GRAPH REPRESENTATIONS AND TOPOLOGY OF REAL AND ANGLE VALUED MAPS

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Abstract. In this paper we review the definition of the invariants "bar codes" and "Jordan cells" of real and angle valued tame maps as proposed in [1] and [4] and prove the homotopy invariance of the sums $\sharp\mathcal{B}^c_r+\sharp\mathcal{B}^o_{r-1}$ and of the set of Jordan cells. Here $\mathcal{B}^c_r$ resp. $\mathcal{B}^o_r$ denote the sets of closed resp. open bar codes in dimension r. In addition we provide calculation of some familiar topological invariants in terms of bar codes and Jordan cells. The presentation provides a different perspective on Morse–Novikov theory based on critical values, bar codes and Jordan cells rather than on critical points instantons and closed trajectories of a gradient of a real or angle valued map.

Contents

1. Introduction

Recently, using graph representations, a new type of invariants, *bar codes* resp. *bar codes* and *monodromy (Jordan cells)*, have been assigned to a *tame real valued map* $f\colon X\to\mathbb{R}$ resp. a *tame angle valued map* $f\colon X\to S^1$ and a field κ [1].They were first introduced in [4] and [1] as invariants for zigzag persistence resp. persistence for circle valued maps based on the changes in the homology of the fibers with coefficients in κ.

In this paper we define these invariants, establish additional results which relate them to familiar topological invariants and prove the homotopy invariance of the

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[1] More recent work which will be detailed in [3] provides a definition of bar codes without any reference to graph representations and extend of the results below to continuous maps f whose X and the levels of f are all compact ANR's.

set of Jordan cells and of the numbers $\sharp\mathcal{B}^c_r + \sharp\mathcal{B}^o_{r-1}$. Here $\mathcal{B}^c_r$ resp.$\mathcal{B}^o_r$ denote the sets of closed resp. open bar codes in dimension r.

The main results are contained in Theorems 2.5, 2.6, 2.9 and Corollary 2.8, presented in section 2. The theory presented below represents an alternative approach to Morse–Novikov theory for real valued and angle valued maps based on critical values instead of critical points. In our approach the topological information about the underlying space is derived from bar codes between critical values, Jordan cells and the canonical long exact sequence associated with a tame map. Morse–Novikov theory cf. [9], [15], derives this information from instantons (isolated trajectories) between critical points, closed trajectories and the Morse complex associated with the gradient of a Morse (real or circle valued) map on the underlying Riemannian manifold. Our approach applies to a considerably larger class of continuous maps than the maps considered by Morse–Novikov theory.

The tame real valued maps are tame angle valued maps and all results about them are particular cases of results about angle valued maps. Rather than consider only angle valued maps, considerably more complex, we decided to discuss both cases, simply because the Morse theory of real valued maps is more familiar than Novikov theory of the circle valued maps and restricting the attention only to the second apparently does not save much space.

The *bar codes* are finite intervals I of real numbers of the type:

(i) closed, $[a, b]$, $a \leq b$,
(ii) open (a, b), $a < b$, and
(iii) mixed $[a, b)$, $(a, b]$, $a < b$.

The *Jordan blocks* J and *Jordan cells* are equivalency classes of pairs (V, T) with V a finite dimensional κ-vector space and $T\colon V \to V$ a linear isomorphism.

An equivalence between (V_1, T_1) and (V_2, T_2) is an linear isomorphism $\omega\colon V_1 \to V_2$ which intertwines T_1 and T_2.

An equivalence class is called a *Jordan block* if indecomposable, i.e. (V, T) is not isomorphic to $(V_1, T_2) \oplus (V_2, T_2)$ with $\dim V_i < \dim V$.

A Jordan block $J = (V, T)$ is called *Jordan cell* if isomorphic to $(\kappa^k, T(\lambda, k))$, $k \in \mathbb{N}$, $\lambda \in \kappa \setminus 0$, where

$$T(\lambda, k) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \lambda & 1 \\ 0 & \cdots & 0 & 0 & \lambda \end{pmatrix} \tag{1}$$

in which case will be denoted by $J = (\lambda, k)$.

If κ is algebraically closed then a Jordan block is a Jordan cells and the two concepts are the same [2].

For a tame real valued map $f\colon X \to \mathbb{R}$ and $r \leq \dim X$ we associate (see section 2) the collection of bar codes $\mathcal{B}_r(f)$. The set $\mathcal{B}_r(f)$ can be written as $\mathcal{B}_r(f) = \mathcal{B}^c_r(f) \sqcup \mathcal{B}^o_r(f) \sqcup \mathcal{B}^m_r(f)$ with $\mathcal{B}^c_r(f)$, $\mathcal{B}^o_r$ and $\mathcal{B}^m_r$ the subset of closed, open and mixed bar codes.

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[2] If (V, T) is a Jordan block then $(V \otimes \overline{\kappa}, T \otimes \overline{\kappa})$ is not necessary a Jordan cell but decomposes as a sum of Jordan cells.

For a tame angle valued map f, in addition to bar codes as above, one associates the collections of Jordan blocks $\mathcal{J}_r(f)$, equivalently of Jordan cells $\overline{\mathcal{J}}_r(f)$ if one consider $\overline{\kappa}$ the algebraic closure of κ. The sum $(V_r(f), T_r(f))$ of all Jordan blocks in $\mathcal{J}_r(f)$ or of all Jordan cells in $\overline{\mathcal{J}}_r(f)$ is referred to as the *monodromy* of f.

If f is only a continuous map, in view of Theorem 2.8, $\mathcal{J}_r(f)$ resp. $\overline{\mathcal{J}}_r(f)$ can still be defined. It is expected (and will be shown in [3]) that the sets $\mathcal{B}^c_r(f)$, $\mathcal{B}^o_r(f)$.

Theorem 2.8 states that the numbers $N_r(f) := \sharp\mathcal{B}^c_r(f) + \sharp\mathcal{B}^o_{r-1}(f)$ are homotopy invariants of the pair (X, ξ_f) where $\xi_f \in H^1(X;\mathbb{Z})$ is the cohomology class determined by f. We say that the pairs $(X_i, \xi_i \in H^1(X,\mathbb{Z}))$, $i = 1, 2$, are homotopy equivalent, if there exists a homotopy equivalence $\theta\colon X_1 \to X_2$ so that $\theta^*(\xi_2) = \xi_1$.

Theorem 2.8, also states the homotopy invariance for the monodromy. In view of these facts we might want to get a homotopy-theoretic description of the numbers $N_r(f)$ and of the monodromy $(V_r(f), T_r(f))$.

For this purpose consider $(X, \xi \in H^1(X;\mathbb{Z}))$ and denote by $\tilde{X} \to X$ the infinite cyclic cover associated to ξ. Note that $H_r(\tilde{X}) := H_r(\tilde{X};\kappa)$ is not only a κ-vector space but is actually a $\kappa[T^{-1}, T]$-module where the multiplication by T is induced by the deck transformation $\tau\colon \tilde{X} \to \tilde{X}$. Here $\kappa[T^{-1}, T]$ denotes the ring of Laurent polynomials. Let $\kappa[T^{-1}, T]]$ be the field of Laurent power series. Define

$$H^N_r(X;\xi) := H_r(\tilde{X}) \otimes_{\kappa[T^{-1},T]} \kappa[T^{-1}, T]]$$

and let

$$H_r(\tilde{X}) \to H^N_r(X;\xi)$$

be the $\kappa[T^{-1}, T]$-linear map induced by taking the tensor product of $H_r(\tilde{X})$ with $\kappa[T^{-1}, T]]$ over $\kappa[T^{-1}, T]$.

The $\kappa[T^{-1}, T]]$-vector spaces $H^N_r(X;\xi)$ are called Novikov homology[3] and their dimensions, the numbers $N_r(X;\xi) := \dim H^N_r(X;\xi)$, Novikov–Betti numbers.

It X is a compact ANR then the κ-vector space $V(\xi) := \ker(H_r(\tilde{X}) \to H^N_r(X;\xi))$ is finite dimensional and when equipped with $T(\xi)\colon V(\xi) \to V(\xi)$ induced by the multiplication by T defines the pair $(V(\xi), T(\xi))$ called the monodromy of (X, ξ). We show that the numbers $N_r(f)$ are exactly $N_r(X;\xi_f)$ (cf. Theorem 2.5), and the pair $(V_r(f), T_r(f))$ described using graph representations is exactly the monodromy $(V(\xi_f), T(\xi_f))$.

The monodromy can be defined for an arbitrary continuous map $f\colon X \to S^1$, using instead of graph representations the regular part of a linear relation provided by the map f in the homology of any fiber of f, as described in section 6.

The plan of this paper is the following.

In section 2 we remind the reader the concepts of tame real and angle valued maps and formulate the main results.

In sections 3 we discuss the representation theory for the two graphs, $\mathcal{Z}$ and G_{2m}, used in the proof of the main theorems. The reader can skip section 4 unless he wants to understand the calculations of the bar codes and the Jordan cells for the example presented in section 7 via an implementable algorithm.

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[3] instead of $\kappa[T^{-1}, T]]$ one can consider the field $\kappa[[T^{-1}, T]$ of Laurent power series in T^{-1}, which is isomorphic to $\kappa[T^{-1}, T]]$ by an isomorphism induced by $T \to T^{-1}$. The (Novikov) homology defined using this field has the same Novikov–Betti numbers as the the one defined using $\kappa[T^{-1}, T]]$.

In sections 5 and 6 we prove the main results. Section 6 can be read independently of the rest of the paper. It does provide the necessary background on linear relations and does not use concepts previously defined.

In section 7 we give an example of a tame angle valued map and derive its bar codes and Jordan cells using the algebraic observations made in section 3.

Acknowledgements: The relationship between the topology of a space to the information extracted from the real or angle valued map as presented in this paper was influenced by the *persistence theory* introduced in [8] and motivated by the interest that computer scientists and data analysts have shown for *persistent homology* and associated concepts. It also owns to the apparently forgotten efforts and ideas of R. Deheuvels to extend Morse theory to all continuous real valued functions (fonctionelles) cf. [5].

2. The main results

2.1. **Tame maps and its r-invariants.**

Definition 2.1. A continuous map $f\colon X \to \mathbb{R}$ resp. $f\colon X \to S^1$, X a compact ANR, is *tame* if:

1. Any fiber $X_\theta = f^{-1}(\theta)$ is the deformation retract of an open neighborhood.
2. Away from a finite set of numbers/angles $\Sigma = \{\theta_1, \dots, \theta_m\} \subset \mathbb{R}$, resp. S^1 the restriction of f to $X \setminus f^{-1}(\Sigma)$ is a fibration (Hurewicz fibration).

Note that:

- Any smooth real or angle valued map on a compact smooth manifold M whose all critical points are isolated, in particular any Morse function, is tame.

- Any real or angle valued simplicial map on a finite simplicial complex is tame.

- The space of tame maps with the induced topology has the same homotopy type as the space of all continuous maps with compact open topology.[4]

- The set of tame maps is dense the the space of all continuous maps with respect to the compact open topology.[5]

Given a tame map $f\colon X \to \mathbb{R}$ resp. $f\colon X \to S^1$ consider the critical values resp. the critical angles $\theta_1 < \theta_2 < \cdots < \theta_m$. In the second case we have $0 < \theta_1 < \cdots < \theta_m \leq 2\pi$. Choose t_i, $i = 1, 2, \dots, m$, with $\theta_1 < t_1 < \theta_2 < \cdots < t_{m-1} < \theta_m < t_m$. In the second case choose t_m s.t. $2\pi < t_m < \theta_1 + 2\pi$. The tameness of f induces the diagram of continuous maps:[6]

$$\begin{tikzcd}
 & \boldsymbol{X}_{t_m} \arrow[dl, dotted, "a_1"'] \arrow[dr, "b_m"] & & \boldsymbol{X}_{t_{m-1}} \arrow[dl, "a_m"'] & & \cdots \boldsymbol{X}_{\omega_2} \arrow[dr, "b_2"] & & \boldsymbol{X}_{t_1} \arrow[dl, "a_2"'] \arrow[dr, "b_1"] & \\
\boldsymbol{X}_{\theta_1} & & \boldsymbol{X}_{\theta_m} & & & & \boldsymbol{X}_{\theta_2} & & \boldsymbol{X}_{\theta_1} .
\end{tikzcd}$$

Diagram 1

Different choices of t_i lead to different diagrams but all homotopy equivalent.

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[4] While (i) and (ii) are simple exercises, we can not locate a reference for statement (iii), but since all ANR's of interest for this paper are homeomorphic to simplicial complexes, for them the statement follows from (ii).

[5] The same comment as in footnote 3.

[6] The doted arrow a_1 in Diagram 1 appears only in the case of an angle valued map.

We will use two graphs, $\mathcal{Z}$ for real valued maps, and G_{2m} for angle valued maps. The graph $\mathcal{Z}$ has vertices x_i, $i \in \mathbb{Z}$, and edges a_i from x_{2i-1} to x_{2i} and b_i from x_{2i+1} to x_{2i}, see picture (The graph $\mathcal{Z}$) in section 3.

The graph G_{2m} has vertices x_i with edges a_i and b_i, $i = 1, 2, \dots, (m-1)$, as before and b_m from x_1 to x_{2m}, see picture (The graph G_{2m}) in section 3.

Let κ be a field. A graph representation ρ is an assignment which to each vertex x assigns a finite dimensional vector space V_x and to each oriented arrow from the vertex x to the vertex y a linear map $V_x \to V_y$.

As stated in section 3 a finitely supported $\mathcal{Z}$-representation[7], resp. an arbitrary G_{2m}-representation can be uniquely decomposed as a sum of indecomposable representations. In the case of the graph $\mathcal{Z}$ the indecomposable representations are indexed by one of the three types of intervals (bar codes) described in the introduction, with ends $i, j \in \mathbb{Z}$, $i \leq j$ for type (i) and $i < j$ for type (ii) and (iii). We refer to both the inedecomposable representation and the interval as *bar code.* In the case of the graph G_{2m} the indecomposable representations are indexed by similar intervals (bar codes) with ends $i, j + mk$, $1 \leq i, j \leq m, k \in \mathbb{Z}_{\leq 0}$, $i \leq j$ with $1 \leq i \leq m$ and by Jordan blocks J (or Jordan cells) as described in the introduction. We refer to both the indecomposable representation and the interval resp. the Jordan block as *bar code* resp. *Jordan block* or Jordan cells.

For a $\mathcal{Z}$-representation or a G_{2m}-representation ρ one denotes by $\mathcal{B}(\rho)$ the set of all bar codes and write $\mathcal{B}(\rho)$ as $\mathcal{B}(\rho) = \mathcal{B}^c(\rho) \sqcup \mathcal{B}^o(\rho) \sqcup \mathcal{B}^m(\rho)$ where $\mathcal{B}^c(\rho)$, $\mathcal{B}^o(\rho)$ and $\mathcal{B}^m(\rho)$ are the subsets of closed, open and mixed bar codes.

For a G_{2m} representation ρ one denotes by $\mathcal{J}(\rho)$ resp. $\overline{\mathcal{J}}(\rho)$ the set of all Jordan blocks resp. Jordan cells.

For any $r \leq \dim X$ let $\rho_r = \rho(f)$ be the $\mathcal{Z}$- resp. G_{2m}-representation associated to the tame map f defined by

$$V_{2i} = H_r(X_{\theta_i}), V_{2i+1} = H_r(X_{t_i}), \quad \alpha_i : V_{2i-1} \to V_{2i}, \quad \beta_i : V_{2i+1} \to V_{2i}$$

with α_i and β_i the linear maps induced by the continuous maps a_i and b_i in Diagram 1. Here and below $H_r(Y)$ denotes the singular homology in dimension r with coefficients in a fixed field κ which will not appear in the notation.

In order to relate the indecomposable components of ρ_r to the critical vales or angles of f, for a real valued map one converts the intervals $\{i, j\}$ into $\{\theta_i, \theta_j\}$ and for an angle value map the intervals $\{i, j + km\}$, $1 \leq i, j \leq m$, into the intervals $\{\theta_i, \theta_j + 2\pi k\}$ [8].

