b)}$$

and denote $\varinjlim \pi^{B(a,b;\epsilon)}_{B,r}$ by

$$\boxed{p^{(a,b)}_{B,r} : \mathbb{F}_r^h(B) \to \delta^h_r(a,b)}.$$

Note that δ^h_r and $\hat{\delta}^h_r$ are not configurations since their support, although discrete, is not finite. Nevertheless, as noticed in part I the assignments $\mathbb{R}^2 \ni (a,b) \rightsquigarrow \hat{\delta}^h_r(a,b)$ has the following properties:

Proposition 3.2

1. *If* $\hat{\delta}^h_r(a,b) \neq 0$ *then both* a, b *are homological critical values, hence* *supp* $\hat{\delta}^h_r =$ *supp* $\delta^h_r \subset CR(h) \times CR(h)$.

2. *If* a, b *are both homological critical values and* $\epsilon < \epsilon(f)$ *then* $\hat{\delta}^h_r(a,b) = \mathbb{F}_r^h(B(a,b;\epsilon))$.

3. For any box $B = (a', a] \times [b, b')$ with $a' < a, b < b'$ one has

$$\sum_{(a,b) \in B \,\cap\, supp\ \hat{\delta}^f_r} \delta^h_r(a,b) = \dim \mathbb{F}^h_r(B).$$

Proof: Items 1. and 2. follow from Observation 3.9 [2] and item 3. from Observation 3.11 in [2].

■

As in part I for each (a,b) one considers the surjective maps

$$\pi^B_r(a,b) : \mathbb{F}^h_r(a,b) \to \mathbb{F}^h_r(B), \;\; \pi_r(a,b) : \mathbb{F}_r(a,b) \to \hat{\delta}^h_r(a,b), \text{ and } \mathrm{p}^{(\mathrm{ab})}_{\mathrm{B,r}} : \mathrm{\mathbb{F}^h_r(B)} \to \mathrm{\hat{\delta}^h_r(a,b)}$$

and call *splitting* any liner map

$$i_r(a,b) : \hat{\delta}^h_r(a,b) \to \mathbb{F}^h_r(a,b)$$

which is a right inverse of $\pi_r(a,b)$, i.e. $\pi_r(a,b) \cdot i_r(a,b) = id$. We keep the same notation for the composition of $i_r(a,b)$ with the inclusion $\mathbb{F}^h_r(a,b) \subseteq H_r(Y)$.

If $(a,b) \in B'$, i.e. B' has to be of the form $B' = (a'', a'] \times [b', b'')$ with $a'' \leq a \leq a',\ b' \leq b \leq b''$, denote by

$$i^{B'}_r(a,b) : \hat{\delta}^f_r(a,b) \to \mathbb{F}^h_r(B')$$

the composition

$$\hat{\delta}^h_r(a,b) \xrightarrow{i_r(a,b)} \mathbb{F}^h_r(a,b) \xrightarrow{\subseteq} \mathbb{F}^h_r(a'.b') \xrightarrow{\pi^{B'}_r(a',b')} \mathbb{F}^h_r(B')\ .$$

Note that both $i_r(a,b)$ and $i^{B'}_r(a,b)$ are injective, the first because $\pi_r(a,b) \cdot i_r(a,b) = Id$. To check the second observe the following:

a. The commutativity of the diagram

$$\begin{tikzcd}
\mathbb{F}_r(a,b) \arrow[r, "\subseteq"] \arrow[d, "\pi^B_r(a,b)"] & \mathbb{F}_r(a',b') \arrow[d, "\pi^{B'}_r(a',b')"] \\
\mathbb{F}_r(B) \arrow[r, "i^{B'}_{B,r}"] & \mathbb{F}_r(B')
\end{tikzcd}$$

for $B = (a'', a] \times [b, b'')$ implies that $i^{B'}_r(a,b) = i^{B'}_{B,r} \cdot i^B_r(a,b)$.

b. In view of Proposition 3.1. item 2., $i^{B'}_{B,r}$ is injective.

c. The composition $i^B_r(a,b) = \pi^B_r(a,b) \cdot i_r(a,b)$ is injective since $p^{ab}_{B,r} \cdot i^B_r(a,b) = p^{(ab)}_{B,r} \cdot \pi^B_r(a,b) \cdot i_r(a,b) = \pi_r(a,b) \cdot i_r(a,b) = id$, (cf diagram (3)).
Clearly a., b. and c. above imply $i^{B'}_r(a,b)$ is injective.

One summarizes the above maps in the diagram (3) below.

$$\begin{tikzcd}
H_r(Y) & \mathbb{F}^h_r(a,b) \arrow[l, "\supseteq"'] \arrow[r, "\pi_r(a,b)"'] \arrow[d, "\pi^B_r(a,b)"'] & \hat{\delta}^h_r(a,b) \arrow[l, bend right, "i_r(a,b)"'] \arrow[dl, bend right, "i^B_r(a,b)"'] \arrow[d, "i^{B'}_r(a,b)"] \\
 & \mathbb{F}^h_r(B) \arrow[ur, "p^{ab}_{B,r}"'] \arrow[r, "i^{B'}_{B,r}"'] & \mathbb{F}^h_r(B').
\end{tikzcd} \tag{3}$$

To simplify the writing, untill the end of this section we will write $\oplus_{(a,b)}$ resp. $\oplus_{(a,b)\in B}$ instead of $\oplus_{(a,b)\in supp\ \delta^h_r}$ resp. $\oplus_{(a,b)\in supp\ \delta^h_r \cap B}$.

Choose a collection of splittings $\mathcal{S} = \{i_r(a,b) \mid (a,b) \in supp\ \delta^h_r\}$, and consider the sum

$$\boxed{{}^{\mathcal{S}}I_r = \bigoplus_{(a,b)} i_r(a,b) : \bigoplus_{(a,b)} \hat{\delta}^h_r(a,b) \to H_r(\tilde{M}),}$$

and for a box B the sums

$$\boxed{{}^{\mathcal{S}}I^B_r = \bigoplus_{(a,b)\in B} i^B_r(a,b) : \bigoplus_{(a,b)\in B} \hat{\delta}^h_r(a,b) \to \mathbb{F}^h_r(B).}$$

Theorem 3.3 *Suppose $\mathcal{S}$ is a collection of splittings.*

1. *For any box $B = (a', a] \times [b, b')$ the linear map ${}^{\mathcal{S}}I^B_r$ is an isomorphism.*
2. *For any (a,b) the space ${}^{\mathcal{S}}I_r(\bigoplus_{\begin{cases}\alpha \le a\\ \beta \ge b\end{cases}} \hat{\delta}^h({}_r\alpha,\beta))$ is contained in $\mathbb{F}^h_r(a,b)$ and ${}^{\mathcal{S}}I^{B(a,b;R)}_r$ defined as ${}^{\mathcal{S}}I_r$ composed with the projection $\mathbb{F}^h_r(a,b) \to \mathbb{F}^h_r(a,b)/(\mathbb{I}_{-\infty}\cap \mathbb{I}^b + \mathbb{I}_a \cap \mathbb{I}^\infty)$ is an isomorphism from $\bigoplus_{\begin{cases}\alpha \le a\\ \beta \ge b\end{cases}} \hat{\delta}^h(\alpha,\beta)$ to $\mathbb{F}^h_r(B)$.*
3. *If $\mathbb{I}^h_{-\infty}(r) := \bigcap_{a\in\mathbb{R}} \mathbb{I}^h_a(r)$, $\mathbb{I}^\infty_h(r) := \cap_{b\in\mathbb{R}}\mathbb{I}^b_h(r)$ [9] and $\pi(r) : H_r(Y) \to H_r(Y)/(\mathbb{I}^h_{-\infty} + \mathbb{I}^\infty_h)$ is the canonical projection then the composition $\pi(r)\cdot({}^{\mathcal{S}}I_r)$ is an isomorphism.*

Proof:

Item 1. As in [2], it suffices to check the result for boxes B with $supp\ \delta^h_r \cap B$ consisting of only one element. This is indeed the case cf Proposition 3.2 item 2. for a box $B(a,b;\epsilon)$ with ϵ small enough.

One introduces the vector spaces $\hat{\mathbb{F}}^h_r(B) := \oplus_{(a,b)\in B}\hat{\delta}^h_r(a,b)$ and $\hat{\mathbb{F}}^h_r := \oplus_{(a,b)}\hat{\delta}^h_r(a,b)$ and for the collection of splittings $\mathcal{S}$ one regards ${}^{\mathcal{S}}I^B_r$ and ${}^{\mathcal{S}}I_r$ as maps

$${}^{\mathcal{S}}I^B_r : \hat{\mathbb{F}}^h_r(B) \to \mathbb{F}^h_r(B)$$

$${}^{\mathcal{S}}I_r : \hat{\mathbb{F}}^h_r \to H_r(Y).$$

For $B = B_1 \sqcup B_2$ with $B_1 = B''_1, B_2 = B'_1$ or $B_1 = B'', B_2 = B'$ as in Figure 1., one has the commutative diagram

$$\begin{tikzcd}
\mathbb{F}^h_r(B_1) \arrow[r] & \mathbb{F}^h_r(B) \arrow[r] & \mathbb{F}^h_r(B_2) \\
\hat{\mathbb{F}}^h_r(B_1) \arrow[r] \arrow[u, "{}^{\mathcal{S}}I^{B_1}_r"'] & \hat{\mathbb{F}}^h_r(B) \arrow[r] \arrow[u, "{}^{\mathcal{S}}I^{B}_r"] & \hat{\mathbb{F}}^h_r(B_2). \arrow[u, "{}^{\mathcal{S}}I^{B_2}_r"']
\end{tikzcd}$$

Manipulation with this diagram as in [2] (decomposition of B as a disjoint union of smaller boxes) permits to establish inductively the result.

