LECTURES ON WITTEN HELFFER SJÖSTRAND THEORY

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Abstract.

Witten- Helffer-Sjöstrand theory is a considerable addition to the De Rham-Hodge theory for Riemannian manifolds and can serve as a general tool to prove results about comparison of numerical invariants associated to compact manifolds analytically, i.e. by using a Riemannian metric, or combinatorially, i.e by using a triangulation. In this presentation a triangulation, or a partition of a smooth manifold in cells, will be viewed in a more analytic spirit, being provided by the stable manifolds of the gradient of a nice Morse function. WHS theory was recently used both for providing new proofs for known but difficult results in topology, as well as new results and a positive solution for an important conjecture about L_2-torsion, cf [BFKM]. This presentation is a short version of a one quarter course I have given during the spring of 1997 at OSU.

Contents

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0. Introduction.

Witten Helffer Sjöstrand theory, or abbreviated, WHS -theory, is a substantial addition to the De Rham -Hodge theory cf [DR] and a powerful tool for comparing numerical invariants associated to compact manifolds analytically (i.e by using a Riemannian metric,) and combinatorially (i.e by using a triangulation), cf[BZ1], [BZ2], [BFKM], [BFK1], [BFK2]. It states in a precise way the relationship between the De Rham complex of a manifold and the cochain complex provided by a smooth triangulation when described with the help of a Riemannian metric and of a Morse function. While there are other results which relate these two complexes, WHS-theory provides a connection between these two complexes with scalar producs and permits to relate some of the spectral properities of the Riemannian Laplacians given by the metric and the combinatorial Laplacians given by the triangulation.

The intuition behind the WHS -theory is provided by physics and consists in regarding a compact smooth manifold equipped with a Riemannian metric and a Morse function as an interacting system of harmonic oscillators. This intuition was first noticed and exploited by E. Witten, cf[Wi], in order to provide a short "physicist's proof " of Morse inequalities, a rather simple but very useful result in topology.

Helffer and Sjöstrand have completed Witten's picture with their results on Schrödinger operators and have considerably strengthened Witten's mathematical statements, cf [HS2]. The work of Helffer and Sjöstrand on the Witten theory can be substantially simplified by using simple observations familiar to topologists, cf [BZ2] and [BFKM]. As presented in [HS], their work, although very appealing, is not very accessible to topologists because of a large amount of estimates and preliminary results about Schrödinger operators. It turns out that not all of them are necessary and the Witten Helffer Sjöstrand work, at least as needed by topologists, can be presented and explained in a selfcontained manner and on a reasonable number of pages, cf. section 5, [BFKM]. This survey is a presentation of the WHS-theory with these simplifications, hopefully accessible to a graduate student in geometry and topology and in a way appropriate to topological applications.

The mathematics behind the WHS-theory is almost entirely based on the following two facts: the existence of a gap in the spectrum of the Witten Laplacians detected by elementary mini-max characterization of the spectrum of selfadjoint positive operators and simple estimates involving the equations of the harmonic oscillator. The Witten Laplacians in the neighborhood of critical points in "admissible coordinates" are given by such equations.

Witten's ideas presented below were used by Witten to provide a new proof of Morse type inequalities and holomorphic Morse inequalities. WHS-theory was also used to provide a new proof of the equality of analytic and Reidemeister torsion and the L_2-version of WHS-theory to provide a new proof of the equality of Novikov-Shubin invariants defined analytically and combinatorially,cf [BFKM].

The L_2-version of WHS theory turned out to be not only important but so far an unavoidable ingredient in the proof of the equality of L_2 -analytic and Reidemeister torsion and of generalizations of this result, cf [BFKM].

1.Triangulations from an analytic point of view.

Let M^n be a compact closed smooth manifold of dimension n. A generalized triangulation is provided by a pair (h, g), $h : M \to \mathbb{R}$ a smooth function, g a Riemannian metric so that :

C1. For any critical point x of h there exists a coordinate chart in the neighborhood of x so that in these coordinates h is quadratic and g is Euclidean.