Definition: The sets $\mathcal{B}_r(f) := \mathcal{B}(\rho_r)$, with the intervals I converted into ones with ends θ_i's and $(\theta_i + 2\pi k)'$s and $\mathcal{J}_r(f) := \mathcal{J}(\rho_r)$ resp. $\overline{\mathcal{J}}_r = \overline{\mathcal{J}}(\rho_r)$ are the r-invariants of the map f.

For a real valued map one has only bar codes, for an angle valued map one has bar codes and Jordan blocks or Jordan cells.

We refer to

$$(V_r(f), T_r(f)) = \bigoplus_{(V,J) \in \mathcal{J}_r(f)} (V, T) = \bigoplus_{(\lambda,k) \in \overline{\mathcal{J}}_r(f)} (\kappa^k, T(\lambda, k))$$

as the *r-monodromy* of the angle valued f.

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[7] i.e. all but finitely many vector spaces V_x have dimension zero

[8] we use the symbol "{" for both "(" and "[" or "}" for both ")" or "]".

Recall that the homotopy classes of continuous maps $f\colon X \to S^1$ are in bijective correspondence to $H^1(X;\mathbb{Z})$ so any such map f defines $\xi := \xi_f \in H^1(X;\mathbb{Z})$ and any homotopy class can be viewed as an element in $H^1(X;\mathbb{Z})$.

Definition 2.2. 1. Two maps $f_1\colon X_1 \to S^1$ and $f_2\colon X_2 \to S^1$ or $f_1\colon X_1 \to \mathbb{R}$ and $f_2\colon X_2 \to \mathbb{R}$ are fiber wise homotopy equivalent if there exists $\omega\colon X_1 \to X_2$ so that $f_2 \cdot \omega = f_1$ and for any $\theta \in S^1$ the restriction $\omega_\theta\colon (X_1)_\theta \to (X_2)_\theta$ is a homotopy equivalence.

2. Two maps $f_1\colon X_1 \to S^1$ and $f_2\colon X_2 \to S^1$ are homotopy equivalent if there exists $\omega\colon X_1 \to X_2$ so that $f_2 \cdot \omega$ is homotopic to f_1, equivalently $\omega^*(\xi_2) = \xi_1$, $\xi_i = \xi_{f_i}$. If so we say that the pairs (X_1, ξ_1) and (X_2, ξ_2) are homotopy equivalent.

The following statement follows from definitions.

Proposition 2.3. *If $f_i\colon X_i \to \mathbb{R}$ resp. $f_i\colon X_i \to S^1$, $i = 1, 2$, are two tame maps and $\omega\colon X_1 \to X_2$ is a fiber wise homotopy equivalence then $\mathcal{B}_r(f_1) = \mathcal{B}_r(f_2)$ resp. $\mathcal{B}_r(f_1) = \mathcal{B}_r(f_2)$ and $\mathcal{J}_r(f_1) = \mathcal{J}_r(f_2)$ (equivalently $\overline{\mathcal{J}}_r(f_1) = \overline{\mathcal{J}}_r(f_2)$).*

2.2. **The results.** Fix a field κ and denote by $H_*(Y)$ the singular homology of Y with coefficients in the field κ. The following result was established in [1].

Theorem 2.4 ([1])**.** *1. If $f\colon X \to S^1$ is a tame map then:*

$$\begin{aligned}
&a. \dim H_r(X_\theta) &&= \sum_{I \in \mathcal{B}_r(f)} n_\theta(I) + \sum_{J \in \mathcal{J}_r(f)} k(J)\\
&b. \dim H_r(X) &&= \begin{cases} \sharp\mathcal{B}^c_r(f) + \sharp\mathcal{B}^o_{r-1}(f) + \\ \sharp\big\{(\lambda,k) \in \overline{\mathcal{J}}_r(f) \bigm| \lambda(J) = 1\big\} + \\ \sharp\big\{(\lambda,k) \in \overline{\mathcal{J}}_{r-1}(f) \bigm| \lambda(J) = 1\big\}\end{cases}\\
&c. \dim \operatorname{im}\big(H_r(X_\theta) \to H_r(X)\big) &&= \sharp\big\{I \in \mathcal{B}^c_r(f) \bigm| \theta \in I\big\}
\end{aligned}$$

where $n_\theta(I) = \sharp\{k \in \mathbb{Z} \mid \theta + 2\pi k \in I\}$ and for $J = (V, T)$, $k(J) = \dim V$.

2. If $f\colon X \to \mathbb{R}$ is a tame map[9] then:

$$\begin{aligned}
&a. \dim H_r(X_t) &&= \sharp\big\{I \in \mathcal{B}_r(f) \bigm| I \ni t\big\}\\
&b. \dim H_r(X) &&= \sharp\mathcal{B}^c_r(f) + \sharp\mathcal{B}^o_{r-1}(f)\\
&c. \dim \operatorname{im}\big(H_r(X_t) \to H_r(X)\big) &&= \sharp\big\{I \in \mathcal{B}^c_r(f) \bigm| I \ni t\big\}.
\end{aligned}$$

Consider $\xi_f \in H^1(X;\mathbb{Z})$ the cohomology class represented by f and for any $u \in \kappa \setminus 0$ denote by $u\xi_f$ the rank one representation

$$u\xi_f\colon H_1(M;\mathbb{Z}) \to \mathbb{Z} \to \kappa \setminus 0 = \mathrm{GL}_1(\kappa) \tag{2}$$

with the last arrow given by $n \to u^n$. Denote by $H_r(X; u\xi_f)$ the r-homology with coefficients in this representation which is a κ-vector space. Theorem 2.4 can be extended to the following theorem.

Theorem 2.5. *If $f\colon X \to S^1$ is a tame map then:*

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[9] A real valued map can be considered an angle valued by identifying $\mathbb{R}$ with $S^1 \setminus 1$.

1.
$$\dim H_r(X; u\xi_f) = \begin{cases} \sharp\mathcal{B}^c_r(f) + \sharp\mathcal{B}^o_{r-1}(f) + \\ \sharp\{J \in \overline{\mathcal{J}}_r(f) | u\lambda(J) = 1\} + \\ \sharp\{J \in \overline{\mathcal{J}}_{r-1}(f) | u^{-1}\lambda(J) = 1\}. \end{cases}$$

2.
$$N_r(X; \xi_f) = \sharp\mathcal{B}^c_r(f) + \sharp\mathcal{B}^o_{r-1}(f).$$

Denote by: $\tilde{f}\colon \tilde{X} \to \mathbb{R}$ the infinite cyclic cover of $f\colon X \to S^1$.

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "\psi"'] & \mathbb{R} \arrow[d, "p"'] \\ X \arrow[r, "f"] & \mathbb{S}^1 \end{tikzcd}$$

Let $\tilde{X}_{[a,b]} := \tilde{f}^{-1}[a,b]$, $\tilde{X}_t = f^{-1}(t)$. Clearly one has $\tilde{X}_t = X_{p(t)}$.
Denote by:

$$\begin{aligned} \tilde{\mathcal{B}}_r(f) &= \big\{I + 2\pi k \;\big|\; k \in \mathbb{Z}, I \in \mathcal{B}_r(f)\big\}, \\ \tilde{\mathcal{B}}^c_r(f) &= \big\{I + 2\pi k \;\big|\; k \in \mathbb{Z}, I \in \mathcal{B}^c_r(f)\big\}, \\ \tilde{\mathcal{B}}^o_r(f) &= \big\{I + 2\pi k \;\big|\; k \in \mathbb{Z}, I \in \mathcal{B}^o_r(f)\big\}. \end{aligned}$$

We have

Theorem 2.6. *If $f\colon X \to S^1$ is a tame map then:*
1.

$$a. \dim H_r(\tilde{X}_{[a,b]}) = \begin{cases} \sharp\{I \in \tilde{\mathcal{B}}_r(f), I \cap [a,b] \neq \emptyset\} + \\ \sharp\{I \in \tilde{\mathcal{B}}^o_{r-1}(f), I \subset [a,b]\} + \\ \sum_{J \in \mathcal{J}_r(f)} k(J). \end{cases}$$

$$b. \dim \operatorname{im}\big(H_r(\tilde{X}_{[a,b]}) \to H_r(\tilde{X})\big) = \begin{cases} \sharp\{I \in \tilde{\mathcal{B}}^c_r(f), I \cap [a,b] \neq \emptyset\} + \\ \sharp\{I \in \tilde{\mathcal{B}}^o_{r-1}(f), I \subset [a,b]\} + \\ \sum_{J \in \mathcal{J}_r(f)} k(J). \end{cases}$$

$$c. \dim \operatorname{im}\big(H_r(\tilde{X}_{[a,b]}) \to H_r(X)\big) = \begin{cases} \sharp\{I \in \mathcal{B}^c_r, [a,b] \cap (I + 2\pi k) \neq 0\} + \\ \sharp\{I \in \mathcal{B}^o_{r-1} \mid I + 2\pi k \subset [a,b]\} + \\ \sharp\{J \in \overline{\mathcal{J}}_r(f) | \lambda(J) = 1\} \end{cases}$$

2. $V_r(\xi_f) := \ker(H_r(\tilde{X}) \to H^N_r(X; \xi_f))$ *is a finite dimensional κ-vector space and* $(V_r(\xi_f), T_r(\xi_f)) = (V_r(f), T_r(f))$

3. $H_r(\tilde{X}) = \kappa[T^{-1}, T]^N \oplus V_r(\xi_f)$ *as $\kappa[T^{-1}, T]$-modules with* $N = N_r(f) = \sharp\mathcal{B}^c_r(f) + \sharp\mathcal{B}^o_{r-1}(f)$.

Observation 2.7. *Theorem 2.6 (1) remains true if one replaces a closed interval by a finite union of closed intervals (possibly points).*

As a consequence we have the main result of this paper:

Corollary 2.8. *If $f_1, f_2\colon X_i \to S^1$ are two homotopy equivalent tame maps, then:*
1. $\sharp\mathcal{B}^c_r(f_1) + \sharp\mathcal{B}^o_{r-1}(f_1) = \sharp\mathcal{B}^c_r(f_2) + \sharp\mathcal{B}^o_{r-1}(f_2)$.
2. $\mathcal{J}_r(f_1) = \mathcal{J}_r(f_2)$.

One can provide an alternative geometric description of the equivalence class of pairs $(V_r(f), T_r(f))$. Start with the tame map $X \to S^1$ representing the cohomology class $\xi \in H^1(X;\mathbb{Z})$ and choose an angle θ. Consider the compact space $X^{f,\theta}$ the cut of X along X_θ. Precisely as a set this is the disjoint union $X^-_\theta \sqcup (X \setminus X_\theta) \sqcup X^+_\theta$ with the $X^\pm_\theta$ copies of X_θ. The topology of $X^{f,\theta}$ is the obvious topology.[10]

We have the two inclusions $i^- \colon X_\theta \to X^{f,\theta}$ and $i^+ \colon X_\theta \to X^{f,\theta}$, which induce the linear maps $(i^-)_r \colon H_r(X_\theta) \to H_r(X^{f,\theta})$ resp. $(i^+)_r \colon H_r(X_\theta) \to H_r(X^{f,\theta})$. These two linear maps define the linear relation $\mathcal{R}^\theta_r \subset H_r(X^-_\theta) \oplus H_r(X^+_\theta)$ defined by $(i^-)_r(x^-) = (i^+)_r(x^+)$, $x^\pm \in H_r(X^\pm_\theta)$, cf. section 6 for definitions.

Theorem 2.9. *The regular part* [11] *of the relation* $\mathcal{R}^\theta$ *is isomorphic to* $(V_r(f), T_r(f))$.

As a consequence of Theorems 2.5 and 2.8 and of the fact that any homotopy class contains tame maps we have:

Corollary 2.10. *1. If a tame angle valued map is homotopic to a fibration there are no closed and no open bar codes.*

2. For any tame map $f : X \to S^1$ *one has* $\beta_r(X) - \mathcal{J}^1_r(f) - \mathcal{J}^1_{r-1}(f) = N_r(f)$.

2.3. **Organizing the closed and open bar codes.** For $f \colon X \to \mathbb{R}$ resp. $f \colon X \to \mathbb{S}^1$ a tame map Theorem 2.4(4.) resp. Theorem 2.5(2.) suggest to collect together the closed r-bar codes and the open $(r-1)$-bar codes as configuration of points in $\mathbb{R}^2$ resp. $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}$. Precisely $\mathbb{T}$ is the quotient space of $\mathbb{R}^2$ the Euclidean plane, by the additive group of integers $\mathbb{Z}$, w.r. to the action $\mu : \mathbb{Z} \times \mathbb{R}^2 \to \mathbb{R}^2$ given by $\mu(n;(x,y)) = (x + 2\pi n, y + 2\pi n)$.

One denotes by $\Delta \subset \mathbb{R}^2$ resp.$\Delta \subset \mathbb{T}$ the diagonal of $\mathbb{R}^2$ resp. the quotient of the diagonal of $\mathbb{R}^2$ by the group $\mathbb{Z}$. The points above or on diagonal, (x,y), $x \leq y$, will be used to record closed bar codes $[x,y]$ and the points below the diagonal, (x,y), $x > y$, to record open bar codes (y,x). This convention comes from the observation that continuous deformation of tame maps can produce deformation of an r-closed bar code $[x,y]$ into an $(r-1)$-open bar code (y',x') but not without passing through a closed bar codes with equal ends (located on Δ) $[x'',y'']$, $x'' = y''$.

One can identify $\mathbb{T}$ with $\mathbb{C} \setminus 0$ sending the point of $\mathbb{T}$ represented by the pair (x,y) to $e^{(y-x)+ix} \in \mathbb{C}$. Clearly, Δ became the circle of radius 1.

For an integer k and X a space, in our case $X = \mathbb{T}$ or $\mathbb{R}^2$, denote by $S^k(X)$ the k-th symmetric power of X, i.e. the quotient space X^n/Σ_n with Σ_n the symmetric group acting on X^n by permutations and $X^n = \underbrace{X \times \cdots \times X}_{n}$.

In view of Theorem 2.4(4.) resp. Theorem 2.5(2.) for a tame real resp. angle valued map f and any r we will collect the closed r-barcodes and the open $(r-1)$-bar codes as a point $C_r(f) \in S^{\beta_r(X)}(\mathbb{R}^2)$ where $\beta_r(X) = \dim H_r(X)$ resp. as a point $Cr(f) \in S^{\mathbb{N}_r(X;\xi_f)}(\mathbb{T})$. If we identify a point in $(x,y) \in R^2$ with $z = x + iy$ it is convenient to regard $C_r(f)$ as the monic polynomial $P^f_r(z)$ of degree $\beta_r(X)$ whose roots are the elements of $C_r(f)$. Similarly, using the identification of $\mathbb{T}$ with $\mathbb{C} \setminus 0$ it is convenient to regard $C_r(f)$ as a monic polynomial of degree $N_r(X;\xi_f)$.

---

[10] This is the unique topology which induces on $X \setminus X_\theta$, $X^\pm_\theta$ the same topology as X and makes of $f^{-1}((\theta - \epsilon, \theta])$ resp. $f^{-1}([\theta, \theta + \epsilon)$ for ϵ small, neighborhoods of X^-_θ resp. X^+_θ in $X^{f,\theta}$.

[11] The regular part of a linear relation $\mathcal{R} \subset W \times W$ is a sub relation $\mathcal{R}^{reg} \subset V \times V$ which is given by an isomorphism $T \colon V \to V$ and is maximal, cf. section 6.

For a generic set of continuous maps $f\colon X \to \mathbb{C} \setminus 0$ both $|f|\colon X \to \mathbb{R}$ and $f/|f|\colon X \to S^1$ are tame and consequently one obtains for any r the pair of polynomials $(P_r(f), P_r(f/|f|))$ which can be viewed as refinements of Betti numbers of X and of Novikov–Betti numbers of $(X; \xi_{f/|f|})$.