Item 2. follows from Item 1. by passing to projective limit as follows.

For $R > 0$ denote by $B(a,b;R)$ the box $B(a,b;R) = (-R+a, a] \times [b, b+R)$, and observe that $B(a,b;\infty) := (-\infty, a] \times [b,\infty) = \cup_{R>0}(B(a,b;R)$ and that $supp\ \delta^h_r \cap B(a,b;R)$ remains constant when R is large enough. This because of Proposition 3.2 item 3. and because $\mathbb{F}^h_r(B(a,b;R))$ is a quotient of the finite dimensional space $\mathbb{F}^h_r(a,b)$.

[9]Note that the definitions of $\mathbb{I}^h_{-\infty}$ and $\mathbb{I}^\infty_h$ are consistent with $\mathbb{I}^h_a = \cap_{a'>a}\mathbb{I}^h_{a'}$ and $\mathbb{I}^b_h = \cap_{b'>b}\mathbb{I}^{b'}_h$

Consider : $\hat{\mathbb{F}}^h_r(B(a,b;\infty)) := \bigoplus_{(a,b)\in(B(a,b:\infty))} \hat{\delta}^h_r(a,b)$. Since the set "$supp\delta^h_r \cap B(a,b;R)$ is constant when R is large" one has $\hat{\mathbb{F}}^h_r(B(a,b;\infty)) = \varprojlim_{R\to\infty} \hat{\mathbb{F}}^h_r(B(a,b;R))$.

Consider $\mathbb{F}^h_r(B(a,b;\infty)) := \mathbb{F}^h_r(a,b)/(\mathbb{I}^h_{-\infty} \cap \mathbb{I}^b_h + \mathbb{I}^h_a \cap \mathbb{I}^\infty_h)$. By the same reason $\mathbb{F}^h_r(B(a,b;\infty)) = \varprojlim_{R\to\infty} \mathbb{F}^h_r(B(a,b;R))$.

Since ${}^{\mathcal{S}}I^{B(a,b;R)}_r$ is an isomorphism for any R and ${}^{\mathcal{S}}I^{B(a,b;\infty)} := \varprojlim^{\mathcal{S}}_{R\to\infty} I^{B(a,b;R)}_r$ one has ${}^{\mathcal{S}}I^{B(a,b;\infty)}$ is an isomorphism.

Item 3.: Note that $\mathbb{R}^2 = \cup_L B(-L,L;\infty)$ and ${}^{\mathcal{S}}I^{\mathbb{R}^2}_r = \varinjlim_{L\to-\infty} {}^{\mathcal{S}}I^{B(-L,L;\infty)}_r$. Since ${}^{\mathcal{S}}I^{B(-L,L;\infty)}_r$ is an isomorphism for any L so is ${}^{\mathcal{S}}I^{\mathbb{R}^2}_r$ which the reader will recognize to be $\pi(r)\cdot({}^{\mathcal{S}}I_r)$.

■

3.2 Definition and properties of δ^f_r and $\hat{\delta}^f_r$.

We apply the previous considerations to $\tilde{f} : \tilde{X} \to \mathbb{R}$ an infinite cyclic cover of $f : X \to \mathbb{S}^1$. In this case we have the deck transformation $\tau : \tilde{X} \to \tilde{X}$ which induces the isomorphism $t_r : H_r(\tilde{X}) \to H_r(\tilde{X})$ and therefore a structure of $\kappa[t^{-1},t]-$module this $\kappa-$vector space. The diagram (4) with the vertical arrows induced by t_r implies Observation 3.4.

$$
\begin{tikzcd}
H_r(\tilde{X}) \arrow[d, "t_r"] & \mathbb{F}^{\tilde{f}}_r(a,b) \arrow[l, "\supseteq"'] \arrow[rr, "\pi_r(a,b)"] \arrow[dr, "\pi^B_{ab,r}"'] \arrow[dd, "t_r(a,b)"] & & \hat{\delta}^{\tilde{f}}_r(a,b) \arrow[dd, "\hat{t}_r(a,b)"] \\
& & \mathbb{F}^{\tilde{f}}_r(B) \arrow[ur, "pr^B_{r,ab}"'] \arrow[dd, "tr(B)" near start] & \\
H_r(\tilde{X}) & \mathbb{F}^{\tilde{f}}_r(a+2\pi,b+2\pi) \arrow[l, "\supseteq"'] \arrow[rr, "\pi_r(a+2\pi,b+2\pi)"] \arrow[dr, "\pi^B_{a'b',r}"'] & & \hat{\delta}^{\tilde{f}}_r(a+2\pi,b+2\pi) \\
& & \mathbb{F}^{\tilde{f}}_r(B+2\pi) \arrow[ur, "pr^B_{r,a'b'}"'] &
\end{tikzcd}
\tag{4}
$$

Observation 3.4 *1. The isomorphism t_r satisfies $t_r(\mathbb{F}^{\tilde{f}}_r(a,b)) = \mathbb{F}^{\tilde{f}}_r(a+2\pi,b+2\pi)$ and $t^{-1}_r(\mathbb{F}^h_r(a,b)) = \mathbb{F}^h_r(a-2\pi,b-2\pi)$.*

2. *For any box $B = (a',a]\times[b,b')$ consider the box $B+2\pi = (a'+2\pi,a+2\pi]\times[b+2\pi,b'+2\pi)$. The isomorphism t_r induces the isomorphisms $t_r(B) : \mathbb{F}^{\tilde{f}}_r(B) \to \mathbb{F}^{\tilde{f}}_r(B+2\pi)$ and then $\hat{t}_r(a,b) : \hat{\delta}^{\tilde{f}}_r(a,b) \to \hat{\delta}^{\tilde{f}}_r(a+2\pi,b+2\pi)$.*

3. *$\mathbb{I}^{\tilde{f}}_{-\infty}(r)$ and $\mathbb{I}^{\infty}_{\tilde{f}}(r)$ are invariant, hence $\kappa[t^{-1},t]-$submodules, therefore $H_r(\tilde{M})/(\mathbb{I}^{\tilde{f}}_{\infty}(r)+\mathbb{I}^{\infty}_{\tilde{f}})(r)$ is a $\kappa[t^{-1},t]-$ module.*

Proposition 3.5 $\mathbb{I}^{\tilde{f}}_{-\infty}(r) = \mathbb{I}^{\infty}_{\tilde{f}}(r) = T(H_r(\tilde{X}))$.

Proof: If $x \in T(H_r(\tilde{X}))$ then there exists an integer $k \in \mathbb{Z}$ and a polynomial $P(t) = \alpha_n t^n + \alpha_{n-1}t^{n-1}\cdots\alpha_1 t + \alpha_0$, $\alpha_i \in \kappa$, $\alpha_0 \neq 0$ such that $P(t)\, t^k x = 0$. Let $y = t^k x$. Since $H_r(\tilde{X}) = \cup_b \mathbb{I}^b_{\tilde{f}}(r)$ one has $y \in \mathbb{I}^b(r)$ for some $b \in \mathbb{R}$. Since $P(t)y = 0$ one concludes that
$y = -(\alpha_n/\alpha_0)t^{n-1}\cdots-(\alpha_1/\alpha_0)ty$ and therefore $y \in \mathbb{I}^{b+2\pi}(r)$. Repeating the argument one concludes that $y \in I^{b+2\pi k}$ for any k, hence $y \in \mathbb{I}^\infty(r)$. Since $x = t^{-k}y$, one has $x \in \mathbb{I}^\infty.(r)$ Hence $T(H_r(\tilde{X})) \subseteq I^\infty(r)$.

Let $x \in \mathbb{I}^{\infty}(r)$. Since $H_r(\tilde X) = \cup_a \mathbb{I}^{\tilde f}_a(r)$ then $x \in \mathbb{I}_a(r)$ for some $a \in \mathbb{R}$, and if in addition $x \in \mathbb{I}^{\infty}(r)$ then by Observation 3.4 3. all $x, t^{-1}x, t^{-2}x, \cdots t^{-k}x, \cdots \in \mathbb{I}_a(r) \cap \mathbb{I}^{\infty}(r)$. Since by Proposition 3.1 (1.) the dimension of $\mathbb{I}_a(r) \cap \mathbb{I}^{\infty}(r)$ is finite, there exists $\alpha_{i_1}, \cdots \alpha_{i_k}$ such that $(\alpha_{i_1} t^{-i_1} + \cdots \alpha_{i_k} t^{-i_k})x = 0$. This makes $x \in T(H_r(\tilde X)$. Hence $\mathbb{I}^{\infty}(r) \subseteq T(H_r(\tilde X))$. Therefore $\mathbb{I}^{\infty}(r) = T(H_r(\tilde X))$. By a similar argument one concludes that $H_r(\tilde X) = \mathbb{I}_{-\infty}(r)$.