Precisely, for any x critical point of h, $(x \in Cr(h))$, there exists a coordinate chart $\varphi : (U, x) \to (D_\epsilon, 0)$, U an open neighborhood of x in M, D_ϵ an open disc of radius ϵ in $\mathbb{R}^n$, φ a diffeomorphism with $\varphi(x) = 0$, so that :

$$(i) : h \cdot \varphi^{-1}(x_1, x_2, \cdots, x_n) = c - 1/2(x_1^2 + \cdots x_k^2) + 1/2(x_{k+1}^2 + \cdots x_n^2)$$

$$(ii) : (\varphi^{-1})^*(g) \text{ is given by } g_{ij}(x_1, x_2, \cdots, x_n) = \delta_{ij}$$

Coordinates so that (i) and (ii) hold are called admissible.

It follows that any critical points has a well defined index, the number k of the negative squares in the expression (i), which is independent of the choice of a coordinate system with respect to which h has the form (i).

C2. h is self indexing, i.e. for any critical point $x \in Cr(h)$ $h(x) =$ index x.

Consider the vector field $-grad_g(h)$ and for any $y \in M$, denote by $\gamma_y(t), -\infty < t < \infty$, the unique trajectory of $-grad_g(h)$ which satisfies the condition $\gamma_y(0) = y$. For $x \in Cr(h)$ denote by W_x^- resp. W_x^+ the sets

$$W_x^\pm = \{y \in M | \lim_{t \to \pm\infty} \gamma_y(t) = x\}.$$

In view of (i), (ii) and of the theorem of existence, unicity and smooth dependence on the initial condition for the solutions of ordinary differential equations, W_x^- resp. W_x^+ is a smooth submanifold diffeomorphic to $\mathbb{R}^k$ resp. to $\mathbb{R}^{n-k}$, with $k =$index$x$. This can be verified easily based on the fact that:

$$\varphi(W_x^- \cap U_x) = \{(x_1, x_2, \cdots, x_n) \in D(\epsilon) | x_{k+1} = x_{k+2} = \cdots = x_n = 0\},$$

and

$$\varphi(W_x^+ \cap U_x) = \{(x_1, x_2, \cdots, x_n) \in D(\epsilon) | x_1 = x_2 = \cdots = x_k = 0\}.$$

Since M is compact and C1 holds, the set $Cr(h)$ is finite and since M is closed (i.e. compact and without boundary), $M = \bigcup_{x \in Cr(h)} W_x^-$. As already observed each W_x^- is a smooth submanifold diffeomorphic to $\mathbb{R}^k$, $k =$index x, i.e. an open cell.

C3. The vector field $-grad_g h$ satisfies the Morse-Smale condition if for any $x, y \in Cr(h)$, W_x^- and W_y^+ are transversal.

C3 implies that $\mathcal{M}(x,y) := W_x^- \cap W_y^+$ is a smooth manifold of dimension equal to Index $x -$ Index y. $\mathcal{M}(x,y)$ is equipped with the action $\mu : \mathbb{R} \times \mathcal{M}(x,y) \to \mathcal{M}(x,y)$, defined by $\mu(t,z) = \gamma_z(t)$.

If Index $x \leq$ Index y, and $x \neq y$, in view of the transversality requested by the Morse Smale condition, $\mathcal{M}(x,y) = \emptyset$.

If $x \neq y$ and $\mathcal{M}(x,y) \neq \emptyset$, the action μ is free and we denote the quotient $\mathcal{M}(x,y)/\mathbb{R}$ by $\tilde{\mathcal{M}}(x,y)$; $\tilde{\mathcal{M}}(x,y)$ is a smooth manifold of dimension Index $x -$ Index $y-1$, diffeomorphic to the submanifold $\mathcal{M}(x,y) \cap h^{-1}(c)$, for any real number c in the open interval (Index x, Index y). The elements of $\tilde{\mathcal{M}}(x,y)$ are the trajectories from "x to y" and such an element will be denoted by γ.

If $x = y$, then $W_x^- \cap W_x^+ = x$.