One expects that the assignment $f \rightsquigarrow C_r(f)$ defined on the space $T(X;\mathbb{R})$ resp. $T(X;\mathbb{S}^1)$ of tame maps be continuous with respect to the compact open topology and extends by continuity to $C(X;\mathbb{R})$ resp. $C(X;S^1)$. In particular one expects that the closed and open bar codes as read off $Cr(f)$ be defined for any continuous map $f\colon X \to \mathbb{R}$ resp. $f\colon X \to S^1$, hence the polynomials considered above for tame maps can be defined for any continuous map f whose source and levels are compact ANR's and depend continuously on f. This will be shown to be true in [3].

3. Graphs representations

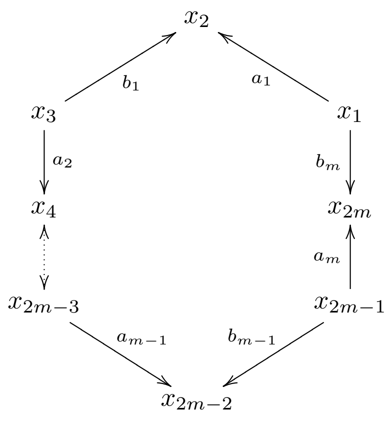
In this section we summarize known facts about the representations of two graphs, $\mathcal{Z}$ and G_{2m} and formulate some technical used in the proof of Theorems 2.4, 2.5, 2.6.

We consider two oriented graphs, $\Gamma = \mathcal{Z}$ whose vertices are x_i, $i \in \mathbb{Z}$, and arrows $a_i\colon x_{2i-1} \to x_{2i}$ and $b_i\colon x_{2i+1} \to x_{2i}$

$$\begin{tikzcd}
\cdots & x_{2i-1} \arrow[l, "b_{i-1}"'] \arrow[r, "a_i"] & x_{2i} & x_{2i+1} \arrow[l, "b_i"'] \arrow[r, "a_{i+1}"] & x_{2i+2} & \cdots \arrow[l, "b_{i+1}"']
\end{tikzcd}$$

The graph $\mathcal{Z}$

and $\Gamma = G_{2m}$ whose vertices are $x_1, x_2, \ldots, x_{2m}$ and arrows a_i, $1 \leq i \leq m$, and b_i, $1 \leq i \leq m-1$, as above and $b_M\colon x_1 \to x_{2m}$.



The graph G_{2m}

Let κ a fixed field.

A Γ-representation ρ is an assignment which to each vertex x of Γ assigns a finite dimensional vector space V_x and to each oriented arrow from the vertex x to the vertex y a linear map $V_x \to V_y$. The concepts of morphism, isomorphism= equivalence, sum, direct summand, zero and nontrivial representations are obvious.

A $\mathcal{Z}$-representation is given by the collection

$$\rho := \begin{cases} V_r, \quad \alpha_i : V_{2i-1} \to V_{2i}, \quad \beta_i : V_{2i+1} \to V_{2i} \\ \qquad\qquad r, i \in \mathbb{Z} \quad , \end{cases}$$

abbreviated to $\rho = \{V_r, \alpha_i, \beta_i\}$, while a G_{2m} representation by the collection

$$\rho := \begin{cases} V_r, \quad \alpha_i : V_{2i-1} \to V_{2i}, \quad \beta_i : V_{2i+1} \to V_{2i} \\ 1 \le r \le 2m, \quad 1 \le i \le m, \quad V_{2m+1} = V_1 \end{cases}$$

also abbreviated to $\rho = \{V_r, \alpha_i, \beta_i\}$.

A representation ρ is *regular* if all the linear maps α_i and β_i are isomorphisms.

Any regular G_{2m}-representation $\rho = \{V_r, \alpha_i, \beta_i\}$ is equivalent to the representation

$$\rho(V, T) = \{V'_r = V, \alpha'_1 = T, \alpha'_i = Id \ i \neq 1, \ \beta'_i = Id\} \tag{3}$$

with $T = \beta_m^{-1} \cdot \alpha_m^{-1} \cdots \beta_1^{-1} \cdot \alpha_1$ [12].

A $\mathcal{Z}$-representation ρ has *finite support* if $V_i = 0$ for all but finitely many i. There are no nontrivial regular $\mathcal{Z}$-representations with finite support.

For $\mathcal{Z}$-representation $\rho = \{V_r, \alpha_i, \beta_i\}$ we denote by $T_{k,l}(\rho)$, $k \le l$ the representation with finite support $T_{i,j}(\rho) = \{V'_r, \alpha'_i, \beta'_i\}$ defined by

$$\begin{aligned} V'_r &= \begin{cases} V_r & 2k \le r \le 2l \\ 0 & \text{otherwise} \end{cases} \\ \alpha'_r &= \begin{cases} \alpha_r & k+1 \le i \le l \\ 0 & \text{otherwise} \end{cases} \\ \beta'_r &= \begin{cases} \beta_r & k \le r \le l-1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{4}$$

A representation ρ is *indecomposable* if not the sum of two nontrivial representations. It is well known and not hard to prove that any $\mathcal{Z}$-representation with finite support and any G_{2m}-representation can be uniquely decomposed in a finite sum of indecomposable representations (the Remack–Schmidt theorem) and these indecomposables are unique up to isomorphism cf. [6].

The indecomposable $\mathcal{Z}-$representations with finite support are indexed by four type of intervals I with ends i and j and denoted by $\rho(I)$ or more precisely by:

1. $\rho([i,j])$, 2. $\rho([i,j))$, 3. $\rho((i,j])$ and 4. $\rho(i,j)$

with $i \le j$ in case (1.) and $i < j$ for the cases (2., 3., 4.) above. They have all vector spaces either one dimensional or zero dimensional and the linear maps α_i, β_j the identity if both the source and the target are nontrivial and zero otherwise. Both the indexing interval I and the representation $\rho(I)$ will be called *bar codes.*

Precisely,

---

[12] The isomorphism is provided by the linear maps $\omega_r \colon V_r \to V_r$ given by

$$\begin{cases} \omega_1 = Id \\ \omega_2 = \beta_m^{-1} \cdots \beta_2^{-1} \cdots \alpha_2 \cdot \beta_1^1 \\ \omega_3 = \beta_m^{-1} \cdots \beta_2^{-1} \cdots \alpha_2 \\ \qquad \cdots \\ \omega_{2m} = \beta^{-1} \end{cases}$$

.

(i) $\rho([i,j]), i \leq j$ has $V_r = \kappa$ for $r = \{2i, 2i{+}1, \cdots 2j\}$ and $V_r = 0$ if $r \neq [2i, 2j]$
(ii) $\rho([i,j)), i < j$ has $V_r = \kappa$ for $r = \{2i, 2i+1, \cdots 2j\}$ and $V_r = 0$ if $r \neq [2i, 2j-1]$
(iii) $\rho((i,j]), i < j$ has $V_r = \kappa$ for $r = \{2i, 2i+1, \cdots 2j\}$ and $V_r = 0$ if $r \neq [2i+1, 2j]$
(iv) $\rho((i,)]), i < j$ has $V_r = \kappa$ for $r = \{2i, 2i+1, \cdots 2j\}$ and $V_r = 0$ if $r \neq [2i+1, 2j-1]$

with all α_i and β_i the identity provided that the source and the target are both non zero.

The above description is implicit in [10].

Denote by $\mathcal{B}(\rho)$ the collection of bar codes which appear as direct summands of ρ, and by $\mathcal{B}^c(\rho)$, resp. $\mathcal{B}^o(\rho)$, resp, $\mathcal{B}^m(\rho)$ the subsets of $\mathcal{B}(\rho)$ consisting of bar codes with both ends closed, resp. open resp. one open one closed. By Remack-Schmidt theorem any $\mathcal{Z}-$ representation ρ can be uniquely written as

$$\rho = \sum_{I \in \mathcal{B}(\rho)} \rho(I). \tag{5}$$

The indecomposable $G_{2m}-$ representations are of two types, type I and type II.

Type I: (*bar codes*) For any triple of integers $\{i, j, k\}$, $1 \leq i, j \leq m$, $k \geq 0$, we have the representations denoted by

(i) $\rho^I([i,j];k) \equiv \rho^I([i, j+mk]), \quad 1 \leq i, j \leq m, k \geq 0$
(ii) $\rho^I((i,j];k) \equiv \rho^I((i, j+mk]), \quad 1 \leq i, j \leq m, k \geq 0$
(iii) $\rho^I([i,j);k) \equiv \rho^I([i, j+mk)), \quad 1 \leq i, j \leq m, k \geq 0$
(iv) $\rho^I((i,j);k) \equiv \rho^I((i, j+mk)), \quad 1 \leq i, j \leq m, k \geq 0$

described as follows.

Suppose the vertices of G_{2m} are located counter-clockwise on the unit circle with evenly indexed vertices $\{x_2, x_4, \cdots x_{2m}\}$ corresponding to the angles $0 < s_1 < s_2 < \cdots < s_m \leq 2\pi$. Draw the spiral curve for $a = s_i$ and $b = s_j + 2\pi k$ with the ends a black or an empty circle if the end is closed or open (see picture below for $k = 2$).

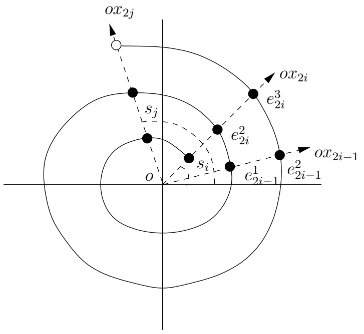


FIGURE 1. The spiral for $[i, j+2m)$.

Denote by V_i the vector space generated by the intersection points of the spiral with the radius corresponding to the vertex x_i and let α_i resp.β_i be defined on bases in an obvious manner ; a generator e of $V_{2i\pm1}$ is sent to the generator e' of V_{2i} if connected by a piece of spiral and to 0 otherwise.

Type II: The representations of Type II are regular representations associated to a Jordan block $J = (V, T)$ cf. formula (3) and denoted by $\rho^{II}(J)$. They are clearly indecomposable.

In consistency with the above conventions we refer to both $J = (V, T)$ and the representation $\rho^{II}(J)$ as *Jordan block*. If the eigenvalues of T are in κ, in particular if κ is algebraically closed, $J = (V, T)$ is indecomposable iff T is conjugate to $T(\lambda, k)$ defined by formula (1) for some $\lambda \in \kappa \setminus 0$ and in this case we will write $\rho^{II}(\lambda, k)$ for the representation $\rho^{II}(\kappa^k, T(\lambda, k))$. In consistency with the above convention we refer to both, the representation $\rho^{II}(\lambda, k)$ and the pair (λ, k) as *Jordan cell.* If κ is not algebraically closed and (V, T) is a Jordan block then $(V \otimes \overline{\kappa}, T \otimes \overline{\kappa})$, $\overline{\kappa}$ the algebraic closure of κ, does not necessary remain a Jordan block. However it decomposes uniquely as a finite sum of Jordan cells. Two Jordan blocks are equivalent iff they remain equivalent after tensored by $\overline{\kappa}$, equivalently the associated Jordan cells over $\overline{\kappa}$ are the same.

By Remack-Schmidt theorem any $G_{2m}-$representation ρ can be uniquely decomposed as

$$\rho = \bigoplus_{I \in \mathcal{B}(\rho)} \rho^I(I) \oplus \bigoplus_{J \in \mathcal{J}(\rho)} \rho^{II}(J). \tag{6}$$

The above description is implicit in [12] and [7].

Introduce

$$\rho_{reg} = \bigoplus_{J \in \mathcal{J}(\rho)} \rho^{II}(J)$$

with $\rho_{reg} = \rho(V_{reg}(\rho), T_{reg}(\rho))$. The pair $(V_{reg}(\rho), T_{reg}(\rho))$ is also referred to as the *monodromy* of ρ.

In [1] an algorithm to provide the decomposition of a $G_{2m}-$representation as a sum of indecomposable is described. The algorithm holds for $\mathcal{Z}-$representations too and is based on four elementary transformations $T_1(i), T_2(i), T_3(i), T_4(i)$ described for the reader convenience in the Appendix. They will be used in section 7. Performing any of these transformations one passes from a representation ρ to a representation ρ' of strictly smaller dimension (of the total vector space $\oplus_i V_i$ or $\oplus_{1 \leq i \leq 2m} V_i$), with the same monodromy (in case of $G_{2m}-$representation) and with bar codes changed in a specified way. After applying such transformations finitely many time one ends up with a regular representation and by backward book keeping, one can reconstruct the initial collection of bar codes too.

To a $\mathcal{Z}-$ representation $\rho = \{V_r, \alpha_i, \beta_i\}$ $r, i \in \mathbb{Z}$ one associates the linear transformation $M(\rho) : \oplus V_{2i-1} \to \oplus V_{2i}$ given by the infinite block matrix with entries

$$M(\rho)_{2r-1,2s} = \begin{cases} \alpha_r, & \text{if s = r} \\ \beta_{r-1}, & \text{if s = r} - 1 \\ 0 & \text{otherwise .} \end{cases} \tag{7}$$

To a $G_{2m}-$ representation $\rho = \{V_r, \alpha_i, \beta_i\}$ $1 \leq r \leq 2m, 1 \leq i \leq m.$ one associates the block matrix $M(\rho) : \bigoplus_{1\leq i\leq m} V_{2i-1} \to \bigoplus_{1\leq i\leq m} V_{2i}$ defined by:

$$\begin{pmatrix} \alpha_1 & -\beta_1 & 0 & \dots & \dots & 0 \\ 0 & \alpha_2 & -\beta_2 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & \dots & \dots & \dots & \dots \alpha_{m-1} & -\beta_{m-1} \\ -\beta_m & \dots & \dots & \dots & \dots & \alpha_m \end{pmatrix}.$$

For a $\Gamma = \mathcal{Z}$ or $\mathrm{G_{2m}}-$ representation ρ denote by:

(i) $\dim(\rho) : \Gamma \to \mathbb{Z}_{\geq 0}$ the function defined by $r \rightsquigarrow \dim(V_r)$
(ii) $n_i := \dim(V_{2i-1})$ and $r_i := \dim(V_{2i})$.
(iii) $d\ker(\rho) = \dim\ker M(\rho)$ and
(iv) $d\,\mathrm{coker}(\rho) = \dim\mathrm{coker}\, M(\rho)$.

For a $G_{2m}-$ representation $\rho = \{V_r, \alpha_i, \beta_i\}$ and $u \in \kappa\backslash 0$ denote by $\rho_u = \{V'_r, \alpha'_i, \beta'_i\}$ the representation with $V'_r = V_r$, $\alpha'_1 = u\alpha_1$, $\alpha'_i = \alpha_i$ for $i \neq 1$ and $\beta'_i = \beta_i$. Clearly $(\rho_1 \oplus \rho_2)_u = (\rho_1)_u \oplus (\rho_2)_u$, $\dim(\rho) = \dim(\rho_u)$ and the block matrix $M(\rho_u)$ is given by

$$\begin{pmatrix} u\alpha_1 & -\beta_1 & 0 & \dots & \dots & 0 \\ 0 & \alpha_2 & -\beta_2 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & \dots & \dots & \dots & \dots \alpha_{m-1} & -\beta_{m-1} \\ -\beta_m & \dots & \dots & \dots & \dots & \alpha_m \end{pmatrix}.$$

One has:

Proposition 3.1. ([1])

(i) $\dim(\rho_1 \oplus \rho_2) = \dim(\rho_1) + \dim(\rho_2)$,
(ii) $d\ker(\rho_1 \oplus \rho_2) = d\ker(\rho_1) + d\ker(\rho_2)$,
(iii) $d\,\mathrm{coker}(\rho_1 \oplus \rho_2) = d\,\mathrm{coker}(\rho_1) + d\,\mathrm{coker}(\rho_2)$,
(iv) $d\ker(\rho) = d\ker(\rho_u),\ d\,\mathrm{coker}(\rho) = d\,\mathrm{coker}(\rho_u)$.

For the indecomposable $\mathcal{Z}-$ representations one has:

Proposition 3.2.