■

Recall from Introduction that $\langle\rangle : \mathbb{R}^2 \to \mathbb{T} = \mathbb{R}^2/\mathbb{Z}$ denotes the map which assigns to $(a,b) \in \mathbb{R}^2$ its equivalence class $\langle a,b\rangle \in \mathbb{T}$. We use the following notations:

– For $(a,b) \in \mathbb{R}^2$ denote by $\mathbb{T}\langle a,b\rangle$ the image by $\langle\rangle$ of the set $\{(x,y) \in \mathbb{R}^2 \mid x \leq a, y \geq b\}$; clearly for $a' \leq a, b' \geq b$ one has $\mathbb{T}\langle a',b'\rangle \subseteq \mathbb{T}\langle a,b\rangle$.

– For a box $B = (a-\alpha, a] \times [b, b+\beta)$ denote by $(B+c)$ the box $(B+c) := (a-\alpha+c, a+c] \times [b+c, b+c+\beta)$, and by $\langle B\rangle \subseteq \mathbb{T}$ the image of B by the map $\langle\rangle$.

– One calls the box B *small* if $0 < \alpha, \beta < 2\pi$, in which case the restriction of $\langle\rangle$ to B is one to one; clearly if B is a small box so is any $(B+c)$ and $(B+2\pi k) \cap (B + 2\pi(k+1)) = \emptyset$ for $k \neq k'$.

– For $\langle a,b\rangle \in \mathbb{T}$ and $\langle B\rangle \subseteq \mathbb{T}$ with B a small box introduce:

$$\begin{aligned} \mathbb{F}^{\tilde f}_r(\langle a,b\rangle) &:= \sum_{k\in\mathbb{Z}} \mathbb{F}^{\tilde f}_r(a+2\pi k, b+2\pi k) \subseteq H_r(\tilde X) \\ \mathbb{F}^{f}_r(\langle a,b\rangle) &:= \pi(r)(\mathbb{F}^{\tilde f}_r(\langle a,b\rangle)) \subseteq H_r(\tilde X)/TH_r(\tilde X) \end{aligned} \tag{5}$$

$$\begin{aligned} \mathbb{F}'^{\tilde f}_r(\langle B\rangle) := \sum_{k\in\mathbb{Z}} \mathbb{F}'^{\tilde f}_r(B+2\pi k) &= (\mathbb{F}^{\tilde f}_r(\langle a',b\rangle) + \mathbb{F}^{\tilde f}_r(\langle a,b'\rangle)) \subseteq \mathbb{F}^{\tilde f}_r(\langle a,b\rangle) \subseteq H_r(\tilde X) \\ \mathbb{F}'^{f}_r(\langle B\rangle) &:= \pi(r)(\mathbb{F}'^{\tilde f}_r(\langle B\rangle)) \subseteq H^N_r(X;\xi) \end{aligned} \tag{6}$$

Note the difference between $(\cdots)^{\tilde f}$ and $(\cdots)^f$; the first has the image in $H_r(\tilde X)$ the second in $H^N_r(X;\xi)$. In view of Proposition (3.5) $T(H_r(\tilde X) \subseteq \mathbb{F}^{\tilde f}(a,b)$, one has:

$$\frac{\mathbb{F}^{\tilde f}_r(\langle a,b\rangle)}{\mathbb{F}'^{\tilde f}_r(\langle B\rangle)} = \frac{\mathbb{F}^{f}_r(\langle a,b\rangle)}{\mathbb{F}'^{f}_r(\langle B\rangle)}.$$

Denote by $\mathbb{F}^f_r(\langle B\rangle)$ the quotient

$$\mathbb{F}^f_r(\langle B\rangle) := \frac{\mathbb{F}^{\tilde f}_r(\langle a,b\rangle)}{\mathbb{F}'^{\tilde f}_r(\langle B\rangle)} = \frac{\mathbb{F}^{f}_r(\langle a,b\rangle)}{\mathbb{F}'^{f}_r(\langle B\rangle)}. \tag{7}$$

There is no reason to use separate exponent $(\cdots)^{\tilde f}$ and $(\cdots)^f$ since the quotients do not naturally lie in any of $H_r(\tilde X)$ or $H^N_r(X;\xi)$.

Introduce

$$\hat\delta^f_r(\langle a,b\rangle) = \bigoplus_{k\in\mathbb{Z}} \hat\delta^{\tilde f}_r(a+2\pi k, b+2\pi k) \tag{8}$$

and for a choosen collection of splittings $\mathcal{S}$ consider the diagrams

$$\begin{array}{ccccc}
\bigoplus_{\langle\alpha,\beta\rangle}\hat\delta^f(\langle\alpha,\beta\rangle) & \xrightarrow{s_{I_r}} & H_r(\tilde X) & \xrightarrow{\pi(r)} & H^N_r(X;\xi) \\
\uparrow\subseteq & & \uparrow\subseteq & & \uparrow\subseteq \\
\bigoplus_{\langle\alpha,\beta\rangle\in\mathbb{T}(a,b)}\hat\delta^f(\alpha,\beta) & \longrightarrow & \mathbb{F}^{\tilde f}_r(\langle a,b\rangle) & \longrightarrow & \mathbb{F}^f_r(\langle a,b\rangle) \\
\uparrow\subseteq & & \uparrow\subseteq & & \uparrow\subseteq \\
\bigoplus_{\langle\alpha,\beta\rangle\in\mathbb{T}(a',b)\cup\mathbb{T}(a,b')}\hat\delta^f(\langle\alpha,\beta\rangle) & \longrightarrow & \mathbb{F'}^{\tilde f}_r(\langle B\rangle) & \longrightarrow & \mathbb{F'}^f_r(\langle B\rangle)
\end{array} \tag{9}$$

and

$$\begin{array}{ccccc}
\bigoplus_{\langle\alpha,\beta\rangle\in\mathbb{T}(a,b)}\hat\delta^f(\langle\alpha,\beta\rangle) & \xrightarrow{s_{I_r}} & \mathbb{F}^{\tilde f}_r(\langle a,b\rangle) & \longrightarrow & \mathbb{F}^f_r(\langle a,b\rangle) \\
\downarrow\pi^{\langle B\rangle}_r & & \downarrow\pi^{\langle B\rangle}_r & & \downarrow\pi^{\langle B\rangle}_r \\
\bigoplus_{\langle\alpha,\beta\rangle\in\langle B\rangle}\hat\delta^f(\langle\alpha,\beta\rangle) & \longrightarrow & \mathbb{F}^{\tilde f}_r(\langle B\rangle) & \xrightarrow{=} & \mathbb{F}^f_r(\langle B\rangle).
\end{array} \tag{10}$$

In view of Observation 3.4 one has:

1. $\hat\delta^f(\langle\alpha,\beta\rangle)$is a free $\kappa[t^{-1},t]-$module with the multiplication by t given by the isomorphism $\bigoplus_{k\in\mathbb{Z}}\hat t_r(\alpha+2\pi k,\beta+2\pi k)$,
2. the vector spaces involved in the above diagrams are all $\kappa[t^{-1},t]-$modules with those located on the left and right columns free $\kappa[t^{-1},t]-$modules,
3. all arrows but the left side horizontal ones in both diagrams are $\kappa[t^{-1},t]-$linear.

In view of Theorem 3.3 (for B a small box) the composition of horizontal arrows in each row of both diagrams are isomorphisms.

Definition 3.6 *The collection of splittings $\mathcal{S}=\{i_r(a,b):\hat\delta^{\tilde f}_r(a,b)\to\mathbb{F}^{\tilde f}_r(a,b)\}$ which satisfy $t_r\cdot i_r(a,b)=i_r(a+2\pi,b+2\pi)\cdot\hat t_r(a,b)$ is called a collection of* **compatible** *splittings.*

Such collections exist. Indeed, it suffices to chose *splittings* only for $\{(a,b)\in supp\ \delta^{\tilde f}_r, 0\leq a<2\pi\}$, observe that any $(a',b')\in supp\ \delta^{\tilde f}$ is of the form $a'=a+2\pi k, b'=b+2\pi k$ for some integer $k\in\mathbb{Z}$ with $0\leq a<2\pi$ and take $i_r(a',b'):=(\hat t_r)^k\cdot i_r(a,b)(\hat t_r)^{-k}$.

If the splittings are compatible then all arrows in the diagrams (9) and (10) are $\kappa[t^{-1},t]-$linear therefore we have the following.