The condition C3 implies that the partition of M into open cells is actually a smooth cell complex. To formulate this fact precisely we recall that an $n-$dimensional manifold X with corners is a paracompact Hausdorff space equipped with a maximal smooth atlas with charts $\varphi : U \to \varphi(U) \subseteq \mathbb{R}_+^n$ with $\mathbb{R}_+^n = \{(x_1, x_2, \cdots x_n) | x_i \geq 0\}$. The collection of points of X which correspond (by some and then by any chart) to points in $\mathbb{R}^n$ with exactly k coordinates equal to zero is a well defined subset of X and it will be denoted by X_k. It has a structure of a smooth $(n-k)-$dimensional manifold. $\partial X = X_1 \cup X_2 \cup \cdots X_n$ is a closed subset which is a topological manifold and $(X, \partial X)$ is a topological manifold with boundary ∂X. A compact smooth manifold with corners, X, with interior diffeomorphic to the Euclidean space, will be called a compact smooth cell.

For any string of critical points $x = y_0, y_1, \cdots, y_k$ with

$$\text{index } y_0 > \text{index } y_1 >, \cdots, > \text{index } y_k,$$

consider the smooth manifold of dimension index $y_0 - k$,

$$\tilde{\mathcal{M}}(y_0, y_1) \times \cdots \tilde{\mathcal{M}}(y_{k-1}, y_k) \times W_{y_k}^-,$$

and the smooth map

$$i_{y_0, y_1, \cdots, y_k} : \tilde{\mathcal{M}}(y_0, y_1) \times \cdots \times \tilde{\mathcal{M}}(y_{k-1}, y_k) \times W_{y_k}^- \to M,$$

defined by $i_{y_0,y_1,\cdots,y_k}(\gamma_1, \cdots, \gamma_k, y) := i_{y_k}(y)$, for $\gamma_i \in \tilde{\mathcal{M}}(y_{i-1}, y_i)$ and $y \in W_{y_k}^-$, with $i_x : W_x^- \to M$ the inclusion of W_x^- in M.

Theorem1.1. *Let $\tau = (h, g)$ be a generalized triangulation. For any critical point $x \in Cr(h)$ the smooth manifold W_x^- has a canonical compactification $\hat{W}_x^-$ to a compact manifold with corners and the inclusion i_x has a smooth extension $\hat{i}_x : \hat{W}_x^- \to M$ so that :*

(a): $(\hat{W}_x^-)_k = \bigcup_{(x, y_1, \cdots, y_k)} \tilde{\mathcal{M}}(x, y_1) \times \cdots \times \tilde{\mathcal{M}}(y_{k-1}, y_k) \times W_{y_k}^-$,

(b): the restriction of $\hat{i}_x$ *to* $\tilde{\mathcal{M}}(x,y_1)\times\cdots\times\tilde{\mathcal{M}}(y_{k-1},y_k)\times W^-_{y_k}$ *is given by* $i_{x=y_0,y_1\cdots,y_k}$.

This theorem was probably well known to experts before it was formulated by Floer in the framework of $\infty-$dimensional Morse theory cf. [F]. In fact, a weaker version of this theorem, e.g. Proposition 2 in [L], suffices to conclude that the linear maps Int^q's defined in section 2 provide a morphism of cochain complexes. This is the only fact one needs in order to formulate the WHS-theory. However, Theorem 1.1 is a statement worth to be known. As formulated Theorem 1.1 is proven in [AB].

The name of generalized triangulation for $\tau=(h,g)$ is justified by the fact that any simplicial smooth triangulation can be obtained as a generalized triangulation, cf [Po]. We also point out that given a selfindexing Morse function h and a Riemannian metric g, one can perform arbitrary small C^0- perturbations to g, so that the pair consisting of h and the perturbed metric is a generalized triangulation, cf [Sm].

Given a generalized triangulation $\tau=(h,g)$, and for any critical point $x\in Cr(h)$ an orientation $\mathcal{O}_x$ of W^-_x, one can associate a cochain complex of vector spaces over the field $\mathbb{K}$ or real or complex numbers, $(C^*(M,\tau),\partial^*)$. The diferential ∂^* depends on the choosen orientations $\mathcal{O}_x$. To describe this complex we introduce the incidence numbers