(i)

$$\dim\rho([i,j]) = \begin{cases} n_l = 1, i+1 \leq l \leq j, \ = 0 \text{ otherwise} \\ r_l = 0, i \leq l \leq j, \ = 0 \text{ otherwise} \end{cases}$$

(ii)

$$\dim\rho((i,j)) = \begin{cases} n_l = 1, i+1 \leq l \leq j, \ = 0 \text{ otherwise} \\ r_l = 0, i+1 \leq l \leq j-1, \ = 0 \text{ otherwise} \end{cases}$$

(iii)

$$\dim\rho([i,j)) = \begin{cases} n_l = 1, i+1 \leq l \leq j, \ = 0 \text{ otherwise} \\ r_l = 0, i \leq l \leq j-1, \ = 0 \text{ otherwise} \end{cases}$$

(iv)

$$\dim \rho((i,j]) = \begin{cases} n_l = 1, i+1 \leq l \leq j, \ = 0 \text{ otherwise} \\ r_l = 0, i+1 \leq l \leq j, \ = 0 \text{ otherwise} \end{cases}$$

Proposition 3.3.

- *(i)* $d \ker \rho([i,j]) = 0,\ d \operatorname{coker} \rho([i,j]) = 1,$
- *(ii)* $d \ker \rho([i,j)) = 0,\ d \operatorname{coker} \rho([i,j)) = 0,$
- *(iii)* $d \ker \rho((i,j]) = 0,\ d \operatorname{coker} \rho((i,j]) = 0,$
- *(iv)* $d \ker \rho((i,j)) = 1,\ d \operatorname{coker} \rho((i,j)) = 0.$

For indecomposable $G_{2m}-$representations one has:

Proposition 3.4. ([1])

- *(i) If* $i \leq j$ *then*
  - *(a)* $\dim \rho^I([i,j];k)$ *is given by:*
    $n_l = k+1$ *if* $(i+1) \leq l \leq j$ *and* k *otherwise,*
    $r_l = k+1$ *if* $i \leq l \leq j$ *and* k *otherwise*
  - *(b)* $\dim \rho^I((i,j];k)$ *is given by:*
    $n_l = k+1$ *if* $(i+1) \leq l \leq j$ *and* k *otherwise,*
    $r_l = k+1$ *if* $(i+1) \leq l \leq j$ *and* k *otherwise,*
  - *(c)* $\dim \rho^I([i,j);k)$ *is given by:*
    $n_l = k+1$ *if*
    $(i+1) \leq l \leq j$ *and* k *otherwise,*
    $r_l = k+1$ *if* $i \leq l \leq (j-1)$ *and* k *otherwise,*
  - *(d)* $\dim \rho^I((i,j);k)$ *is given by:*
    $n_l = k+1$ *if* $(i+1) \leq l \leq j$ *and* k *otherwise,*
    $r_l = k+1$ *if* $(i+1) \leq l \leq (j-1)$ *and* k *otherwise*
- *(ii) If* $i > j$ *then similar statements hold.*
  - *(a)* $\dim \rho^I([i,j];k)$ *is given by:*
    $n_l = k$ *if* $(j+1) \leq l \leq i$ *and* $k+1$ *otherwise;*
    $r_l = k$ *if* $(j+1) \leq l \leq (i-1)j$ *and* $k+1$ *otherwise*
  - *(b)* $\dim \rho^I((i,j];k)$ *is given by:*
    $n_l = k$ *if* $(j+1) \leq l \leq i$ *and* $k+1$ *otherwise.*
    $r_l = k$ *if* $(j+1) \leq l \leq i$ *and* $k+1$ *otherwise,*
  - *(c)* $\dim \rho^I([i,j);k)$ *is given by:*
    $n_l = k$ *if* $(j+1) \leq l \leq i$ *and* $k+1$ *otherwise;*
    $r_l = k$ *if* $j \leq l \leq (i-1)$ *and* $k+1$ *otherwise,*
  - *(d)* $\dim \rho^I((i,j);k)$ *is given by:*
    $n_l = k$ *if* $(j+1) \leq l \leq i$ *and* $k+1$ *otherwise;*
    $r_l = k$ *if* $j \leq l \leq i$ *and* $k+1$ *otherwise.*

Proposition 3.5. ([1])

- *(i)* $d \ker \rho^I([i,j];k) = 0,\ d \operatorname{coker} \rho^I([i,j];k) = 1,$
- *(ii)* $d \ker \rho^I([i,j);k) = 0,\ d \operatorname{coker} \rho^I([i,j);k) = 0,$
- *(iii)* $d \ker \rho^I((i,j];k) = 0,\ d \operatorname{coker} \rho^I((i,j];k) = 0,$
- *(iv)* $d \ker \rho^I((i,j);k) = 1,\ d \operatorname{coker} \rho^I((i,j);k) = 0,$
- *(v)* $d \ker \rho^{II}(\lambda,k) = 0$ (resp. $= 1$) *if* $\lambda \neq 1$ (resp. $= 1$),
- *(vi)* $d \operatorname{coker} \rho^{II}(\lambda,k) = 0$ (resp. $= 1$) *if* $\lambda \neq 1$ (resp. $= 1$).

The proof of Propositions 3.1 (1,2,3), 3.2, 3.4 are straightforward. Items (i- vi) in Proposition 3.5 follow from the calculation of the kernel of $M(\rho)$ and from Proposition 3.4 while Proposition 3.3 can be viewed as a particular case of Proposition 3.5. Proposition 3.1 (iv) has to be verified first for indecomposable representations and then in view of Proposition 3.1 the statements hold for an arbitrary representation.

The calculation of kernel of $M(\rho)$ for ρ of Type I or II boils down to the description of the space of solutions of the linear system

$$\begin{aligned}\alpha_1(v_1) =& \beta_1(v_3)\\ \alpha_2(v_3) =& \beta_2(v_5)\\ &\cdots\\ \alpha_m(v_{2m-1}) =& \beta_m(v_1)\end{aligned}$$

which were explicitly described above.

Proposition 3.3 and 3.5 can be refined.

For this purpose let us choose once for all for any open resp. closed interval I an isomorphism between $\ker \rho(I)$ resp. $\operatorname{coker} \rho(I)$ and κ and for any Jordan cell $(1,k)$ an isomorphism between $\ker \rho^{II}(1,k)$ resp. $\operatorname{coker} \rho^{II}(1,k)$ and κ.

For a set S let $\kappa[S]$ denote the vector space generated by S. Recall that for a representation ρ we have denoted by $\mathcal{B}^c(\rho)$ the collection of closed bar codes and by $\mathcal{B}^o(\rho)$ the collection of open bar codes. The following propositions follows immediately from Propositions 3.1, 3.3 and 3.5.

Proposition 3.6. *If for a $\mathcal{Z}-$ representation with finite support ρ a decomposition of $\rho = \sum_{I\in\mathcal{B}(\rho)} \rho(I)$ is given, then Proposition 3.2 provides canonical isomorphisms*

$$\Psi^c : \kappa[\mathcal{B}^c(\rho)] \to \operatorname{coker} M(\rho)$$

and

$$\Psi^o : \kappa[\mathcal{B}^o(\rho)] \to \ker M(\rho).$$

Let us write $\overline{\mathcal{J}}^\lambda$ for the collection of Jordan cells cells whose eigenvalue is exactly λ.We have:

Proposition 3.7. *If ρ is a G_{2m} representation Proposition 3.5 provides the canonical isomorphisms*

$$\begin{aligned}\Psi^c &: \kappa[\mathcal{B}^c(\rho) \sqcup \overline{\mathcal{J}}^1(\rho)] \to \operatorname{coker} M(\rho)\\ \Psi^o &: \kappa[\mathcal{B}^o(\rho) \sqcup \overline{\mathcal{J}}^1(\rho)] \to \ker M(\rho).\end{aligned}$$

More general for any $u \in \kappa \setminus 0$ it provides the canonical isomorphisms

$$\begin{aligned}\Psi^c &: \kappa[\mathcal{B}^c(\rho) \sqcup \overline{\mathcal{J}}^{(u^{-1})}(\rho)] \to \operatorname{coker} M(\rho_u))\\ \Psi^o &: \kappa[\mathcal{B}^o(\rho) \sqcup \overline{\mathcal{J}}^{(u^{-1})}(\rho)] \to \ker M(\rho_u).\end{aligned}$$

For $\rho = \{V_r, \alpha_i, \beta_i\}$ a $G_{2m}-$ representation, consider the $\mathcal{Z}-$ representation $\tilde\rho := \{V'_{2mk+r} = V_r, \alpha'_{mk+i} = \alpha_i, \beta'_{mk+i} = \beta_i\}$ and denote by:

$\tilde{\mathcal{B}}(\rho) := \{I + 2\pi k \mid k \in \mathbb{Z}, I \in \mathcal{B}\}$,

$\tilde{\mathcal{B}}^c(\rho) := \{I + 2\pi k \mid k \in \mathbb{Z}, I \in \mathcal{B}^c\}$,

$\tilde{\mathcal{B}}^o(\rho) := \{I + 2\pi k \mid k \in \mathbb{Z}, I \in \mathcal{B}^o\}$.

Let $\tilde{\mathcal{J}}(\rho)$ be the set which contains $\dim(V)$ copies of J for any Jordan block $J=(V,T)\in\mathcal{J}(\rho)$, equivalently k copies of each Jordan cell $(\lambda,k)\in\overline{\mathcal{J}}(\rho)$.

In section 5 we will need the following observation.

Observation 3.8.

$$\begin{aligned}
\mathcal{B}(T_{i,j}(\tilde{\rho})) =&\{I\in\tilde{\mathcal{B}}_r(\rho)\mid I\cap[2i,2j]\neq\emptyset\}\sqcup\tilde{\mathcal{J}}(\rho)\\
\mathcal{B}^c(T_{i,j}(\tilde{\rho})) =&\{I\in\tilde{\mathcal{B}}^c_r(\rho)\mid I\cap[2i,2j]\text{a closed nonempty interval}\}\sqcup\tilde{\mathcal{J}}(\rho) \qquad (8)\\
\mathcal{B}^o(T_{i,j}(\tilde{\rho})) =&\{I\in\tilde{\mathcal{B}}^o_r(\rho), I\subset(2i,2j)).
\end{aligned}$$

The above statement can be easily verified for representations of Type I and II and then follows for arbitrary representations.

4. Appendix to Graph Representations

The Elementary transformations.

We discuss here only G_{2m} representations since $\mathcal{Z}-$ representations with finite support can be viewed as particular cases. We convene that for $i>2m$ $V_i=V_{i-2m}$, $\alpha_i=\alpha_{i-2m}$ and $\beta_i=\beta_{i-2m}$.

Each transformation takes an index i and a representation

$$\rho=\{V_j|1\leq j\leq 2m,\,\alpha_s:V_{2s-1}\to V_{2s},\,\beta_s:V_{2s+1}\to V_{2s}|1\leq s\leq m\}$$

and produces a new representation

$$\rho'=\{V'_j|1\leq j\leq 2m,\,\alpha'_s:V'_{2s-1}\to V'_{2s},\,\beta'_s:V'_{2s+1}\to V'_{2s}|1\leq s\leq m\}$$

as follows:

(i) If $\rho'=T_1(i)\rho$ then $V'_{2i-1}=V_{2i-1}/\ker(\beta_{i-1})$, $V'_{2i}=V_{2i}/\alpha_i(\ker(\beta_{i-1}))$, $V'_j=V_j$ for $j\neq\{2i-1,2i\}$ with α'_s,β'_s being induced from α_s,β_s for $s\in[1,m]$.

(ii) If $\rho'=T_2(i)\rho$ then $V'_{2i+1}=V_{2i+1}/\ker(\alpha_{i+1})$, $V'_{2i}=V_{2i}/\beta_i(\ker\alpha_{i+1})$, $V'_j=V_j$ for $j\neq\{2i+1,2i\}$ with α'_s,β'_s being induced from α_s,β_s for $s\in[1,m]$.

(iii) If $\rho'=T_3(i)\rho$ then $V'_{2i}=\alpha_i(V_{2i-1})$, $V'_{2i+1}=\beta_i^{-1}(\alpha_i(V_{2i-1}))$, $V'_j=V_j$ for $j\neq\{2i,2i+1\}$ with α'_s,β'_s being the restrictions of α_s,β_s for $s\in[1,m]$.

(iv) If $\rho'=T_4(i)\rho$ then $V'_{2i}=\beta_i(V_{2i+1})$, $V'_{2i-1}=\alpha_i^{-1}(\beta_i(V_{2i+1}))$, $V'_j=V_j$ for $j\neq\{2i,2i-1\}$ with α'_s,β'_s being the restrictions of α_s,β_s for $s\in[1,m]$.

The following diagrams [13] indicate the constructions described above. The indices increase from right to left to signify that the vector spaces are laid counterclockwise with increasing indices around a quiver.

Transformation $T_1(i)\rho$:

$$\begin{tikzcd}
\cdots & V_{2i+1} \arrow[l, "\alpha_{i+1}"'] \arrow[r, "\beta_i"] \arrow[dr, "\beta'_i"'] & V_{2i} \arrow[d] & V_{2i-1} \arrow[l, "\alpha_i"'] \arrow[r, "\beta_{i-1}"] \arrow[d] & V_{2i-2} & \cdots \arrow[l] \\
 & & V'_{2i} & V'_{2i-1} \arrow[l, "\alpha'_i"] \arrow[ur, "\beta'_{i-1}"'] & &
\end{tikzcd}$$

---

[13] in these diagrams $V_0=V_{2m}$ and $\beta_0=\beta_m$

$$V'_{2i-1} = V_{2i-1}/\ker(\beta_{i-1}) \quad V'_{2i} = V_{2i}/\alpha_i(\ker(\beta_{i-1}))$$

Transformation $T_2(i)\rho$:

$$\begin{tikzcd}
\cdots \arrow[r, "\beta_{i+1}"] & V_{2i+2} & V_{2i+1} \arrow[l, "\alpha_{i+1}"'] \arrow[r, "\beta_i"] \arrow[d] & V_{2i} \arrow[d] & V_{2i-1} \arrow[l, "\alpha_i"'] \arrow[r, "\beta_i"] \arrow[dl, "\alpha'_i"] & \cdots \\
 & & V'_{2i+1} \arrow[ul, "\alpha'_{i+1}"] \arrow[r, "\beta'_i"] & V'_{2i} & &
\end{tikzcd}$$

$$V'_{2i+1} = V_{2i+1}/\ker(\alpha_{i+1}), \quad V'_{2i} = V_{2i}/\beta_i(\ker(\alpha_{i+1}))$$

Transformation $T_3(i)\rho$:

$$\begin{tikzcd}
\cdots \arrow[r, "\beta_{i+1}"] & V_{2i+2} & V_{2i+1} \arrow[l, "\alpha_{i+1}"'] \arrow[r, "\beta_i"] & V_{2i} & V_{2i-1} \arrow[l, "\alpha_i"'] \arrow[r, "\beta_i"] \arrow[dl, "\alpha'_i"] & \cdots \\
 & & V'_{2i+1} \arrow[ul, "\alpha'_{i+1}"] \arrow[u] \arrow[r, "\beta'_i"] & V'_{2i} \arrow[u] & &
\end{tikzcd}$$

$$V'_{2i} = \alpha_i(V_{2i-1}) \quad V'_{2i+1} = \beta_i^{-1}(\alpha_i(V_{2i-1}))$$

Transformation $T_4(i)\rho$:

$$\begin{tikzcd}
\cdots & V_{2i+1} \arrow[l, "\alpha_{i+1}"'] \arrow[r, "\beta_i"] \arrow[dr, "\beta'_i"'] & V_{2i} & V_{2i-1} \arrow[l, "\alpha_i"'] \arrow[r, "\beta_{i-1}"] & V_{2i-2} & \cdots \arrow[l] \\
 & & V'_{2i} \arrow[u] & V'_{2i-1} \arrow[l, "\alpha'_i"'] \arrow[u] \arrow[ur, "\beta'_{i-1}"'] & &
\end{tikzcd}$$

$$V'_{2i} = \beta_i(V_{2i+1}) \quad V'_{2i-1} = \alpha_i^{-1}(\beta_i(V_{2i+1})).$$

The following observations follow straightforwardly from the definitions.