Proposition 3.7 *Both $\mathbb{F}^f_r(\langle a,b\rangle)$ and $\mathbb{F'}^f_r(\langle B\rangle)$ are split free submodules of $H^N_r(X;\xi)$, and $\mathbb{F}^f_r(\langle B\rangle)$ is a quotient of split free submodules hence also free. In particular $\hat\delta^f_r(\langle a,b\rangle)$, which is canonically isomorphic to $\mathbb{F}^f_r(\langle B(a,b;\epsilon)\rangle)$ for $\epsilon<\epsilon(f)$, is a quotient of split free submodules.*

Definition of δ^f_r, $\hat\delta^f_r$, $\hat{\hat\delta}^f_r$ and $P^f_r(z)$.
In view of Proposition 3.7 one considers the configurations δ^f_r defined by the formula:

$$\delta^f_r(\langle a,b\rangle):=\delta^{\tilde f}_r(a,b) \tag{11}$$

and the configuration

$$\hat{\delta}^f_r \text{ defined by (8).}$$

We use the identification of $\mathbb{T}$ with $\mathbb{C}\setminus 0$ provided by the map $\langle a,b\rangle = e^{ia+(b-a)}$ and if $z_1, z_2, \cdots z_k \in \mathbb{C}\setminus 0$ are the points in the support of δ^f_r, define the polynomial

$$P^f_r(z) := \prod (z-z_i)^{(-1)^i \delta^f_r(z_i)}.$$

When $\kappa = \mathcal{C}$ the von Neumann completion converts $\mathbb{C}[t^{-1},t]-$ into the von Neumann algebra $L^\infty(\mathbb{S}^1)$ and an $\mathbb{C}[t^{-1},t]-$valued inner product converts $H^N_r(M;\xi)$, $\mathbb{F}'^f_r(\langle B\rangle)$ $\mathbb{F}^f_r(\langle B\rangle)$ and $\hat{\delta}(\langle a,b\rangle)$ into Hilbert submodules. The von Neumann completion discussed in section 2 leads to the configuration $\hat{\hat{\delta}}^f_r$ of mutually orthogonal $L^\infty(\mathbb{S}^1)-$Hilbert modules.

One takes

$$\hat{\hat{\delta}}^f_r = \text{ von Neumann completion of } \hat{\delta}^{\mathrm{f}}_{\mathrm{r}}.$$

4 Proof of Theorem 1.1 and 1.2

Theorem 1.1:

Item 1. is verified by Proposition 3.2 (1.).

Item 2. follows from the fact that $\pi(r)\cdot({}^{\mathcal{S}}I_r)$ is an isomorphism as established in Theorem 3.3 above. The configuration $\hat{\delta}^f_r$ is actually a special configuration as described in section 2 with $\hat{\delta}^f_r(\langle ab\rangle) = \mathbb{F}^f_r(\langle a,b\rangle)/\mathbb{F}'^f_r(\langle B(a,b;\epsilon)\rangle)$ for any $\epsilon < \epsilon(f)$.

For Item 3. one proceeds as in the proof of Theorem 4.1 item 4. in [2]. For example in case X is a smooth manifold, possibly with boundary, any angle valued map is arbitrary closed to a Morse angle valued map f which takes different values on different critical points. Then the same remains true for $\tilde{f}: \tilde{X} \to \mathbb{R}$, an infinite ciclic cover of this Morse map; this guarantees that for the sequence of critical values $\cdots c_{i-1} < c_i < c_{i+1} < \cdots$, the inclusion induced linear maps $H_*(\tilde{X}_{c_{i-1}}) \to H_*(\tilde{X}_{c_i})$ have co-kernel of dimension at most one. As argued in [2], in the proof of Theorem 1 (4.), this implies that $\delta^{\tilde{f}}_r$ and then δ^f_r takes as values only 0 or 1. In the same way as in [2], with the help of compact Hilbert cube manifolds, one derives Item 3. in the generality stated.

Theorem 1.2.

Proof: In view of Observations 2.1, 2.2 and 2.3 the proof of item 1. is the same as of Theorem 4.2 in [2] provided we replace $f: X \to \mathbb{R}$, by $\tilde{f}: \tilde{X} \to R$ a lift of $f: X \to \mathbb{S}^1$ representing ξ. The basic ingredients, Proposition 3.16, (based on Lemma 3.17 and Lemma 3.18) in [2] holds for $h: Y \to \mathbb{R}$, Y a locally compact ANR and h a proper map, instead of $f: X \to \mathbb{R}$, X compact. The steps of the proof are similar to the one described in subsection 4.2 in [2] and are summarized below.

1. For a pair (X,ξ), X compact ANR let $\tilde{X}$ be an infinite cyclic cover associated to ξ and let $C_\xi(X,\mathbb{S}^1)$, denote the set of maps in the homotopy class defined by ξ equipped with the compact open topology. Observe that:

   – the compact open topology is induced from the complete metric $D(f,g)$ and $D(f,g) = D(\tilde{f},\tilde{g})$ for appropriate liftings.

   – for $f,g \in C_\xi(X,\mathbb{S}^1)$ with $D(f,g) < \pi$ and any sequence $0 = t_0 < t_1 \cdots t_{N-1} < t_N = 1$ by Observation 2.1 item 3. the canonical homotopy h_t from f to g satisfies

$$D(f,g) = \sum_{0\le i<N} D(h_{t_{i+1}}, h_{t_i}). \tag{12}$$

2. For X is a simplicial complex let $\mathcal{U} \subset C_\xi(X, \mathbb{S}^1)$ be the subset of p.l. maps. One can verify that:

   – $\mathcal{U}$ is a dense subset in $C_\xi(X, \mathbb{S}^1)$,

   – if $f, g \in \mathcal{U}$, $D(f,g) < \pi$ then for the canonical homotopy each $h_t \in \mathcal{U}$, hence $\epsilon(h_t) > 0$, hence for any $t \in [0,1]$ there exists $\delta(t) > 0$ s.t. $t', t'' \in (t - \delta(t), t + \delta(t))$ implies $D(h_{t'}, h_t) < \epsilon(h_t)/3$. Both statements 2. and 3. are argued as in [2].

3. Consider the space of configurations $\mathcal{C}_{b_r}(\mathbb{T})$ viewed as the $b_r = \beta_r^N(X;\xi)-$ symmetric product of $\mathbb{T}$ equipped with the induced metric, $\underline{D}$, which is complete. Since any map in $\mathcal{U}$ is tame, in view Proposition (3.16) in [2], $f, g \in \mathcal{U}$ with $D(f,g) < \epsilon(f)/3$ imply

$$\underline{D}(\delta_r^f, \delta_r^g) \leq 2D(f,g). \tag{13}$$

This suffices to conclude the continuity of the assignment $f \rightsquigarrow \delta_r^f$.

To finalize the proof of Theorem (1.2) item 1. we check first (step 1.) that the inequality (13) extends to all $f, g \in \mathcal{U}$, second (step 2.) that the inequality (13) extends to all $f, g \in C_\xi(X, \mathbb{S}^1)$ for X a finite simplicial complex, third (step 3.) that the inequality (13) extends to all $f, g \in C_\xi(X, \mathbb{S}^1)$ for X an arbitrary compact ANR.

- *Step 1.:* In view of the continuity of the assignment $f \rightsquigarrow \delta_r^f$ it suffices to verify step 1. for $f, g \in U$ s.t. $D(f,g) < \pi$ since $D(f,g) \leq \pi$. Start with $f, g \in \mathcal{U}$ and consider the canonical homotopy $\tilde{h}_t = t\tilde{f} + (1-t)\tilde{g}$, $t \in [0,1]$ between two lifts lifts $\tilde{f}, \tilde{g}$ of f and g which satisfy $D(f,g) = D(\tilde{f}, \tilde{g})$. Note that each $\tilde{h}_t$ satisfies $\tilde{h}_t(\mu(n,x)) = \tilde{h}_t(x) + 2\pi n$, hence is a lift, and each h_t is p.l.

  Choose a sequence $0 < t_1 < t_3 < t_5, \cdots t_{2N-1} < 1$ such that for $i = 1, \cdots, (2N-1)$ the intervals $(t_{2i-1} - \delta(t_{2i-1}), t_{2i-1} + \delta(t_{2i-1}))$ cover $[0,1]$ and $(t_{2i-1}, t_{2i-1} + \delta(t_{2i-1})) \cap (t_{2i+1} - \delta(t_{2i+1}), t_{2i+1}) \neq \emptyset$. This is possible in view of the compacity of $[0,1]$.