$$I_q: Cr(h)_q\times Cr(h)_{q-1}\to\mathbb{Z}$$

defined as follows:
If $\tilde{\mathcal{M}}(x,y)=\emptyset$, we put $I_q(x,y)=0$.
If $\tilde{\mathcal{M}}(x,y)\neq\emptyset$, for any $\gamma\in\tilde{\mathcal{M}}(x,y)$, the set $\gamma\times W^-_y$ appears as an open set of the boundary $\partial\hat{W}^-_x$ and the orientation $\mathcal{O}_x$ induces an orientation on it. If this is the same as the orientation $\mathcal{O}_y$, we set $\epsilon(\gamma)=+1$, otherwise we set $\epsilon(\gamma)=-1$. Define $I_q(x,y)$ by

$$I_q(x,y)=\sum_{\gamma\in\tilde{\mathcal{M}}(x,y)}\epsilon(\gamma).$$

In the case M is an oriented manifold, the orientation of M and the orientation $\mathcal{O}_x$ on W^-_x induce an orientation $\mathcal{O}^+_x$ on the stable manifold W^+_x. For any $c\in(\text{index } y,\text{index } x)$, $h^{-1}(c)$ carries a canonical orientation induced from the orientation of M. One can check that $I_q(x,y)$ is the intersection number of $W^-_x\cap h^{-1}(c)$ with $W^+_y\cap h^{-1}(c)$ inside $h^{-1}(c)$ and is also the incidence number of the open cells W^-_x and W^-_y in the $CW-$ complex structure provided by τ.

Denote by $(C^*(M,\tau),\partial^*)$ the cochain complex of $\mathbb{K}-$vector spaces defined by
(1) $C^q(M,\tau):=Maps(Cr_q(h),\mathbb{K})$
(2) $\partial^{q-1}: C^{q-1}(M,\tau)\to C^q(M,\tau),\ \partial^{q-1}f(x)=\sum_{y\in Cr_{q-1}(h)}I_q(x,y)f(y)$, where $x\in Cr_q(h)$.

Since $C^q(M,\tau)$ is equipped with a canonical base provided by the maps E_x defined by $E_x(y) = \delta_{x,y}$, $x, y \in Cr_q(h)$, it carries a natural scalar product which makes E_x, $x \in Cr_q(h)$ orthonormal.

Proposition 1.2. *For any q, $\partial^q \cdot \partial^{q-1} = 0$.*

A geometric proof of this Proposition follows from Theorem 1.1 (cf [F] or [AB]), The reader can also derive it by noticing that $(C^*(M,\tau), \partial^*)$ as defined is nothing but the cochain complex associated to the $CW-$complex structure provided by τ.

2. De Rham theory and Integration theory.

Let M be a closed smooth manifold and $\tau = (h, g)$ be a generalized triangulation. Denote by $(\Omega^*(M), d^*)$ the De Rham complex of M. This is a cochain complex whose component $\Omega^r(M)$ is the (Frechet) space of smooth differential forms of degree r and whose differential $d^r : \Omega^r(M) \to \Omega^{r+1}(M)$ is given by the exterior differential. Recall that Stokes theorem can be formulated as follows:

Theorem 2.1. *Let P be a compact $r-$dimensional oriented smooth manifold with corners and $f : P \to M$ be a smooth map. Denote by $\partial f : P_1 \to M$ the restriction of f to the smooth oriented manifold P_1 (P_1 defined as above). If $\omega \in \Omega^{r-1}(M)$ is a smooth form then $\int_{P_1} \partial f^*(\omega)$ is convergent and*

$$\int_P f^*(d\omega) = \int_{P_1} \partial f^*(\omega).$$

Consider the linear map $Int^q : \Omega^q(M) \to C^q(M, \tau)$, with
$C^q(M, \tau) = \mathrm{Maps}(Cr(h)_q, \mathcal{K})$, $Cr_q(h) = \{x \in Cr(h) | \text{index } x = q\}$ defined by

$$Int^q(\omega)(x) = \int_{\hat{W}_x^-} \omega,$$

The collection of the linear maps Int^q' s defines a morphism

$$Int^* : (\Omega^*(M), d^*) \to (C^*(M, \tau), \partial^*)$$

of cochain complexes.

Theorem 2.2. *(De Rham) Int^* induces an isomorphism in cohomology.*

Theorem 3.2, one of the two main results of the WHS theory,whose proof will be sketched below, is a considerable strengthening of this theorem.