$T_1(i)\rho$ eliminates all bar codes of the form $(i-1, i)$ and $(i-1, i]$, if the case, shrinks each bar code of the form $(i-1, k]$ and $(i-1, k)$, $k \geq (i+2)$, into bar codes $(i, k]$ and (i, k) respectively with the convention that $(i-1, k\}$ is $(m, m+k\}$ when $i = 1$, and leaves all other barcodes and Jordan cells unchanged. If β_{i-1} is injective then $T_1(i)\rho = \rho$.

$T_2(i)\rho$ eliminates all bar codes of the form $(i, i+1)$ and $[i, i+1)$,if the case, shrinks each bar code of the form $[l, i+1)$ and $(l, i+1)$, $l \leq i-1$, into bar codes $[l, i)$ and (l, i) respectively, and leaves any other barcodes and Jordan cells unchanged. If α_{i+1} is injective then $T_2(i)\rho = \rho$.

Type $T_3(i)\rho$ eliminates all bar codes of the form $[i, i]$ and $[i, i+1)$,if the case,shrinks each bar code of the forms $[i, k)$ and $[i, k]$, $k \geq i+1$, into the bar codes $[i+1, k)$ and $[i+1, k]$ respectively, and leaves all other type of barcodes and Jordan cells unchanged. If α_i is surjective then $T_3(i)\rho = \rho$.

Type $T_4(i)\rho$ eliminates all bar codes of the form $[i, i]$ and $(i-1, i]$, if the case, shrinks each bar code of the forms $(l, i]$ and $[l, i]$, $l \leq i-1$, into the bar codes $(l, i-1]$ and $[l, i-1]$ respectively with the convention that $\{l, 0\}$ is identified to

$\{l+m, m\}$, and leaves all other type of barcodes and Jordan cell unchanged.If β_i is surjective then $T_4(i)\rho = \rho$.

In deciding "if the case " the following proposition is of use. Let $\sharp\{i,j\}_\rho$ denote the number of bar codes of type $\{i,j\}$ for a representation ρ. We have the following proposition which can be derived using the inspection of the transformations described above.

Proposition 4.1. ([1])

(i) $\sharp(i,i+1)_\rho = \dim\ker\beta_i \cap \ker\alpha_{i+1}$
(ii) $\sharp[i,i]_\rho = \dim(V_{2i}/((\beta_i(V_{2i+1}) + \alpha_i(V_{2i-1}))$
(iii) $\sharp(i,i+1]_\rho = \dim(\beta_i(V_{2i+1}) + \alpha_i(\ker\beta_{i-1})) - \dim(\beta_i(V_{2i+1}))$
(iv) $\sharp[i,i+1)_\rho = \dim(\alpha_i(V_{2i-1}) + \beta_i(\ker\alpha_{i+1})) - \dim(\alpha_i(V_{2i-1}))$

Note that unless at least one elimination is performed each of these transformation is ineffective (i.e.$=$ *Id.*) so an algorithm based on successive applications of the transformations eventually stops.

5. PROOF OF THE MAIN RESULTS BUT THEOREM 2.9

Since a tame real valued map can be regarded as a tame angle valued map by identifying $\mathbb{R}$ to an open subset of S^1, we will consider only tame angle valued maps.

Let $f\colon X \to S^1$ be a tame map with m critical angles $s_1, s_2, \ldots, s_m$ and regular angles $t_1, t_2, \ldots, t_m$. First observe that, up to homotopy, the space X and the map $f\colon X \to S^1$ can be regarded as the iterated mapping torus $\mathcal{T}$ and the map $f^{\mathcal{T}} \to [0,m]/\!\sim$ described below. Consider the collection of spaces and continuous maps:

$$X_m = X_0 \xleftarrow{b_0 = b_m} R_1 \xrightarrow{a_1} X_1 \xleftarrow{b_1} R_2 \xrightarrow{a_2} X_2 \leftarrow \cdots \to X_{m-1} \xleftarrow{b_{m-1}} R_m \xrightarrow{a_m} X_m \tag{9}$$

with $R_i := X_{t_i}$ and $X_i := X_{s_i}$ and denote by $\mathcal{T} = T(\alpha_1 \cdots \alpha_m; \beta_1 \cdots \beta_m)$ the space obtained from the disjoint union

$$\Big(\bigsqcup_{1\le i\le m} R_i \times [0,1]\Big) \sqcup \Big(\bigsqcup_{1\le i\le m} X_i\Big)$$

by identifying $R_i \times \{1\}$ to X_i by α_i and $R_i \times \{0\}$ to X_{i-1} by β_{i-1}. Denote by $f^{\mathcal{T}} : \mathcal{T} \to [0,m]/\!\sim\, = S^1$ where $f^{\mathcal{T}} : R_i \times [0,1] \to [i-1,i]$ is the projection on $[0,1]$ followed by the translation of $[0,1]$ to $[i-1,i]$ and $[0,m]/\!\sim$ the space obtained from the segment $[0,m]$ by identifying the ends. This map is a *homotopical reconstruction* of $f\colon X \to S^1$ provided that, with the choice of angles t_i, s_i, the maps a_i, b_i are those described in section 2 for $X_i := f^{-1}(s_i)$ and $R_i := f^{-1}(t_i)$.

Let $\mathcal{P}'$ denote the space obtained from the disjoint union

$$\Big(\bigsqcup_{1\le i\le m} R_i \times (\epsilon,1]\Big) \sqcup \Big(\bigsqcup_{1\le i\le m} X_i\Big)$$

by identifying $R_i \times \{1\}$ to X_i by α_i, and $\mathcal{P}''$ denote the space obtained from the disjoint union

$$\Big(\bigsqcup_{1\le i\le m} R_i \times [0,1-\epsilon\Big) \sqcup \Big(\bigsqcup_{1\le i\le m} X_i\Big)$$

by identifying $R_i \times \{0\}$ to X_{i-1} by β_{i-1}.

Let $\mathcal{R} = \bigsqcup_{1\le i\le m} R_i$ and $\mathcal{X} = \bigsqcup_{1\le i\le m} X_i$. Then, one has:

(i) $\mathcal{T} = \mathcal{P}' \cup \mathcal{P}''$,

(ii) $\mathcal{P}' \cap \mathcal{P}'' = \left(\bigsqcup_{1\leq i\leq m} R_i \times (\epsilon, 1-\epsilon)\right) \sqcup \mathcal{X}$, and

(iii) the inclusions $\left(\bigsqcup_{1\leq i\leq m} R_i \times \{1/2\}\right) \sqcup \mathcal{X} \subset \mathcal{P}' \cap \mathcal{P}''$ as well as the obvious inclusions $\mathcal{X} \subset \mathcal{P}'$ and $\mathcal{X} \subset \mathcal{P}''$ are homotopy equivalences.

The Mayer–Vietoris long exact sequence applied to $\mathcal{T} = \mathcal{P}' \cup \mathcal{P}''$ leads to the diagram:

$$\begin{tikzcd}
 & & H_r(\mathcal{R}) \arrow[r, "M_r(\rho_r)"] & H_r(\mathcal{X}) \arrow[dr] & & \\
\cdots \arrow[r] & H_{r+1}(\mathcal{T}) \arrow[ur] \arrow[r, "\partial_{r+1}"] & H_r(\mathcal{R}) \oplus H_r(\mathcal{X}) \arrow[u, "pr_1"] \arrow[r, "N"] & H_r(\mathcal{X}) \oplus H_r(\mathcal{X}) \arrow[u, "(Id,-Id)"] \arrow[r, "(i^r,-i^r)"] & H_r(\mathcal{T}) \arrow[r] & {} \\
 & & H_r(\mathcal{X}) \arrow[u, "in_2"] \arrow[r, "Id"] & H_r(\mathcal{X}) \arrow[u, "\Delta"] & &
\end{tikzcd}.$$

Diagram 2

Here Δ denotes the diagonal, in_2 the inclusion on the second component, pr_1 the projection on the first component, i^r the linear map induced in homology by the inclusion $\mathcal{X} \subset \mathcal{T}$. The matrix $M_r(\alpha, \beta)$ is defined by

$$\begin{pmatrix}
\alpha_1^r & -\beta_1^r & 0 & \cdots & 0 \\
0 & \alpha_2^r & -\beta_2^r & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \alpha_{m-1}^r & -\beta_{m-1}^r \\
-\beta_m^r & 0 & \cdots & 0 & \alpha_m^r
\end{pmatrix}.$$

with $\alpha_i^r \colon H_r(R_i) \to H_r(X_i)$ and $\beta_i^r \colon H_r(R_{i+1}) \to H_r(X_i)$ induced by the maps α_i and β_i and the matrix N is defined by

$$\begin{pmatrix}
\alpha^r & \mathrm{Id} \\
-\beta^r & \mathrm{Id}
\end{pmatrix}$$

where α^r and β^r are the matrices

$$\begin{pmatrix}
\alpha_1^r & 0 & \cdots & 0 \\
0 & \alpha_2^r & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \alpha_{m-1}^r
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
0 & \beta_1^r & 0 & \dots & 0 \\
0 & 0 & \beta_2^r & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & \dots & 0 & \beta_{m-1}^r \\
\beta_m^r & 0 & \dots & 0 & 0
\end{pmatrix}.$$

As a consequence the long exact sequence [14]

$$\cdots \to H_r(\mathcal{R}) \xrightarrow{M(\rho_r)} H_r(\mathcal{X}) \to H_r(\mathcal{T}) \to H_{r-1}(\mathcal{R}) \xrightarrow{M(\rho_{r-1})} H_{r-1}(\mathcal{X}) \to \cdots \tag{10}$$

from Diagram 2 implies the short exact sequence

$$0 \to \operatorname{coker} M(\rho_r) \to H_r(\mathcal{T}) \to \ker M(\rho_{r-1}) \to 0 \tag{11}$$

---

[14] In subsequent papers this long exact sequence is referred to as the *canonical sequence* associated with a tame real or circle valued map. We like to regard it as an analogue of the Morse complex associated to a generic gradient like vector field for a Morse real or circle valued map.

and then the noncanonical isomorphism

$$H_r(\mathcal{T}) = \operatorname{coker} M(\rho_r) \oplus \ker M(\rho_{r-1}). \tag{12}$$

Any splitting $s\colon \ker M(\rho_{r-1}) \to H_r(\mathcal{T})$ in the short exact sequence (11) provides an isomorphism (12). Note that the long exact sequence (10) holds also for homology with local coefficients (i.e. homology with coefficients in a representation). Such sequence can be derived from a similar diagram as Diagram 2, where instead of homology with coefficients in κ one uses homology with local coefficients; in case of interest to us with coefficients in $u\xi_f$. In order to calculate $H_r(X; u\xi_f)$, $u \in \kappa\setminus 0$, we will use this new diagram. Since the local coefficients system $u\xi_f$, when restricted to X_s for any $s \in S^1$ is trivial, in this new diagram all vector spaces and linear maps but $M(\rho_r)$ remain the same as in Diagram 2. The map $M(\rho_r)$ gets replaced by $M((\rho_r)_u)$. In the matrix $M((\rho_r)_u)$ all β_i and all α_i but α_1 are the same as in $M(\rho_r)$ with α_1 replaced by the composition

$$H_r(X_1) \xrightarrow{\alpha_1} H_r(X_2) \xrightarrow{u} H_r(X_2).$$

The second arrow is induced by the multiplication by u on the field κ. As above one obtains the non canonical isomorphism

$$H_r(X; u\xi_f) = \operatorname{coker} M((\rho_r)_u) \oplus \ker M((\rho_{(r-1)})_u) \tag{13}$$

Theorem 2.4: Parts 1 a. and 2a. are a straightforward consequence of Propositions 3.1(i), 3.2, 3.4 and the regularity of the Jordan blocks representations. Parts 1 b. and 2.b are a consequence of equation (12) and of Propositions 3.6 and 3.7. Parts 1 c. and 2 c. are a particular case of Theorem 2.6 parts 1 b.and 1.c.

Theorem 2.5: Part (1) is a consequence of equation (13) and of Proposition 3.7. Part (2) follows from Theorem 2.6 parts 2. and 3. Note that Theorem 2.4(1 b.) is also a consequence of Theorem 2.5 Part 1) for $u = 1$.

A few additional observations are necessary for the proof of Theorem 2.6.

A collection of topological spaces and continuous maps $\{X_i, R_i, a_i\colon R_i \to X_i, b_i\colon R_{i+1} \to X_i, i \in \mathbb{Z}\}$ with $X_i = \emptyset$, $i \leq n-1$, $i \geq m+1$ and $R_i = \emptyset$, $i \leq n$, $i \geq m+1$ can be regarded as a collection (9) considered at the beginning of the section.This is exactly what we obtain from a tame real valued map whose critical points are indexed by the integers between n and m, in particular for $\tilde f\colon \tilde X_{[n,m]} \to \mathbb{R}$, after composing with a homeomorphism of $\mathbb{R}$ to make the critical values of $\tilde f$ indexed by integers. The naturality of the sequence (10) leads, for c, a, b, d critical values with $c \leq a \leq b \leq d$, to the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{coker} M(T_{a,b}(\tilde\rho_r)) \arrow[r] \arrow[d, "v_l"] & H_r(\tilde X_{[a,b]}) \arrow[r] \arrow[d, "v"] & \ker M(T_{a,b}(\tilde\rho_{r-1})) \arrow[r] \arrow[d, "v_r"] & 0 \\
0 \arrow[r] & \operatorname{coker} M(T_{c,d}(\tilde\rho_r)) \arrow[r] & H_r(\tilde X_{[c,d]}) \arrow[r] & \ker M(T_{c,d}(\tilde\rho_{r-1})) \arrow[r] & 0
\end{tikzcd} \tag{14}$$

with v induced by inclusion and v_r injective.

Indeed, given a decomposition of the $G_{2m}-$ representation ρ_r as a sum of bar-codes and Jordan cells, for any $[a,b]$, the $\mathcal{Z}-$ representation with compact support $T_{a,b}(\tilde\rho_{r-1}))$ has a decomposition as a sum of bar codes. The open bar codes in this decomposition, in view of Observation 3.8 are exactly

$$\{I = (\alpha, \beta) \in \tilde{\mathcal{B}}^o_{r-1} \mid I \subset [a,b]\}.$$

In view of of Proposition 3.6 one obtains a base in $\ker M(T_{a,b}(\tilde{\rho}_{r-1}))$ indexed by these open bar codes, say $e^{a,b}_I$. Note that v_r sends $e^{a,b}_I$ into $e^{c,d}_I$ for any I with $I=(\alpha,\beta)\subset[a,b]$. This shows the injectivity of v_r.

If $I=(\alpha,\beta)$, choose $s_I\in H_r(\tilde{X}_{[\alpha,\beta]})$ to be a lift of of $e^{\alpha,\beta}_I\in\ker(M(T_{[\alpha,\beta]})$ w.r. to the surjective map $H_r(\tilde{X}_{[\alpha,\beta]}\to\ker(M(T_{[\alpha,\beta]})$. For each $[a,b]$define the splitting

$$s_{[a,b]}\colon \ker(M(T_{a,b}(\tilde{\rho}_{r-1})))\to H_r(\tilde{X}_{[a,b]})$$

by assigning to $e^{a,b}_I$ $I\in\tilde{\mathcal{B}}^o_{r-1}\mid I\subset(a,b)$ the image of s_I in $H_r(\tilde{X}_{[a,b]}$ by the linear map induced by the inclusion $[\alpha,\beta]\subseteq[a,b]$.

This shows that it is possible to choose splittings

$$s_{[a,b]}\colon \ker(M(T_{a,b}(\tilde{\rho}_{r-1})))\to H_r(\tilde{X}_{[a,b]})$$

and

$$s_{[c,d]}\colon \ker(M(T_{c,d}(\tilde{\rho}_{r-1})))\to H_r(\tilde{X}_{[c,d]}),$$

satisfying $v\cdot s_{[a,b]}=s_{[c,d]}\cdot v_l$, and this for all pairs of critical values.