  Take $t_0 = 0, t_{2N} = 1$ and $t_{2i} \in (t_{2i-1}, t_{2i-1} + \delta(t_{2i-1})) \cap (t_{2i+1} - \delta(t_{2i+1})$. To simplify the notation abbreviate h_{t_i} to h_i. In view of 2. and 3. above (inequality (22)) one has:

  $|t_{2i-1} - t_{2i}| < \delta(t_{2i-1})$ implies $D(\delta^{h_{2i-1}}, \delta^{h_{2i}}) < 2D(h_{2i-1}, h_{2i})$ and

  $|t_{2i} - t_{2i+1}| < \delta(t_{2i+1})$ implies $D(\delta^{h_{2i}}, \delta^{h_{2i+1}}) < 2D(h_{2i}, h_{2i+1})$
  Then we have

$$\underline{D}(\delta^f, \delta^g) \leq \sum_{0 \leq i < 2N-1} D(\delta^{h_i}, \delta^{h_{i+1}}) \leq 2 \sum_{0 \leq i < 2N-1} D(h_i, h_{i+1}) = D(f,g).$$

- *Step 2.:* Suppose X is a simplicial complex. In view of the density of $\mathcal{U}$ and of the completeness of the metrics on $C_\xi(X;\mathbb{S}^1)$ and $\mathcal{C}_{b_r}(\mathbb{T})$, the inequality (13) extends to the entire $C_\xi(X;\mathbb{S}^1)$ in case X is a simplicial complex. Indeed the assignment $\mathcal{U} \ni f \rightsquigarrow \delta_r^f \in C_{b_r}(\mathbb{R}^2)$ preserve the Cauchy sequences.

- *Step 3.:* We verify the inequality (13) for $X = K \times Q$, K simplicial complex and Q the Hilbert cube. One proceed exactly as in [2]. Since by Theorem 2.3 item 2 in [2] any compact Hilbert cube manifold is homeomorphic to $K \times Q$ for some finite simplicial complex K, the inequality (13) continues to hold for X any compact Hilbert cube manifold. Since for any X a compact ANR, by Theorem 2.3 item 1 in [2], $X \times Q$ is a Hilbert cube manifold, $I : C(X;\mathbb{R}) \to C(X \times Q;\mathbb{R})$ defined by $I(f) = \overline{f}_Q$ is an isometric embedding and $\delta^f = \delta^{\overline{f}_Q}$, hence the inequality (13) holds for any X a compact ANR.

For item 2, suppose that we are in the situation to realize $\hat{\delta}^f_r$ as a configuration of free submodules of $H^N(X, \xi)$, as it is the case in the presence of a collection of compatible splittings or, in the case $\kappa = \mathbb{C}$, after the von Newman completion into the $L^\infty(\mathbb{S}^1)-$Hilbert module $H^{L_1}(\tilde{X})$. Note that Lemma 3.18 and Lemma 3.17 in [2] imply that for a given f and $(a, b) \in CR(f) \times CR(f)$ and $\epsilon < \epsilon(f)$ and any g with $||g - f||_\infty < \epsilon/3$, inside $H^N_r(X; \xi)$, the following two spaces are exactly the same.

The first is

$$\sum_{(a',b')\in D(a,b;\epsilon)\cap supp\delta^{\tilde{g}}} (\hat{i}_r(\langle a', b'\rangle)(\hat{\delta}^g(\langle a', b'\rangle)$$

with $D(a, b; \epsilon) := (a - \epsilon, a + \epsilon] \times [b - \epsilon, b + \epsilon)$, the second is

$$\hat{i}_r(\langle a, b\rangle)(\hat{\delta}^f(\langle a, b\rangle).$$

This is explained in [2] in the first three lines after the proof of Lemma 3.18. This insures the continuity in the fine collision topology and therefore in the natural collision topology.

■

5 Proof of Theorem 1.3

In view of Theorem 1.2 it suffices to establish the result for tame maps. We will suppose that both a, b are regular values in strong sense. In case X is a compact manifold a number c is such regular value if $f^{-1}(c)$ is a codimension one sub manifold and for a small neighborhood $U \ni c$, the map $f^{-1}(U) \to U$ is a locally trivial bundle.

To establish $\hat{\delta}^{\tilde{f}}_r(\langle a, b\rangle) \simeq \hom_{\kappa[t^{-1},t]}(\hat{\delta}^{\tilde{f}}_{n-r}(\langle b, a\rangle), \kappa[t^{-1}, t])$ for $\tilde{f} : \tilde{M} \to \mathbb{R}$ the infinite cyclic cover of $f : M \to \mathbb{S}^1$ we apply the same strategy as in section 6 of [2]. Note that if items 1.and 2. are established for such regular values a, b, in view of Theorem 1.2, they will hold for any any a, b and any continuous map.

First one shows that the diagrams (17) and (18) in section 6 part I continue to hold if one replaces M, M_a, M^a by $\tilde{M}, \tilde{M}_a, \tilde{M}^a$ and $H_r(M), H_r(M_a), H_r(M^a), H_r(M, M_a), H_r(M, M^a)$ by $H^{BM}_r(\tilde{M})$, $H^{BM}_r(\tilde{M}_a), H^{BM}_r(\tilde{M}^a), H^{BM}_r(\tilde{M}, \tilde{M}_a), H^{BM}_r(\tilde{M}, \tilde{M}^a)$. They become the diagrams (15) and (16) below. Here H^{BM}_r denotes Borel–Moore homology. For the readers familiar with Borel Moore homology these diagrams are familiar, for those who are not one can regard H^{BM}_r as notation for the right side of the equalities (14).

$$\begin{aligned}
H^{BM}_r(\tilde{M}) &= \varprojlim_{0<l\to\infty} H_r(\tilde{M}, \tilde{M}_{-l} \sqcup \tilde{M}^t), \\
H^{BM}_r(\tilde{M}_a) &= \varprojlim_{0<l,t\to\infty} H_r(\tilde{M}_a, \tilde{M}_{a-l}), \\
H^{BM}_r(\tilde{M}^a) &= \varprojlim_{0<l\to\infty} H_r(\tilde{M}, \tilde{M}^{a+l}), \\
H^{BM}_r(\tilde{M}, \tilde{M}_a) &= \varprojlim_{0<l\to\infty} H_r(\tilde{M}, \tilde{M}_a \sqcup \tilde{M}^{a+l}), \\
H^{BM}_r(\tilde{M}, \tilde{M}^a) &= \varprojlim_{0<l\to\infty} H_r(\tilde{M}, \tilde{M}^a \sqcup \tilde{M}_{a-l}).
\end{aligned} \tag{14}$$

$$\begin{array}{ccccc} H_r^{BM}(\tilde M_a) & \xrightarrow{i_a(r)} & H_r^{BM}(\tilde M) & \xrightarrow{j_a(r)} & H_r^{BM}(\tilde M, \tilde M_a) \\ \downarrow{\scriptstyle PD_a^1} & & \downarrow{\scriptstyle PD} & & \downarrow{\scriptstyle PD_a^2} \\ H^{n-r}(\tilde M, \tilde M^a) & \xrightarrow{s^a(n-r)} & H^{n-r}(\tilde M) & \xrightarrow{r^a(n-r))} & H^{n-r}(\tilde M^a) \\ \downarrow & & \downarrow & & \downarrow \\ (H_{n-r}(\tilde M, \tilde M^a))^* & \xrightarrow{(j^a(n-r))^*} & (H_{n-r}(\tilde M))^* & \xrightarrow{(i^a(n-r))^*} & (H_{n-r}(\tilde M^a))^* \end{array} \tag{15}$$

$$\begin{array}{ccccc} H_r^{BM}(\tilde M^b) & \xrightarrow{i^b(r)} & H_r^{BM}(\tilde M) & \xrightarrow{j^b(r)} & H_r^{BM}(\tilde M, \tilde M^b) \\ \downarrow{\scriptstyle PD_1^b} & & \downarrow{\scriptstyle PD} & & \downarrow{\scriptstyle PD_2^b} \\ H^{n-r}(\tilde M, \tilde M_b) & \xrightarrow{s_b(n-r)} & H^{n-r}(\tilde M)) & \xrightarrow{r_b(n-r)} & H^{n-r}(\tilde M_b) \\ \downarrow & & \downarrow & & \downarrow \\ (H_{n-r}(\tilde M, \tilde M_b))^* & \xrightarrow{(j_b(n-r))^*} & (H_{n-r}(\tilde M))^* & \xrightarrow{(i_b(n-r))^*} & (H_{n-r}(\tilde M_b))^* \end{array} \quad . \tag{16}$$

If one uses $H_r^{BM}(\cdots)$ instead of $H_r(\cdots)$ one can also consider ${}^{BM}\mathbb{F}_r^{\tilde f}(a,b)$ and ${}^{BM}\hat\delta_r^{\tilde f}(a,b)$ instead of $\mathbb{F}_r^{\tilde f}(a,b)$ and $\hat\delta_r^{\tilde f}(a,b)$. We fix the attention to the Poincaré Duality isomorphism, (the composition of the vertical arrows in the middle of diagram 15 or 16)

$$PD_r^{BM} : H_r^{BM}(\tilde M) \xrightarrow{PD_r} H^{n-r}(\tilde M) \xrightarrow{=} (H_{n-r}(\tilde M))^*$$

Note that all three terms of this sequence are $\kappa[t^{-1}, t]-$modules and the two arrows are $\kappa[t^{-1}, t]-$linear.

Let us remind a couple of definitions from [2].

- For $a, b \in \mathbb{R}$ denote by $\mathbb{G}_r^{\tilde f}(a,b) := H_r(\tilde M)/\mathbb{I}_a^{\tilde f} + \mathbb{I}_{\tilde f}^b$ and by $p(r) : H_r(\tilde M) \to \mathbb{G}_r^{\tilde f}(a,b)$ the canonical projection on the quotient space.
- For a box $B = (a', a] \times [b, b')$ denote by $\mathbb{G}_r^{\tilde f}(B) := \ker(\mathbb{G}_r^{\tilde f}(a', b') \to \mathbb{G}_r^{\tilde f}(a', b) \times_{\mathbb{G}_r^{\tilde f}(a,b)} \mathbb{G}_r^{\tilde f}(a, b')$, by $u_r : \mathbb{G}_r^{\tilde f}(B) \to \mathbb{G}_r^{\tilde f}(a', b')$ the canonical inclusion and by $\theta_r(B) : \mathbb{F}_r^{\tilde f}(B) \to \mathbb{G}_{n-r}^{\tilde f}(B)$ the canonical isomorphism described in [2] Proposition 4.5.
- For the box $B = (a', a] \times [b, b')$ denote by B' the box $B' = (b, b'] \times [a', a)$.

Proposition 5.1

1. *For any a, b regular values the Poincaré Duality isomorphism restricts to an isomorphism*

$$PD_r^{BM}(a,b) :{}^{BM} \mathbb{F}_r^{\tilde f}(a,b) \to (\mathbb{G}_{n-r}^{\tilde f}(b,a))^*.$$

2. *For any box B with all a, a', b, b' regular values PD_r^{BM} induces the isomorphisms $PD_r^{BM}(a,b)$, $PD_r^{BM}(B)$, making the diagram below commutative.*

$$
\begin{tikzcd}
{}^{BM}\mathbb{F}_r^{\tilde f}(a',b') \arrow[rr, "PD_r^{BM(a',b')}"] \arrow[d] & & (\mathbb{G}_{n-r}^{\tilde f}(b',a'))^* \arrow[d] & & \\
{}^{BM}\mathbb{F}_r^{\tilde f}(a,b) \arrow[rr, "PD_r^{BM}"] \arrow[dr] \arrow[dd] & & (\mathbb{G}_{n-r}^{\tilde f}(b,a))^* \arrow[dr, "u_{n-r}^*"] \arrow[dd] & & \\
 & {}^{BM}\mathbb{F}_r^{\tilde f}(B) \arrow[rr, "p_r^*"] & & (\mathbb{G}_{n-r}^{\tilde f}(B'))^* \arrow[r, "\theta_{n-r}^*"] & (\mathbb{F}_{n-r}^{\tilde f}(B')^* \\
H_r^{BM}(\tilde M) \arrow[rr, "PD_r^{BM}"] & & (H_{n-r}(\tilde M))^* & &
\end{tikzcd}
\tag{17}
$$

Proof:

Item 1.: In view of diagrams (15) and (16) one has $img\ i_a(r) \cap img\ i^b(r) = \ker j_a(r) \cap \ker j^b(r) = \ker(i^a(n-r))^* \cap \ker(i_b(n-r))^* = (coker(i_b(n-r) \oplus i^a(n-r))^* = (\mathbb{G}_{n-r}^{\tilde f}(b,a))^*$. The first equality holds by exactness of the first rows in the diagrams, the second by the equality of the top and bottom right horizontal arrows and the third by linear algebra duality.

Item 2.: Note that the image of the diagram

$$
{}^{BM}\mathcal{F}(B)^* := \left\{
\begin{tikzcd}
{}^{BM}\mathbb{F}_r^{\tilde f}(a',b') \arrow[r] \arrow[d] & {}^{BM}\mathbb{F}_r^{\tilde f}(a,b') \arrow[d] \\
{}^{BM}\mathbb{F}_r^{\tilde f}(a',b) \arrow[r] & {}^{BM}\mathbb{F}_r^{\tilde f}(a,b)
\end{tikzcd}
\right.
$$

by ${}^{BF}PD_r$ is the diagram

$$
\left\{
\begin{tikzcd}
(\mathbb{G}_{n-r}^{\tilde f}(b',a'))* \arrow[r] \arrow[d] & (\mathbb{G}_{n-r}^{\tilde f}(b',a))^* \arrow[d] \\
(\mathbb{G}_r^{\tilde f}(b,a'))^* \arrow[r] & (\mathbb{G}_{n-r}^{\tilde f}(b,a))^*
\end{tikzcd}
\right.
$$

which is the dual of the diagram

$$
\mathcal{G}(B') := \left\{
\begin{tikzcd}
\mathbb{G}_{n-r}^{\tilde f}(b,a) \arrow[r] \arrow[d] & (\mathbb{G}_{n-r}^{\tilde f}(b',a) \arrow[d] \\
\mathbb{G}_r^{\tilde f}(b,a') \arrow[r] & (\mathbb{G}_{n-r}^{\tilde f}(b',a')
\end{tikzcd}
\right.
$$

Therefore ${}^{BM}PD_r$ induces a linear map from ${}^{BM}\mathbb{F}_r(B) = coker\mathcal{F}(B)$ to $(\ker(\mathcal{G}(B'))^* = \mathbb{G}_{n-r}^{\tilde f}(B')$.
■

For c, c' critical values of $\tilde f$, by choosing $a' = c - \epsilon, a = c + \epsilon, b = c' - \epsilon, b' = c + \epsilon, \epsilon < \epsilon(f)$, and in view of Proposition (3.2) item 2., ${}^{BM}\hat\delta_r^{\tilde f}(c,c') = \mathbb{F}_r^{\tilde f}(B)$ and $\hat\delta_{n-r}^{\tilde f}(c',c) = \mathbb{F}_{n-r}^{\tilde f}(B') = \mathbb{G}_{n-r}^{\tilde f}(B')$. Then from diagram (17) one derives

$$\begin{tikzcd}
{}^{BM}\mathbb{F}_r^{\tilde f}(a,b) \arrow[rr, "PD_r^{BM}(a,b)"] \arrow[dr] \arrow[dd] & & (\mathbb{G}_{n-r}^{\tilde f}(b,a))^* \arrow[dr, "u^*_{n-r}"] \arrow[dd] & \\
& {}^{BM}\hat\delta_r^{\tilde f}(a,b) \arrow[rr, "p_r^*"] & & (\hat\delta_{n-r}^{\tilde f}(b,a))^* \\
H_r^{BM}(\tilde M) \arrow[rr, "PD_r^{BM}"] & & (H_{n-r}(\tilde M))^* &
\end{tikzcd} \tag{18}$$

with the horizontal arrows isomorphisms, the vertical injective and the oblique arrows surjective.

The key observation for finalizing items 1. and 2. is the following proposition.

Proposition 5.2 *The $\kappa-$ linear maps $\mathbb{F}_r^{\tilde f}(a,b) \to^{BM} \mathbb{F}_r^{\tilde f}(a,b)$*

1. *are compatible with the deck transformation,*
2. *are surjective,*
3. *have the kernel independent on (a,b), equal to the kernel of the $\kappa[t^{-1},t]-$linear map $H_r(\tilde M) \to H_r^{BM}(\tilde M)$ which is equal to $T(H_r(\tilde M)$.*

Precisely one shows that one has a natural short exact sequence

$$0 \to C_r(M) \to \mathbb{F}_r^{\tilde f}(a,b) \to^{BM} \mathbb{F}_r^{\tilde f}(a,b) \to 0$$

which is compatible with the action provided by the deck transformations and leaving $C_r(M)$ invariant, and $C_r(M)$ is exactly the $\kappa[t^{-1},t]-$torsion of the $H_r(\tilde M)$. Precisely one has the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & C_r(M) \arrow[r] \arrow[d, "t_r"] & \mathbb{F}_r^{\tilde f}(a,b) \arrow[r] \arrow[d, "t_r"] & {}^{BM}\mathbb{F}_r^{\tilde f}(a,b) \arrow[r] \arrow[d, "t_r^{BM}"] & 0 \\
0 \arrow[r] & C_r(M) \arrow[r] & \mathbb{F}_r^{\tilde f}(a+2\pi,b+2\pi) \arrow[r] & {}^{BM}\mathbb{F}_r^{\tilde f}(a+2\pi,b+2\pi) \arrow[r] & 0
\end{tikzcd} \tag{19}$$

with $C_r(M)$ as stated.