Note that:

(i) In view of tameness of f it suffices to prove Theorem 2.6 only for a, b critical values of $\tilde{f}$, i.e. $\pi(a)$, $\pi(b)$ critical angles.

(ii) $H_r(\tilde{X})=\lim_{n\to\infty}H_r(\tilde{X}_{[a(n),b(n)]})$ with $a(n)$, $b(n)$ critical values of $\tilde{f}$ and $\lim_{n\to\infty}a(n)=-\infty$, $\lim_{n\to\infty}b(n)=\infty$.

(iii) $\rho_r(\tilde{f}|_{X_{[a,b]}})=T_{a,b}(\tilde{\rho}_r(f))$ and Observation 3.8 calculates the closed and open bar codes of $T_{a,b}(\tilde{\rho}_r(f))$.

Observation 5.1. *1. Choose a decomposition of ρ_r and ρ_{r-1} in indecomposable components and a splitting $s\colon\ker(M(\rho_{r-1})_u)\to H_r(\mathcal{T};u\xi)$, $\xi=\xi_{f^{\mathcal{T}}}$ in the short exact sequence*

$$0\to\operatorname{coker}M((\rho_r)_u)\to H_r(\mathcal{T})\to\ker M((\rho_{r-1})_u)\to 0,$$

resp. compatible splittings $s_{[a,b]}\colon\ker(M(T_{a,b}(\tilde{\rho}_{r-1}))\to H_r(\tilde{X}_{[a,b]})$ in the short exact sequences

$$0\to\operatorname{coker}M(T_{a,b}(\tilde{\rho}_r))\to H_r(\tilde{X}_{[a,b]})\to\ker M(T_{a,b}(\tilde{\rho}_{r-1}))\to 0.$$

In view of Proposition 3.7 and Observation 3.8 one obtains the canonical isomorphisms

$$\Psi_r\colon\kappa[\{I\in\tilde{\mathcal{B}}_r\mid I\ni t\}\sqcup\tilde{\mathcal{J}}_r]\to H_r(\tilde{X}_t)\tag{15}$$

for any $t\in\mathbb{R}$ and

$$\Psi_r\colon\kappa\big[\mathcal{B}^c_r\sqcup\mathcal{B}^o_{r-1}\sqcup\mathcal{J}^{u^{-1}}_r\sqcup\mathcal{J}^{u^{-1}}_{r-1}\big]\to H_r(\mathcal{T};u\xi)\tag{16}$$

resp.

$$\Psi_r([a,b])\colon\kappa\big[\mathcal{B}^c(T_{a,b}(\tilde{\rho}_r))\sqcup\mathcal{B}^o(T_{a,b}(\tilde{\rho}_{r-1})\big]\to H_r(\tilde{X}_{[a,b]}).\tag{17}$$

for any two critical values a,b.

2. Suppose $c\le a\le b\le d$ are critical values. The following diagram is commutative.

$$\begin{tikzcd}
\kappa\bigl[\mathcal{B}^c(T_{a,b}(\tilde\rho_r)) \sqcup \mathcal{B}^o(T_{a,b}(\rho_{r-1})\bigr] \arrow[r, "\Psi_r({[a,b]})"] \arrow[d, "\varphi"'] & H_r(\tilde X_{[a,b]}) \arrow[d] \\
\kappa\bigl[\mathcal{B}^c(T_{c,d}(\tilde\rho_r) \sqcup \mathcal{B}^o(T_{c,d}(\tilde\rho_{r-1})\bigr] \arrow[r, "\Psi_r({[c,d]})"] & H_r(\tilde X_{[c,d]})
\end{tikzcd}$$

Diagram 3

The right side vertical arrow in Diagram 3 is induced by inclusion and the left side vertical arrow

$$\varphi\colon \kappa[\mathcal{B}^c(T_{a,b}(\tilde\rho_r))] \oplus \kappa[\mathcal{B}^o(T_{a,b}(\tilde\rho_{r-1}))] \to \kappa[\mathcal{B}^c(T_{c,d}(\tilde\rho_r))] \oplus k[\mathcal{B}^o(T_{c,d}(\rho_{r-1}))]$$

is the direct sum of the linear maps

$$\varphi_1\colon \kappa[\mathcal{B}^c(T_{a,b}(\tilde\rho_r))] \to \kappa[\mathcal{B}^c(T_{c,d}(\tilde\rho_r))], \quad \varphi_2\colon \kappa[\mathcal{B}^o(T_{a,b}(\tilde\rho_r))] \to \kappa[\mathcal{B}^o(T_{c,d}(\tilde\rho_r))].$$

The map φ_2 *is induced by inclusion and* φ_1 *is the linear extension of the map defined on* $\mathcal{B}^c(T_{a,b}(\tilde\rho_r))$ *as follows. If an* $I \in \mathcal{B}^c(T_{a,b}(\tilde\rho_r))$ *remains an element in* $\mathcal{B}^c(T_{c,d}(\tilde\rho_r))$ *then* $\varphi_1(I) = I$*, if not* $\varphi_1(I) = 0$.

Observations 5.1 and item (ii) above lead to the commutative diagram

$$\begin{tikzcd}
\kappa\bigl[\mathcal{B}^c(T_{a,b}(\tilde\rho_r)) \sqcup \mathcal{B}^o(T_{a,b}(\tilde\rho_{r-1})\bigr] \arrow[r, "\Psi_r({[a,b]})"] \arrow[d, "\varphi'"'] & H_r(\tilde X_{[a,b]}) \arrow[d] \\
\kappa\bigl[(\tilde B^c_r \sqcup \tilde{\mathcal{J}}_r) \sqcup \tilde{\mathcal{B}}^o_{r-1}\bigr] \arrow[r, "\Psi_r"] \arrow[d, "\varphi''"'] & H_r(\tilde X) \arrow[d] \\
\kappa\bigl[(B^c_r \sqcup \mathcal{J}^1_r) \sqcup (\mathcal{B}^o_{r-1} \sqcup \mathcal{J}^1_{r-1})\bigr] \arrow[r, "\hat\Psi_r"] & H_r(X).
\end{tikzcd}$$

Diagram 4

The right side vertical arrows are induced by inclusion and by the covering map $p\colon \tilde X \to X$, and the left side vertical arrows are defined as follows.

The map

$$\varphi'\colon \kappa[\mathcal{B}^c(T_{a,b}(\tilde\rho_r)) \sqcup \mathcal{B}^o(T_{a,b}(\rho_{r-1}))] \to \kappa[(\tilde B^c_r \sqcup \tilde{\mathcal{J}}_r) \sqcup \tilde{\mathcal{B}}^o_{r-1}]$$

is the direct sum of the linear maps

$$\varphi'_1\colon \kappa[\mathcal{B}^c(T_{a,b}(\tilde\rho_r))] \to \kappa[\tilde B^c_r \sqcup \tilde{\mathcal{J}}_r], \qquad \varphi'_2\colon \kappa[\mathcal{B}^o(T_{a,b}(\rho_{r-1}))] \to \kappa[(B^o_{r-1})],$$

and the map

$$\varphi''\colon \kappa[(\tilde B^c_r \sqcup \tilde{\mathcal{J}}_r) \sqcup \tilde{\mathcal{B}}^o_{r-1}] \to k[(B^c_r \sqcup \mathcal{J}^1_r) \sqcup (\mathcal{B}^o_{r-1} \sqcup \mathcal{J}^1_{r-1})]$$

is the direct sum of the linear maps

$$\varphi''_1\colon \kappa[(\tilde B^c_r \sqcup \tilde{\mathcal{J}}_r)] \to \kappa[\mathcal{B}^c_r \sqcup \mathcal{J}^1_r)], \qquad \varphi''_2\colon \kappa[\tilde{\mathcal{B}}^o_{r-1}] \to \kappa[\mathcal{B}^o_{r-1}].$$

The map φ'_2 is induced by inclusion and φ'_1 is the linear extension on the map defined on $\tilde B^c_r \sqcup \tilde{\mathcal{J}}_r$ as follows. If I is an element of $\tilde B^c_r \sqcup \tilde{\mathcal{J}}_r$ which is actually an element of $\tilde B^c_r$ or an element of $\tilde{\mathcal{J}}_r$ then $\varphi'_1(I) = I$, otherwise $\varphi'_1(I) = 0$. The maps φ''_1 and φ''_2 are the linear extensions of the maps defined on $(\tilde B^c_r \sqcup \tilde{\mathcal{J}}_r)$ and on $\tilde{\mathcal{B}}^o_{r-1}$ as follows.

An element $I + 2\pi k \in \tilde{\mathcal{B}}^c_r$ with $I \in \mathcal{B}^c_r$ is sent by φ''_1 to I and an element in $\tilde{\mathcal{J}}_r$ which corresponds to the Jordan cell in $J \in \mathcal{J}^1_r$ is sent to J. All other elements are sent to zero.

An element $I + 2\pi k \in \tilde{\mathcal{B}}^o_{r-1}$ with $I \in \mathcal{B}^o_{r-1}$ is sent by φ''_2 to I.

Theorem 2.6 (Part 1) follows from Diagram 4 by inspecting its left side. To derive (Part 2) and (Part 3) observe that the additive group of integers $\mathbb{Z}$ acts on the set $\tilde{\mathcal{B}}^c_r \sqcup \tilde{\mathcal{B}}^o_r$ freely by translation with the quotient set $\mathcal{B}^c_r \sqcup \mathcal{B}^o_r$ and trivially on $\tilde{\mathcal{J}}_r$. The $\mathbb{Z}[T^{-1}, T]$-module structure of $H_r(\tilde{X})$ corresponds via Ψ to the module structure on $\kappa[(\tilde{\mathcal{B}}^c_r \sqcup (\tilde{\mathcal{J}}_r) \sqcup (\tilde{\mathcal{B}}^o_{r-1})]$ induced by these actions. Theorem 2.8 (Part 2) follows from Theorem 2.6 and (Part 1) from Theorem 2.8 (Part 2) and Theorem 2.4.

6. Proof of Theorem 2.9

Suppose $f\colon X \to S^1$ is a continuous map. Let $\theta \in S^1$ be a tame value [15] and denote its level by $X_\theta = f^{-1}(\theta)$. Moreover, let $H_*(X_\theta)$ denote its singular homology with coefficients in any fixed unital ring κ which is a $\kappa-$ module (vector space when κ is a field). To this situation we will associate a linear relation,

$$R\colon H_*(X_\theta) \rightsquigarrow H_*(X_\theta),$$

see section 6.2 below. One can think of a linear relation as a partially defined, multivalued linear map, see section 6.1 below. While this relation R depends very much on the tame value θ and the function f, its regular part (a linear isomorphism to be defined in section 6.1),

$$R_{\text{reg}}\colon H_*(X_\theta)_{\text{reg}} \xrightarrow{\cong} H_*(X_\theta)_{\text{reg}},$$

turns out to be independent on θ and a homotopy invariant of f.. More precisely, we will show that R_{reg} coincides with the monodromy induced by the deck transformation on a certain invariant submodule of $H_*(\tilde{X})$, where $\tilde{X}$ denotes the infinite cyclic covering associated with f. For the precise statement see Theorem 6.13 below. As a corollary of these considerations we obtain a proof of Theorem 2.9.

6.1. **Linear relations and their regular part.** Suppose V and W are two modules over a fixed commutative ring. Recall that a linear relation from V to W can be considered as a submodule $R \subseteq V \times W$. Notationally, we indicate this situation by $R\colon V \rightsquigarrow W$. For $v \in V$ and $w \in W$ we write vRw iff v is in relation with w, i.e. $(v, w) \in R$. Every module homomorphism $V \to W$ can be regarded as a linear relation $V \rightsquigarrow W$ in a natural way. If U is another module, and $S\colon W \rightsquigarrow U$ is a linear relation, then the composition $SR\colon V \rightsquigarrow U$ is the linear relation defined by $v(SR)u$ iff there exists $w \in W$ such that vRw and wSu. Clearly, this is an associative composition generalizing the ordinary composition of module homomorphisms. For the identical relations we have $R\,\mathrm{id}_V = R$ and $\mathrm{id}_W\, R = R$. Modules over a fixed commutative ring and linear relations thus constitute a category. If $R\colon V \rightsquigarrow W$ is a linear relation we define a linear relation $R^\dagger\colon W \rightsquigarrow V$ by $wR^\dagger v$ iff vRw. Clearly, $R^{\dagger\dagger} = R$ and $(SR)^\dagger = R^\dagger S^\dagger$.

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[15] i.e. X_θ is a deformation retract of an open neighborhood of X_θ

A linear relation $R\colon V \rightsquigarrow W$ gives rise to the following submodules:

$$\begin{aligned} \operatorname{dom}(R) &:= \{v \in V \mid \exists w \in W : vRw\} \\ \operatorname{img}(R) &:= \{w \in W \mid \exists v \in V : vRw\} \\ \ker(R) &:= \{v \in V \mid vR0\} \\ \operatorname{mul}(R) &:= \{w \in W \mid 0Rw\} \end{aligned}$$

Clearly, $\ker(R) \subseteq \operatorname{dom}(R) \subseteq V$, and $W \supseteq \operatorname{img}(R) \supseteq \operatorname{mul}(R)$. Note that R is a homomorphism (map) iff $\operatorname{dom}(R) = V$ and $\operatorname{mul}(R) = 0$. One readily verifies:

Lemma 6.1. *For a linear relation $R\colon V \rightsquigarrow W$ the following are equivalent:*

(a) R is an isomorphism in the category of modules and linear relations.
(b) $\operatorname{dom}(R) = V$, $\operatorname{img}(R) = W$, $\ker(R) = 0$, *and* $\operatorname{mul}(R) = 0$.
(c) R is an isomorphism of modules.

In this case $R^{-1} = R^\dagger$.

For a linear relation $R\colon V \rightsquigarrow V$, we introduce the following submodules:

$$\begin{aligned} K_+ &:= \{v \in V \mid \exists k\, \exists v_i \in V : vRv_1Rv_2R\cdots Rv_kR0\} \\ K_- &:= \{v \in V \mid \exists k\, \exists v_i \in V : 0Rv_{-k}R\cdots Rv_{-2}Rv_{-1}Rv\} \\ D_+ &:= \{v \in V \mid \exists v_i \in V : vRv_1Rv_2Rv_3R\cdots\} \\ D_- &:= \{v \in V \mid \exists v_i \in V : \cdots Rv_{-3}Rv_{-2}Rv_{-1}Rv\} \\ D := D_- \cap D_+ &= \{v \in V \mid \exists v_i \in V : \cdots Rv_{-2}Rv_{-1}RvRv_1Rv_2R\cdots\}, \end{aligned}$$

Clearly, $K_- \subseteq D_- \subseteq V \supseteq D_+ \supseteq K_+$. Also note that passing from R to $R^\dagger$, the roles of $+$ and $-$ get interchanged. Moreover, we introduce a linear relation on the quotient module

$$V_{\mathrm{reg}} := \frac{D}{(K_- + K_+) \cap D}$$

defined as the composition

$$V_{\mathrm{reg}} = \frac{D}{(K_- + K_+) \cap D} \overset{\pi^\dagger}{\rightsquigarrow} D \overset{\iota}{\rightsquigarrow} V \overset{R}{\rightsquigarrow} V \overset{\iota^\dagger}{\rightsquigarrow} D \overset{\pi}{\rightsquigarrow} \frac{D}{(K_- + K_+) \cap D} = V_{\mathrm{reg}},$$

where ι and π denote the canonical inclusion and projection, respectively. In other words, two elements in V_{reg} are related by R_{reg} iff they admit representatives in D which are in related by R. We refer to R_{reg} as the *regular part* of R.

Proposition 6.2. *The relation $R_{\mathrm{reg}}\colon V_{\mathrm{reg}} \rightsquigarrow V_{\mathrm{reg}}$ is an isomorphism of modules. Moreover, the natural inclusion induces a canonical isomorphism*

$$V_{\mathrm{reg}} = \frac{D}{(K_- + K_+) \cap D} \xrightarrow{\cong} \frac{(K_- + D_+) \cap (D_- + K_+)}{K_- + K_+} \tag{18}$$

which intertwines R_{reg} with the relation induced on the right hand side quotient.

Proof. Clearly, (18) is well defined and injective. To see that it is onto let

$$x = k_- + d_+ = d_- + k_+ \in (K_- + D_+) \cap (D_- + K_+),$$

where $k_\pm \in K_\pm$ and $d_\pm \in D_\pm$. Thus

$$x - k_- - k_+ = d_+ - k_+ = d_- - k_- \in D_- \cap D_+ = D.$$

We conclude $x \in D + K_- + K_+$, whence (18) is onto. We will next show that this isomorphism intertwines R_{reg} with the relation induced on the right hand side. To do so, suppose $xR\tilde{x}$ where

$$x = k_- + d_+ = d_- + k_+ \in (K_- + D_+) \cap (D_- + K_+),$$
$$\tilde{x} = \tilde{k}_- + \tilde{d}_+ = \tilde{d}_- + \tilde{k}_+ \in (K_- + D_+) \cap (D_- + K_+),$$

and $k_\pm, \tilde{k}_\pm \in K_\pm$ and $d_\pm, \tilde{d}_\pm \in D_\pm$. Note that there exist $k'_+ \in K_+$ and $\tilde{k}'_- \in K_-$ such that $k_+ R k'_+$ and $\tilde{k}'_- R \tilde{k}_-$. By linearity of R we obtain

$$\underbrace{(x - k_+ - \tilde{k}'_-)}_{\in D_-} R \underbrace{(\tilde{x} - k'_+ - \tilde{k}_-)}_{\in D_+}.$$

We conclude $d := x - k_+ - \tilde{k}'_- \in D$, $\tilde{d} := \tilde{x} - k'_+ - \tilde{k}_- \in D$, and $dR\tilde{d}$. This shows that the relations induced on the two quotients in (18) coincide. We complete the proof by showing that R_{reg} is an isomorphism. Clearly, $\mathrm{dom}(R_{\mathrm{reg}}) = V_{\mathrm{reg}} = \mathrm{img}(R_{\mathrm{reg}})$. We will next show $\ker(R_{\mathrm{reg}}) = 0$. To this end suppose $dR\tilde{d}$, where

$$d \in D \quad \text{and} \quad \tilde{d} = \tilde{k}_- + \tilde{k}_+ \in (K_- + K_+) \cap D$$

with $\tilde{k}_\pm \in K_\pm$. Note that $\tilde{k}_- = \tilde{d} - \tilde{k}_+ \in K_- \cap D_+$. Thus there exists $k_- \in K_- \cap D_+$ such that $k_- R \tilde{k}_-$. By linearity of R, we get $(d - k_-) R \tilde{k}_+$, whence $d - k_- \in K_+$ and thus $d \in K_- + K_+$. This shows $\ker(R_{\mathrm{reg}}) = 0$. Analogously, we have $\mathrm{mul}(R_{\mathrm{reg}}) = 0$. In view of Lemma 6.1 we conclude that R_{reg} is an isomorphism of modules. ☐

We will now specialize to linear relations on finite dimensional vector spaces and provide another description of V_{reg} in this case. Consider the category whose objects are finite dimensional vector spaces V equipped with a linear relation $R\colon V \rightsquigarrow V$ and whose morphisms are linear maps $\psi\colon V \to W$ such that for all $x, y \in V$ with xRy we also have $\psi(x) Q \psi(y)$, where W is another finite dimensional vector space with linear relation $Q\colon W \rightsquigarrow W$. It is readily checked that this is an abelian category. By the Remak–Schmidt theorem, every linear relation on a finite dimensional vector space can therefore be decomposed into a direct sum of indecomposable ones, $R \cong R_1 \oplus \cdots \oplus R_N$, where the factors are unique up to permutation and isomorphism. The decomposition itself, however, is not canonical.

Proposition 6.3. *Let $R\colon V \rightsquigarrow V$ be a linear relation on a finite dimensional vector space over an algebraic closed field , and let $R \cong R_1 \oplus \cdots \oplus R_N$ denote a decomposition into indecomposable linear relations. Then R_{reg} is isomorphic to the direct sum of factors R_i whose relations are linear isomorphisms.*

Proof. Since the definition of R_{reg} is a natural one, we clearly have

$$R_{\mathrm{reg}} \cong (R_1)_{\mathrm{reg}} \oplus \cdots \oplus (R_N)_{\mathrm{reg}}.$$

Consequently, it suffices to show the following two assertions:

(a) If $R\colon V \rightsquigarrow V$ is an isomorphism of vector spaces, then $V_{\mathrm{reg}} = V$ and $R_{\mathrm{reg}} = R$.
(b) If $R\colon V \rightsquigarrow V$ is an indecomposable linear relation on a finite dimensional vector space which is not a linear isomorphism, then $V_{\mathrm{reg}} = 0$.

The first statement is obvious, in this case we have $K_- = K_+ = 0$ and $D = D_- = D_+ = V$. To see the second assertion, note that an indecomposable linear relation $R \subseteq V \times V$ gives rise to an indecomposable representation $R \rightrightarrows V$ of the quiver G_2. Since R is not an isomorphism, the quiver representation has to be of the bar code type. Using the explicit descriptions of the bar code representations, it is straight forward to conclude $V_{\mathrm{reg}} = 0$. ☐

In the subsequent section we will also make use of the following result:

Proposition 6.4. *Suppose $R\colon V \rightsquigarrow V$ is a linear relation on a finite dimensional vector space. Then:*

$$D_+ = D + K_+, \quad D_- = K_- + D, \quad \textit{and} \tag{19}$$

$$K_- \cap D_+ = K_- \cap K_+ = D_- \cap K_+. \tag{20}$$

For the proof we first establish two lemmas.

Lemma 6.5. *Suppose $R\colon V \rightsquigarrow W$ is a linear relation between vector spaces such that $\dim V = \dim W < \infty$. Then the following are equivalent:*

(a) R is an isomorphism.
(b) $\operatorname{dom}(R) = V$ and $\ker(R) = 0$.
(c) $\operatorname{img}(R) = W$ and $\operatorname{mul}(R) = 0$.

Proof. This follows immediately from the dimension formula

$$\dim \operatorname{dom}(R) + \dim \operatorname{mul}(R) = \dim(R) = \dim \operatorname{img}(R) + \dim \ker(R)$$

and Lemma 6.1. ☐

Lemma 6.6. *If V is finite dimensional, then the composition of relations*

$$D_+/K_+ \overset{\pi^\dagger}{\rightsquigarrow} D_+ \overset{\iota}{\rightsquigarrow} V \overset{R^k}{\rightsquigarrow} V \overset{\iota^\dagger}{\rightsquigarrow} D_+ \overset{\pi}{\rightsquigarrow} D_+/K_+,$$

is a linear isomorphism, for every $k \geq 0$, where ι and π denote the canonical inclusion and projection, respectively. Analogously, the relation induced by R^k on D_-/K_- is an isomorphism, for all $k \geq 0$. Moreover, for sufficiently large k,

$$D_- = \operatorname{img}(R^k) \quad \textit{and} \quad D_+ = \operatorname{dom}(R^k).$$

Proof. One readily verifies $\operatorname{dom}(\pi\iota^\dagger R^k \iota\pi^\dagger) = D_+/K_+$ and $\ker(\pi\iota^\dagger R^k \iota\pi^\dagger) = 0$. The first assertion thus follows from Lemma 6.5 above. Considering $R^\dagger$ we obtain the second statement. Clearly, $\operatorname{dom}(R^k) \supseteq \operatorname{dom}(R^{k+1})$, for all $k \geq 0$. Since V is finite dimensional, we must have $\operatorname{dom}(R^k) = \operatorname{dom}(R^{k+1})$, for sufficiently large k. Given $v \in \operatorname{dom}(R^k)$, we thus find $v_1 \in \operatorname{dom}(R^k)$ such that vRv_1. Proceeding inductively, we construct $v_i \in \operatorname{img}(R^k)$ such that $vRv_1Rv_2R\cdots$, whence $v \in D_+$. This shows $\operatorname{dom}(R^k) \subseteq D_+$, for sufficiently large k. As the converse inclusion is obvious we get $D_+ = \operatorname{dom}(R^k)$. Considering $R^\dagger$, we obtain the last statement. ☐

Proof of Proposition 6.4. From Lemma 6.6 we get $\operatorname{img}(\pi\iota^\dagger R^k) = D_+/K_+$, whence $D_+ \subseteq \operatorname{img}(R^k) + K_+$, for every $k \geq 0$, and thus $D_+ \subseteq D_- + K_+$. This implies $D_+ = D + K_+$. Considering $R^\dagger$ we obtain the other equality in (19). From Lemma 6.6 we also get $\operatorname{mul}(\pi\iota^\dagger R^k) = 0$, whence $\operatorname{mul}(R^k) \cap D_+ \subseteq K_+$, for every $k \geq 0$. This gives $K_- \cap D_+ = K_- \cap K_+$. Considering $R^\dagger$ we get the other equality in (20). ☐

6.2. **Monodromy.** Suppose $f : X \to S^1$ is a continuous map and let

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\tilde{f}"] \arrow[d] & \mathbb{R} \arrow[d] \\
X \arrow[r, "f"] & S^1
\end{tikzcd}$$

denote the associated infinite cyclic covering. For $r \in \mathbb{R}$ we put $\tilde{X}_r = \tilde{f}^{-1}(r)$ and let $H_*(\tilde{X}_r)$ denote its singular homology with coefficients in any fixed module. If $r_1 \leq r_2$ we define a linear relation

$$B_{r_1}^{r_2} \colon H_*(\tilde{X}_{r_1}) \rightsquigarrow H_*(\tilde{X}_{r_2})$$

by declaring $a_1 \in H_*(\tilde{X}_{r_1})$ to be in relation with $a_2 \in H_*(\tilde{X}_{r_2})$ iff their images in $H_*(\tilde{X}_{[r_1,r_2]})$ coincide, where $\tilde{X}_{[r_1,r_2]} = f^{-1}([r_1, r_2])$. If $r_1 \leq r_2 \leq r_3$ we clearly have $B_{r_2}^{r_3} B_{r_1}^{r_2} \subseteq B_{r_1}^{r_3}$. If r_2 is a tame value this becomes an equality of relations:

Lemma 6.7. *Suppose $r_1 \leq r_2 \leq r_3$ and assume r_2 is a tame value. Then, as linear relations, $B_{r_2}^{r_3} B_{r_1}^{r_2} = B_{r_1}^{r_3}$.*

Proof. Since r_2 is a tame value, we have an exact Mayer–Vietoris sequence,

$$H_*(\tilde{X}_{r_2}) \to H_*(\tilde{X}_{[r_1,r_2]}) \oplus H_*(\tilde{X}_{[r_2,r_3]}) \to H_*(\tilde{X}_{[r_1,r_3]}),$$

which immediately implies the statement. $\square$

Fix a tame value $\theta \in S^1$ of f and a lift $\tilde{\theta} \in \mathbb{R}$, $e^{\mathbf{i}\tilde{\theta}} = \theta$. Using the projection $\tilde{X} \to X$, we may canonically identify $\tilde{X}_{\tilde{\theta}} = X_\theta = f^{-1}(\theta)$. Moreover, let $\tau \colon \tilde{X} \to \tilde{X}$ denote the fundamental deck transformation, i.e. $\tilde{f} \circ \tau = \tilde{f} + 2\pi$. Note that τ induces homeomorphisms between levels, $\tau \colon \tilde{X}_r \to \tilde{X}_{r+2\pi}$, and define a linear relation

$$R \colon H_*(X_\theta) \rightsquigarrow H_*(X_\theta)$$

as the composition

$$H_*(X_\theta) = H_*(\tilde{X}_{\tilde{\theta}}) \overset{B_{\tilde{\theta}}^{\tilde{\theta}+2\pi}}{\rightsquigarrow} H_*(\tilde{X}_{\tilde{\theta}+2\pi}) \overset{\tau_*^\dagger}{\rightsquigarrow} H_*(\tilde{X}_{\tilde{\theta}}) = H_*(X_\theta). \tag{21}$$

In other words, for $a, b \in H_*(X_\theta)$ we have aRb iff $aB_{\tilde{\theta}}^{\tilde{\theta}+2\pi}(\tau_* b)$, i.e. iff a and $\tau_* b$ coincide in $H_*(\tilde{X}_{[\tilde{\theta},\tilde{\theta}+2\pi]})$. Particularly:

Lemma 6.8. *If $a, b \in H_*(X_\theta)$ and aRb, then $a = \tau_* b$ in $H_*(\tilde{X})$.*

We will continue to use the notation $K_\pm$, $D_\pm$, and R_{reg} introduced in the previous section for this relation R on $H_*(X_\theta)$. Particularly, its regular part,

$$R_{\mathrm{reg}} \colon H_*(X_\theta)_{\mathrm{reg}} \to H_*(X_\theta)_{\mathrm{reg}},$$

is a module automorphism.

Lemma 6.9. *We have:*

$$K_+ = \ker\big(H_*(X_\theta) \to H_*(\tilde{X}_{[\tilde{\theta},\infty)})\big)$$
$$K_- = \ker\big(H_*(X_\theta) \to H_*(\tilde{X}_{(-\infty,\tilde{\theta}]})\big)$$

Both maps are induced by the canonical inclusion $X_\theta = \tilde{X}_{\tilde{\theta}} \to \tilde{X}$.

Proof. We will only show the first equality, the other one can be proved along the same lines. To see the inclusion $K_+ \subseteq \ker(H_*(X_\theta) \to H_*(\tilde{X}_{[\tilde{\theta},\infty)}))$, let $a \in K_+$. Hence, there exist $a_k \in H_*(X_\theta)$, almost all of which vanish, such that $aRa_1Ra_2R\cdots$. In $H_*(\tilde{X}_{[\tilde{\theta},\tilde{\theta}+2\pi]})$, we thus have:

$$a = \tau_* a_1, \quad a_1 = \tau_* a_2, \quad a_2 = \tau_* a_3, \quad \dots$$

In $H_*(\tilde{X}_{[\tilde{\theta},\infty)})$, we obtain:

$$a = \tau_* a_1 = \tau_*^2 a_2 = \tau_*^3 a_3 = \cdots$$

Since some a_k have to be zero, we conclude that a vanishes in $H_*(\tilde{X}_{[\tilde{\theta},\infty)})$.

To see the converse inclusion, $K_+ \supseteq \ker(H_*(\tilde{X}_\theta) \to H_*(\tilde{X}_{[\tilde{\theta},\infty)}))$, set

$$U := \bigsqcup_{0 \leq k \text{ even}} \tilde{X}_{[\tilde{\theta}+2\pi k,\tilde{\theta}+2\pi(k+1)]}, \qquad V := \bigsqcup_{1 \leq k \text{ odd}} \tilde{X}_{[\tilde{\theta}+2\pi k,\tilde{\theta}+2\pi(k+1)]}$$

and note that $U \cup V = \tilde{X}_{[\tilde{\theta},\infty)}$, as well as $U \cap V = \bigsqcup_{k\in\mathbb{N}} \tilde{X}_{\tilde{\theta}+2\pi k}$. Since θ is a tame value, we have an exact Mayer–Vietoris sequence

$$\bigoplus_{k\in\mathbb{N}} H_*(\tilde{X}_{\tilde{\theta}+2\pi k}) = H_*\Big(\bigsqcup_{k\in\mathbb{N}} \tilde{X}_{\tilde{\theta}+2\pi k}\Big) \to H_*(U) \oplus H_*(V) \to H_*(\tilde{X}_{[\tilde{\theta},\infty)}).$$

For $b \in \ker(H_*(X_\theta) \to H_*(\tilde{X}_{[\tilde{\theta},\infty)}))$ we thus find $b_k \in H_*(\tilde{X}_{\tilde{\theta}+2\pi k})$, almost all of which vanish, such that:

$$b = b_1 \in H_*(\tilde{X}_{[\tilde{\theta},\tilde{\theta}+2\pi]}), \quad b_1 + b_2 = 0 \in H_*(\tilde{X}_{[\tilde{\theta}+2\pi,\tilde{\theta}+4\pi]}), \quad b_2 + b_3 = 0 \in H_*(\tilde{X}_{[\tilde{\theta}+4\pi,\tilde{\theta}+6\pi]}), \quad \dots$$

Putting $c_k := (-1)^{k-1}\tau_*^{-k} b_k \in H_*(\tilde{X}_{\tilde{\theta}})$, we obtain the following equalities in $H_*(\tilde{X}_{[\tilde{\theta},\tilde{\theta}+2\pi]})$:

$$b = \tau_* c_1, \quad c_1 = \tau_* c_2, \quad c_2 = \tau_* c_3, \quad \dots$$

In other words, we have the relations $bRc_1Rc_2Rc_3R\cdots$. Since some c_k has to be zero, we conclude $b \in K_+$, whence the lemma. ☐

Introduce the upwards Novikov complex as a projective limit of relative singular chain complexes,

$$C_*^{\mathrm{Nov},+}(\tilde{X}) := \varprojlim_r C_*(\tilde{X}, \tilde{X}_{[r,\infty)}),$$

and let $H_*^{\mathrm{Nov},+}(\tilde{X})$ denote its homology. Analogously, we define a downwards Novikov complex $C_*^{\mathrm{Nov},-}(\tilde{X}) = \varprojlim_r C_*(\tilde{X}, \tilde{X}_{(-\infty,r]})$ and the corresponding homology, $H_*^{\mathrm{Nov},-}(\tilde{X})$. We will also use similar notation for subsets of $\tilde{X}$.