The proof is derived from the diagram (20) below where $-l < a' < a$ and $b < b' < t$. By passing to limits $l, t \to \infty$, diagram (20) induces diagram (21) which provides the relation between $\mathbb{F}_r(a,b), \mathbb{F}_r(a',b'), H_r(\tilde M)$ and their Borel-Moore versions.

$$
\begin{tikzcd}
H_{r-1}(\tilde M_{-l}) \arrow[r, "="] & H_{r-1}(\tilde M_{-l}) \arrow[r, "\hat{i}_{-l}(r-1)"] & H_{r-1}(\tilde M_{-l} \sqcup \tilde M^t) & H_{r-1}(\tilde M^t) \arrow[l, "\hat{i}^t(r-1)"'] & H_{r-1}(\tilde M^t) \arrow[l, "="'] \\
H_r(\tilde M_{a'}, \tilde M_{-l}) \arrow[u] \arrow[r] & H_r(\tilde M_a, \tilde M_{-l}) \arrow[u] \arrow[r] & H_r(\tilde M, \tilde M_{-l} \sqcup \tilde M^t) \arrow[u] & H_r(\tilde M^b, \tilde M^t) \arrow[u] \arrow[l] & H_r(\tilde M^{b'}, \tilde M^t) \arrow[u] \arrow[l] \\
H_r(\tilde M_{a'}) \arrow[u] \arrow[r] \arrow[rr, bend left, "i_{a'}(r)"] & H_r(\tilde M_a) \arrow[u] \arrow[r, "i_a(r)"'] & H_r(\tilde M) \arrow[u] & H_r(\tilde M^b) \arrow[u] \arrow[l, "i^b(r)"] & H_r(\tilde M^{b'}) \arrow[u] \arrow[l] \arrow[ll, bend right, "i^{b'}(r)"'] \\
H_r(\tilde M_{-l}) \arrow[u] \arrow[r, "="] & H_r(\tilde M_{-l} l) \arrow[u] \arrow[r, "\hat{i}_{-l}(r)"] & H_r(\tilde M_{-l} \sqcup \tilde M^t) \arrow[u] & H_r(\tilde M^t) \arrow[u] \arrow[l, "\hat{i}^t(r)"'] & H_r(\tilde M^t) \arrow[u] \arrow[l, "="']
\end{tikzcd}
\tag{20}
$$

The vertical columns in the diagram (20) are exact sequences.

$$
\begin{tikzcd}
H^{BM}(\tilde M_{a'}) \arrow[r] \arrow[rr, bend left, "i_{a'}^{BM}(r)"] & H_r^{BM}(\tilde M_a) \arrow[r, "i_a^{BM}(r)"'] & H_r^{BM}(\tilde M) & H_r^{BM}(\tilde M^b) \arrow[l, "i_b^{BM}(r)"'] & H^{BM}(\tilde M^{b'}) \arrow[l] \arrow[ll, bend left, "i_{b'}^{BM}(r)"] \\
H(\tilde M_{a'}) \arrow[u] \arrow[r] & H_r(\tilde M_a) \arrow[u] \arrow[r, "i_a(r)"] & H_r(\tilde M) \arrow[u] & H_r(\tilde M^b) \arrow[u] \arrow[l, "i^b(r)"'] & H_r(\tilde M^{b'}) \arrow[u] \arrow[l]
\end{tikzcd}
\tag{21}
$$

The diagram (21) implies Item 1.

Since ${}^{BM}\mathbb{F}_r^{\tilde f}(a,b) = \mathrm{i}mg((i_a^{BM}(r)) \cap \mathrm{i}mg(i^{BM})^b(r))$ and $img(\hat{i}_{-l}(r) \cap img(\hat{i}^t(r)) = 0$ for any r, l, t, a careful analysis of the projective limit and of the diagram (20) implies

$$\mathbb{F}_r^{\tilde f}(a,b) \to^{BM} \mathbb{F}_r^{\tilde f}(a,b)$$

is surjective, (hence Item 2. holds), with kernel isomorphic to

$$\varprojlim_{0<l,t\to\infty} \mathrm{i}mg(H_r(\tilde M_{-l} \sqcup \tilde M^t) \to H_r(\tilde M)) = \mathbb{I}_{-\infty} + \mathbb{I}^{\infty} = C_r(M).$$

In view of Proposition (3.5) $C_r(M)$ is equal to $TH_r(\tilde M)$ hence Item 3. holds too.

q.e.d

The diagram (17) and the above observations induce the diagram (22) with first three horizontal arrows isomorphisms and the last arrow injective and $\kappa[t^{-1}, t]-$linear.

$$
\begin{tikzcd}
\mathbb{F}_r^{\tilde f}(a,b)/T(H_r(\tilde M) =^{BM} \mathbb{F}_r^{\tilde f}(a,b) \arrow[rr, "PD^{BM}(a,b)"] \arrow[dr] \arrow[ddd] & & (\mathbb{G}^{\tilde f}_{n-r}(b,a))^* \arrow[dr, "u^*_{n-r}"] \arrow[ddd, "p_r^*"] & \\
 & \mathbb{F}_r^{\tilde f}(B) \arrow[rr, "PD(B)"] \arrow[d] & & (\mathbb{G}^{\tilde f}_{n-r}(B'))^* \arrow[d, "p_r^*"] \\
 & \hat\delta_r^{\tilde f}(a,b) \arrow[rr, "P\hat D(a,b)"] & & (\hat\delta^{\tilde f}_{n-r}(b,a))^* \\
H_r^N(M;\xi) \arrow[rr, "PD^N"] & & (H^N_{n-r}(M;\xi))^* &
\end{tikzcd}
\tag{22}
$$

The first three horizontal arrows are isomorphisms in view of the isomorphism $\mathbb{F}_r^{\tilde f}(a,b)/T(H_r(\tilde M) \simeq^{BM} \mathbb{F}_r^{\tilde f}(a,b)$. The last arrow is the composition

$$
\begin{aligned}
H_r(\tilde M)/TH_r(\tilde M) = H_r^N(M;\xi) \to H_r^{BM}(\tilde M)/T(H_r^{BM}(\tilde M)) \to \\
\to (H_{n-r}(\tilde M))^*/T((H_{n-r}(\tilde M))^* = (H_r^N(X;\xi))^*
\end{aligned}
\tag{23}
$$

with the first arrow in (23) $\kappa[t^{-1},t]-$linear and injective and the second arrow in (23) $\kappa[t^{-1},t]-$linear and isomorphism. Indeed, the kernel of the first is exactly the $\kappa[t^{-1},t]-$torsion of $H_r(\tilde M)$. The module $H_{n-r}(\tilde M)$ is a f.g., hence $H_{n-r}(\tilde M) = H^N_{n-r}(X;\xi) \oplus T(H_{n-r}(\tilde X))$. Hence $H_{n-r}(\tilde M)^* = (H^N_{n-r}(X;\xi))^* \oplus T(H_{n-r}(\tilde X))^*$. Since $(H^N_{n-r}(X;\xi))^*$ has no torsion elements and $T(H_{n-r}(\tilde X))^*$ consists of torsion elements one has $(H_{n-r}(\tilde M))^*/T((H_{n-r}(\tilde M)^*) = (H_r^N(X;\xi))^*$ which establishes the isomorphism mentioned above. Note that $H^N_{n-r}(M;\xi))^*$ is NOT a f.g $\kappa[t^{-1},t]-$module.

In view of the commutativity of the diagram

$$
\begin{tikzcd}
\hat\delta_r^{\tilde f}(a,b) \arrow[r, "P\hat D_r(a,b)"] \arrow[d] & (\hat\delta^{\tilde f}_{n-r} r(b,a))^* \arrow[d] \\
\hat\delta_r^{\tilde f}(a+2\pi, b+2\pi) \arrow[r, "P\hat D_r(a,b)"] & (\hat\delta^{\tilde f}_{n-r} r(b+2\pi, a+2\pi))^*
\end{tikzcd}
\tag{24}
$$

one defines

$$P\hat D_r\langle a,b\rangle := \oplus_{k\in\mathbb{Z}} P\hat D(a+2\pi k, b+2\pi k)$$

and one observes that

$$\hom_{\kappa[t^{-1},t]}(\hat\delta^f\langle a,b\rangle, \kappa[t^{-1},t]) = \oplus_{k\in\mathbb{Z}}(\hat\delta^{\tilde f}(a+2\pi k, b+2\pi k))^*$$

with the right side a $\kappa[t^{-1},t]-$module, whose module structure is given by the isomorphism $\oplus_{k\in\mathbb{Z}}\tau_r(a+2\pi k,, b+2\pi k)$.

One obtains the isomorphism of $\kappa[t^{-1},t]-$modules

$$P\hat D\langle a,b\rangle : \hat\delta_r^f\langle a,b\rangle \to \hom_{\kappa[t^{-1},t]}(\hat\delta^f_{n-r}\langle a,b\rangle, \kappa[t^{-1},t]),$$

which implies $supp\delta_r^f \subseteq supp\delta_{n-r}$. Because $supp\delta^f_{n-r} \subseteq supp\delta_r$ one obtains that $supp\delta_r = supp\delta_{n-r}$ and in view of the non canonical isomorphism (induced by a compatible collection of splittings)

$$\bigoplus_{\langle a,b\rangle} \hat\delta_r^f\langle a,b\rangle \simeq H_r^N(M;\xi)$$

one has the non canonical isomorphism

$$H_r^N(M;\xi) \simeq \hom_{\kappa[t^{-1},t]}(H^N_{n-r}(M;\xi), \kappa[t^{-1},t] \simeq H^N_{n-r}(M;\xi).$$

The last isomorphism "$\simeq$" is true because both are free modules of rank $\beta_r^N(X;\xi)$.

Item 3. What follows is only an informal presentation of the arguments for Item 3. A detailed presentation will be provided in [5].

In case $\kappa = \mathbb{C}$ and M is a Riemannian manifold, hence $\tilde M$ is also equipped with a Riemannian metric, the Poincaré Duality linear isomorphism

$$PD = \varprojlim_{0<l,t\to\infty} H_r((\tilde M, M_{-l} \sqcup M^t) \to (H_{n-r}(M(-l,t))^*$$

should be regarded in the category of vector spaces with scalar product. Each of these vector spaces are finite dimensional and equipped with a scalar product induced by the Riemannian metric (via the identification of $H_r^N(M;\xi)$ with L_2-harmonic forms in degree $(n-r)$ on $\tilde M$). The appropriate inverse limit in this setting leads to

$$PD^{L_2} : H_r^{L_2}(\tilde M) \to (H^{L_2}_{n-r}(\tilde M))^* \simeq H^{L_2}_{n-r}(\tilde M)$$

where "$*$" here means the Hilbert space dual. The $\mathbb{C}[t^{-1},t]-$linearity of the map PD_r becoming in this setting the morphism PD^{L_2} of $L^\infty(\mathbb{S}^1)-$Hilbert modules. Clearly in this settings the diagram (22) becomes a diagram of finite type Hilbert modules and the linear surjections receiving a canonical splitting provided by the orthogonality inside Hilbert modules. These lead to a canonical realization $\hat{\hat{\delta}}^f_r(a,b)$ as a closed Hilbert module inside $H_r^{L_2}(\tilde M)$. The Poincaré Duality isomorphism in this case, via the identification of $H_r^{L_2}(\tilde M)$ with the L_2-harmonic $(n-r)-$forms is explicitly realized by the Hodge star operator.

6 Some consequences

Observation 6.1

1. A pleasant consequence of Theorem (1.3) is that $(c,c') \in \mathrm{supp}\delta^f_r$ iff $(c'c) \in \mathrm{supp}\delta^f_{n-r}$ and both pairs appear with equal multiplicity $\delta^f_r(c,c') = \delta^f_{n-r}(c',c)$.

2. Theorem 1.3 remains valid by the same proof in case M is a compact manifold with boundary $(M,\partial M)$, provided $H_r^N(\partial M;\xi_{f_{\partial M}})$ [10]vanishes for all r. In particular, under the above hypothesis , $H_r^N(M;\xi_f) \simeq H^N_{n-r}(M;\xi_f)$.

Corollary 6.2 *Suppose $(M^{2n},\partial M^{2n})$ is a compact manifold with boundary which has the homotopy type of a simplicial complex of dimension $\leq n$ and $\xi \in H^1(M;\mathbb{Z})$ s.t $H_r^N(\partial M;\xi_{\partial M}) = 0$ for all r. Then:*

1. $\beta_r^N(X:\xi) = \begin{cases} 0 \text{ if r} \neq \text{n} \\ -1^n\chi(M_n) \text{ if r} \neq \text{n} \end{cases}$, with $\chi(M)$ the Euler -Poincaré characteristic with coefficients in κ.

2. $\beta_r(X) = \begin{cases} \alpha_{r-1}+\alpha_r \text{ if r} \neq \text{n} \\ \alpha_{n-1}+\alpha_n + -1^n\chi(M_n) \end{cases}$, where α_r denotes the number of Jordan cells $J_r(M,\xi_f)$, cf [3] or [4]) with eigenvalue is equal to one.

3. If $V^{2n-1} \subset M^{2n}$ is compact proper sub manifold with Poincaré dual ξ_f (i,e, $V \pitchfork \partial M$,[11] and $V \cap \partial M = \partial V$) and $H_r(V) = 0$ the set of Jordan cells $J_r(M,\xi)$ is empty.

[10]with $f_{\partial M}$ notation for the restriction of f to ∂M

[11]$\pitchfork$= transversal

Item 1. follows from Observation (6.1) and the fact that both Betti numbers and Novikov–Betti numbers calculate the same Euler–Poincaré characteristic. Item 2 follows from Theorem 11 item c. in [4], and Item 3. from the description of Jordan cells in terms of linear relations as provided in [1] .

As pointed out to us by L Maxim, the complement $X = \mathbb{C}^n \setminus V$ of a complex hyper surface $V \subset \mathbb{C}^n, V := \{(z_1, z_2, \cdots z_n) \mid f(z_1, z_2, \cdots z_n) = 0\}$ regular at infinity, equipped with the canonical class $\xi_f \in H^1(X : \mathbb{Z})$ defined by $f : X \to \mathbb{C} \setminus 0$ is an example of an open manifold with an integral cohomology class which has as compactification a manifold with boundary with a cohomology class which satisfies the hypotheses above.

Item 1. recovers a calculation of L Maxim, cf [13] and [14] [12] that the complement of an algebraic hyper surface regular at infinity has vanishing Novikov homologies in all dimension but n.

7 Appendix (on Borel Moore homology)

Recall that $\tilde{M}(a,b)$ and $\tilde{M}(c)$ denote the compact set $\tilde{f}^{-1}([a,b])$ and $\tilde{f}^{-1}(a)$ which for a, b, c regular values are sub manifolds (with boundary in the first case). We also denote by $\tilde{M}_a = \tilde{f}^{-1}((-\infty, a])$ and $\tilde{M}^b = \tilde{f}^{-1}([b, \infty))$.

Note that the Poincaré Duality for bordisms provides the isomorphisms

$$\begin{aligned} PD(-l,a) &: H_r(\tilde{M}(-l,a), \tilde{M}(-l)) \to H^{n-r}(\tilde{M}(-l,a), \tilde{M}(a)),\ -l < a \\ PD(b,t) &: H_r(\tilde{M}(b,t), \tilde{M}(t)) \to H^{n-r}(\tilde{M}(b,b+t), \tilde{M}(b)),\ t > b \\ PD(-l,t) &: H_r(\tilde{M}(-l,t), \tilde{M}(-l) \sqcup \tilde{M}(t)) \to H^{n-r}(\tilde{M}(-l,+t)),\ t, l > 0. \end{aligned} \tag{25}$$

Combining with excision property in homology or cohomology and passing to limit when $0 < l \to \infty$, (or $0 < l, t \to \infty$) one derives the Poincaré Duality isomorphisms

$$\begin{aligned} PD^1_a &: H^{BM}_r(\tilde{M}_a) \to H^{n-r}(\tilde{M}, \tilde{M}^a) \\ PD^b_1 &: H^{BM}_r(\tilde{M}^b) \to H^{n-r}(\tilde{M}, \tilde{M}_b) \\ PD &: H^{BM}_r(\tilde{M}) \to H^{n-r}(\tilde{M}) \\ PD^b_2 &: H^{BM}_r(\tilde{M}, \tilde{M}^b) \to H^{n-r}(\tilde{M}_b) \\ PD^2_a &: H^{BM}_r(\tilde{M}, \tilde{M}_a) \to H^{n-r}(\tilde{M}^a) \end{aligned} \tag{26}$$

where

$$\begin{aligned} PD^1_a &= \varprojlim_{l\to\infty} PD(-l,a),\ PD^b_1 = \varprojlim_{t\to\infty} PD(b,t) \\ PD &= \varprojlim_{l\to\infty, l\to\infty} PD(-l,t) \\ PD^b_2 &= \varprojlim_{l\to\infty, t=b} PD(-l,t),\ PD^a_2 = \varprojlim_{t\to\infty, -l=a} PD(-l,t). \end{aligned} \tag{27}$$

For example, in case of the first isomorphism in (26),

$$\begin{aligned} H^{BM}(\tilde{M}_a) &= \varprojlim_{l\to\infty} (H_r(\tilde{M}(-l,a), \tilde{M}(-l)) \\ H^{n-r}(\tilde{M}, \tilde{M}^a) = H^{n-r}(\tilde{M}_a, \tilde{M}(a)) &= \varprojlim_{l\to\infty} H^{n-r}(\tilde{M}(-l,a), \tilde{M}(a)) \end{aligned}$$

where the passage from l to l' $l' > l$ in the first equality above is derived from the commutative diagram

[12] The Friedl-Maxim results state the vanishing of more general and more sophisticated L_2-homologies and Novikov type homologies. They can be also recovered via the appropriate Poincaré Duality isomorphisms

$$\begin{tikzcd}
H_r(\tilde M(-l,a),\tilde M(-l)) \arrow[r, "="] \arrow[d] & H_r(\tilde M(-l',a),\tilde M(-l',-l)) & H_r(\tilde M(-l',a),\tilde M(-l')) \arrow[l] \arrow[d] \\
H^{n-r}(\tilde M(-l,a),\tilde M(a)) & & H^{n-r}(\tilde M(-l',a),\tilde M(a)) \arrow[ll]
\end{tikzcd}$$

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