Lemma 6.10. *We have:*

$$D_+ = \ker\big(H_*(X_\theta) \to H_*^{\mathrm{Nov},+}(\tilde{X}_{[\tilde{\theta},\infty)})\big)$$
$$D_- = \ker\big(H_*(X_\theta) \to H_*^{\mathrm{Nov},-}(\tilde{X}_{(-\infty,\tilde{\theta}]})\big)$$

Both maps are induced by the canonical inclusion $X_\theta = \tilde{X}_{\tilde{\theta}} \to \tilde{X}$.

Proof. Using the exact Mayer–Vietoris sequence

$$\prod_{k\in\mathbb{N}} H_*(\tilde{X}_{\tilde{\theta}+2\pi k}) = H_*^{\mathrm{Nov},+}\Big(\bigsqcup_{k\in\mathbb{N}} \tilde{X}_{\tilde{\theta}+2\pi k}\Big) \to H_*^{\mathrm{Nov},+}(U) \oplus H_*^{\mathrm{Nov},+}(V) \to H_*^{\mathrm{Nov},+}(\tilde{X}_{[\tilde{\theta},\infty)}),$$

this can be proved along the same lines as Lemma 6.9. ☐

Let us introduce a complex

$$C_*^{\mathrm{l.f.}}(\tilde X) := \varprojlim_r C_*(\tilde X, \tilde X_{(-\infty,-r]} \cup \tilde X_{[r,\infty)})$$

and denote its homology by $H_*^{\mathrm{l.f.}}(\tilde X)$. If f is proper, this is the complex of locally finite singular chains.

Lemma 6.11. *We have:*

$$\begin{aligned} K_- + K_+ &= \ker\bigl(H_*(X_\theta) \to H_*(\tilde X)\bigr) \\ K_- + D_+ &= \ker\bigl(H_*(X_\theta) \to H_*^{\mathrm{Nov},+}(\tilde X)\bigr) \\ D_- + K_+ &= \ker\bigl(H_*(X_\theta) \to H_*^{\mathrm{Nov},-}(\tilde X)\bigr) \\ D_- + D_+ &= \ker\bigl(H_*(X_\theta) \to H_*^{\mathrm{l.f.}}(\tilde X)\bigr) \end{aligned}$$

All maps are induced by the canonical inclusion $X_\theta = \tilde X_{\tilde\theta} \to \tilde X$.

Proof. The first statement follows from the exact Mayer–Vietoris sequence

$$H_*(\tilde X_{\tilde\theta}) \to H_*(\tilde X_{(-\infty,\tilde\theta]}) \oplus H_*(\tilde X_{[\tilde\theta,\infty)}) \to H_*(\tilde X)$$

and Lemma 6.9. The second assertion follows from the exact Mayer–Vietoris sequence

$$H_*(\tilde X_{\tilde\theta}) \to H_*(\tilde X_{(-\infty,\tilde\theta]}) \oplus H_*^{\mathrm{Nov},+}(\tilde X_{[\tilde\theta,\infty)}) \to H_*^{\mathrm{Nov},+}(\tilde X)$$

and Lemma 6.9 and 6.10. Similarly, one can check the third equality. To see the last statement we use the exact Mayer–Vietoris sequence

$$H_*(\tilde X_{\tilde\theta}) \to H_*^{\mathrm{Nov},-}(\tilde X_{(-\infty,\tilde\theta]}) \oplus H_*^{\mathrm{Nov},+}(\tilde X_{[\tilde\theta,\infty)}) \to H_*^{\mathrm{l.f.}}(\tilde X)$$

and Lemma 6.10. □

Lemma 6.12. *We have*

$$\ker\Bigl(H_*(\tilde X) \to H_*^{\mathrm{Nov},-}(\tilde X) \oplus H_*^{\mathrm{Nov},+}(\tilde X)\Bigr) \subseteq \operatorname{img}\bigl(H_*(\tilde X_{\tilde\theta}) \to H_*(\tilde X)\bigr),$$

where all maps are induced by the tautological inclusions.

Proof. This follows from the following commutative diagram of exact Mayer–Vietoris sequences:

$$\begin{tikzcd}
H_{*+1}^{\mathrm{l.f.}}(\tilde X) \arrow[r, "\partial"] \arrow[d, equal] & H_*(\tilde X) \arrow[r] & H_*^{\mathrm{Nov},-}(\tilde X) \oplus H_*^{\mathrm{Nov},+}(\tilde X) \\
H_{*+1}^{\mathrm{l.f.}}(\tilde X) \arrow[r, "\partial"] & H_*(\tilde X_{\tilde\theta}) \arrow[r] \arrow[u] & H_*^{\mathrm{Nov},-}(\tilde X_{(-\infty,\tilde\theta]}) \oplus H_*^{\mathrm{Nov},+}(\tilde X_{[\tilde\theta,\infty)}) \arrow[u]
\end{tikzcd}$$

A similar argument was used in [11, Lemma 2.5]. □

Theorem 6.13. *The inclusion $\iota\colon X_\theta = \tilde X_{\tilde\theta} \to \tilde X$ induces a canonical isomorphism*

$$H_*(X_\theta)_{\mathrm{reg}} = \frac{D}{(K_- + K_+) \cap D} \xrightarrow{\cong} \ker\Bigl(H_*(\tilde X) \to H_*^{\mathrm{Nov},-}(\tilde X) \oplus H_*^{\mathrm{Nov},+}(\tilde X)\Bigr),$$

intertwining R_{reg} with the monodromy isomorphism induced by the deck transformation $\tau\colon \tilde X \to \tilde X$ on the right hand side. Moreover, working with coefficients in *a field, and assuming that $H_*(X_\theta)$ is finite dimensional, the common kernel on the right hand side above coincides with*

$$\ker\bigl(H_*(\tilde X)\to H_*^{\mathrm{Nov},-}(\tilde X)\bigr)=\ker\bigl(H_*(\tilde X)\to H_*^{\mathrm{Nov},+}(\tilde X)\bigr).$$

Particularly, in this case the latter two kernels are finite dimensional too.

Proof. It follows immediately from Lemma 6.11 and 6.12 that $\iota_*\colon H_*(X_\theta)\to H_*(\tilde X)$ induces an isomorphism

$$\frac{(K_-+D_+)\cap(D_-+K_+)}{K_-+K_+}\xrightarrow{\cong}\ker\Bigl(H_*(\tilde X)\to H_*^{\mathrm{Nov},-}(\tilde X)\oplus H_*^{\mathrm{Nov},+}(\tilde X)\Bigr).$$

In view of Lemma 6.8, this isomorphism intertwines the isomorphism induced by R on the left hand side, with the monodromy isomorphism on the right hand side. Combining this with Proposition 6.2 we obtain the first assertion. For the second statement it suffices to show

$$\ker\bigl(H_*(\tilde X)\to H_*^{\mathrm{Nov},+}(\tilde X)\bigr)\subseteq\ker\Bigl(H_*(\tilde X)\to H_*^{\mathrm{Nov},-}(\tilde X)\oplus H_*^{\mathrm{Nov},+}(\tilde X)\Bigr),\qquad(22)$$

as the converse inclusion is obvious, and the corresponding statement for the downward Novikov homology can be derived analogously. To this end, suppose $a\in\ker\bigl(H_*(\tilde X)\to H_*^{\mathrm{Nov},+}(\tilde X)\bigr)$. Then there exists k such that $\tau_*^k a$ is contained in the image of $H_*(\tilde X_{(-\infty,\tilde\theta]})\to H_*(\tilde X)$. Using the exact Mayer–Vietoris sequence

$$H_*(\tilde X_{\tilde\theta})\to H_*(\tilde X_{(-\infty,\tilde\theta]})\oplus H_*^{\mathrm{Nov},+}(\tilde X_{[\tilde\theta,\infty)})\to H_*^{\mathrm{Nov},+}(\tilde X)$$

we conclude, that $\tau_*^k a$ is contained in the image of $H_*(\tilde X_{\tilde\theta})\to H_*(\tilde X)$. Thus $\tau_*^k a$ is contained in $\iota_*(D_+)$, see Lemma 6.11. Since $H_*(X_\theta)$ is assumed to be a finite dimensional vector space, we have $\iota_*(D_-)=\iota_*(D)=\iota_*(D_+)$, see (19). Using Lemma 6.11 we thus conclude $\tau_*^k a$ is contained in the kernel on the right hand side of (22). Since this common kernel is invariant under the isomorphism $\tau_*\colon H_*(\tilde X)\to H_*(\tilde X)$, we conclude that a has to be contained in the common kernel too, whence the theorem. ☐

Clearly, Theorem 6.13 and Proposition 6.3 imply Theorem 2.9.

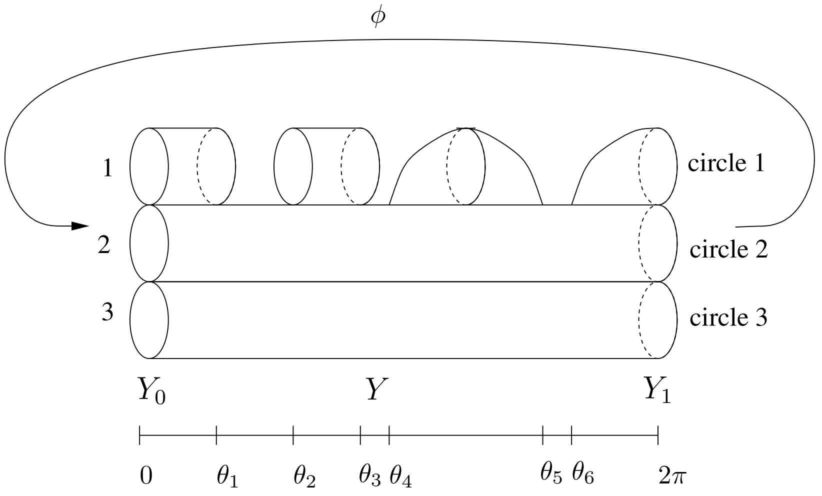
7. EXAMPLE

Figure 2 below, describes a tame angle valued map $f\colon X\to\mathbb{R}$ whose bar codes and Jordan cells are given in the attached table.

The space X is obtained from Y by identifying its right end Y_1 (a union of three circles) to the left end Y_0 (a union of three circles) following the map $\phi\colon Y_1\to Y_0$ explained in the table. The map $f\colon X\to S^1$ is induced by the projection of Y on the interval $[0,2\pi]$. Note that $H_1(Y_1)=H_1(Y_0)=\kappa\oplus\kappa\oplus\kappa$ and ϕ induces a linear map in H_1-homology represented by the matrix

$$\begin{pmatrix}3&0&0\\1&2&-1\\0&0&2\end{pmatrix}.$$

There are no bar codes or Jordan cells in dimension 2 since each fiber of f is one-dimensional and, as all fibers are connected in dimension zero we have only one Jordan cell $\rho^{II}(1;1)$.It remains to describe the bar codes and the Jordan cells in dimension 1. For this example it is not hard to derive them by applying the main theorems:



map ϕ	r-invariants		
	dimension	bar codes	Jordan cells
circle 1: 3 times around circle 1 circle 2: 2 time around circle 2 circle 3: 1 time around 2 counter clockwise and 2 times around circle 3	0		$(1, 1)$
	1	$(\theta_6, \theta_1 + 2\pi]$ $[\theta_2, \theta_3]$ (θ_4, θ_5)	$(2, 2)$

FIGURE 2. Example of r-invariants for a circle valued map

In view of Theorem 2.9 we see that the monodromy identifies to the regular part of the linear relation defined by the linear maps $\omega_1 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} : \kappa^3 \to \kappa^4$ and $\omega_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} : \kappa^3 \to \kappa^4$. This regular part can be calculated using the definition in subsection 6.1 which can be calculated and is the Jordan cell $(2, 2)$.

Theorem 2.6 1. a. implies that there exists an open bar code $(4, 5)$ and one closed bar code $[2, 3]$. This by looking at the homology of various $X_{[a,b]}$ with $0 \leq a \leq b \leq 2\pi$. The same argument implies that we have an other bar code of the form $(\theta_6, \theta_1 + 2\pi k]$. Theorem 2.4 a. implies that $k = 1$ and these are all bar codes.

We explain below how to use the elementary transformations described in section 4 to derive the bar codes and the Jordan cells in the table above.

Note that $m = 7$ and we have three representations to consider: ρ_0, whose all vector spaces are isomorphic to κ and linear maps identity, the representation ρ_2 which is trivial and the representation ρ_1 which has to be described and decomposed.

The G_{14}-representation ρ_1: Choose $t_1, \ldots, t_7$ so that we have $0 < t_7 - 2\pi < \theta_1 < t_1 < \theta_2 < \cdots < \theta_6 < t_6 < \theta_7 = 2\pi$. One has:

$$V_{2i-1} = \begin{cases} \kappa^2 & \text{for } i = 2, 4, 6 \\ \kappa^3 & \text{for } i = 1, 5, 7, 9 \end{cases} \qquad V_{2i} = \begin{cases} \kappa^2 & \text{for } i = 4, 5, 6 \\ \kappa^3 & \text{for } i = 1, 2, 3, 7 \end{cases}$$

$$\beta_i = \begin{cases} \mathrm{Id} & \text{for } i = 2,5,7 \\ \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \text{for } i = 1,3 \\ \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) & \text{for } i = 4,6 \end{cases} \qquad \alpha_i = \begin{cases} \mathrm{Id} & \text{for } i = 1,3,4,6 \\ \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \text{for } i = 2 \\ \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) & \text{for } i = 5 \\ \left(\begin{smallmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{smallmatrix}\right) & \text{for } i = 7 \end{cases}$$

Using the elementary transformation we modify the representation ρ_1 into $\rho(1)$ then into $\rho(2)$ and finally into $\rho(3)$ keeping track of the elimination of bar codes.

Precisely:

1. Apply $T_1(5)$ and get $\rho(1) = T_1(5)(\rho_1)$.
2. Apply $T_4(3) \cdot T_3(2)$ and get $\rho(2) = T_4(3) \cdot T_3(2)(\rho(1))$.
3. Apply $T_1(7) \cdot T_4(1)$ and get $\rho(3) = T_1(7) \cdot T_4(1)(\rho(2))$.

In view of the Appendix to section 3, which describe what each elementary transformation does, it is easy to see that:

(i) $\rho(3)$ has all α_i but α_7 and β_i the identity with $\alpha_7 = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$; hence no bar codes,

(ii) $\rho(2)$ has one bar code $(6, 8]$,

(iii) $\rho(1)$ has two bar codes $(6, 8]$ and $[2, 3]$,

(iv) ρ_1 has the bar codes $(6, 8]$, $[2, 3]$ $(4, 5)$.

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