3. Witten deformation and the main results of WHS-theory.

Let M be a closed manifold and $h : M \to \mathbb{R}$ a smooth function. For $t > 0$ we consider the complex $(\Omega^*(M), d^*(t))$ with differential $d^q(t) : \Omega^q(M) \to \Omega^{q+1}(M)$ given by $d^q(t) = e^{-th} d e^{th}$ or equivalently

$$d^q(t)(\omega) = d\omega + t dh \wedge \omega. \tag{3.1}$$

$d^*(t)$ is the unique differential in $\Omega^*(M)$ which makes the multiplication by the smooth function e^{th} an isomorphism of cochain complexes

$$e^{th} : (\Omega^*(M), d^*(t)) \to (\Omega^*(M), d^*).$$

Recall that for any vector field X on M one defines the zero order differential operator, $\iota_X = \iota_X^* : \Omega^*(M) \to \Omega^{*-1}(M)$, by

$$\iota_X^q \omega(X_1, X_2, \cdots, X_{q-1}) := \omega(X, X_1, \cdots, X_{q-1}) \tag{3.2}$$

and the first order differential operator $L_X = L_X^* : \Omega^*(M) \to \Omega^*(M)$, the Lie derivative in the direction X,

by

$$L_X^q := d^{q-1} \cdot \iota_X^q + \iota_X^{q+1} \cdot d^q. \tag{3.3}$$

They satisfy the following identities:

$$\iota_X(\omega_1 \wedge \omega_2) = \iota_X(\omega_1) \wedge \omega_2 + (-1)^{|\omega_1|} \omega_1 \wedge \iota_X(\omega_2). \tag{3.4}$$

for $\omega_1 \in \Omega^{|\omega_1|}(M)$, and

$$L_X(\omega_1 \wedge \omega_2) = L_X(\omega_1) \wedge \omega_2 + \omega_1 \wedge L_X(\omega_2). \tag{3.5}$$

Given a Riemannian metric g on the oriented manifold M we have the zeroth order operator $R^q : \Omega^q(M) \to \Omega^{n-q}(M)$, known as the star-Hodge operator which, with respect to an oriented orthonormal frame $e_1, e_2, \cdots, e_n$ in the cotangent space at x, is given by

$$R_x^q(e_{i_1} \wedge \cdots \wedge e_{i_q}) = \epsilon(i_1, \cdots, i_q) e_1 \wedge \cdots \wedge \hat{e}_{i_1} \wedge \cdots \wedge \hat{e}_{i_q} \wedge \cdots \wedge e_n, \tag{3.6}$$

$1 \leq i_1 < i_2, \cdots, i_q \leq n$, with $\epsilon(i_1, i_2, \cdots, i_q)$ denoting the sign of the permutation of $(1, 2 \cdots n)$ given by

$$(i_1, i_2, \cdots, i_q, 1, 2, \cdots \hat{i}_1, \cdots, \hat{i}_2, \cdots \hat{i}_2, \cdots, \hat{i}_q, \cdots, n).$$

Here "hat" above symbol means the deletion of this symbol.

The operators R^q's satisfy

$$R^q \cdot R^{n-q} = (-1)^{q(n-q)} Id. \tag{3.7}$$

With the help of the operators R^q of an oriented Riemannian manifold of dimension n, one defines the fiberwise scalar product $\Omega(M)^q \times \Omega^q(M) \to \Omega^0(M)$ and the formal adjoints

$$\delta^{q+1}, \delta^{q+1}(t) : \Omega^{q+1}(M) \to \Omega^q(M),$$

$$(\iota_X^{q-1})^\sharp : \Omega^{q-1}(M) \to \Omega^q(M), \text{and } (L_X^q)^\sharp : \Omega^q(M) \to \Omega^q(M)$$

of $d^q, d^q(t), \iota_X^q, L_X^q$ by:

$$\ll \omega_1, \omega_2 \gg = (R^n)^{-1}(\omega_1 \wedge R^q(\omega_2)), \tag{3.8}$$

$$\begin{aligned} \delta^{q+1} &= (-1)^{nq+1} R^{n-q} \cdot d^{n-q-1} \cdot R^{q+1}, \\ \delta^{q+1}(t) &= (-1)^{nq+1} R^{n-q} \cdot d^{n-q-1}(t) \cdot R^{q+1}, \\ (\iota_X^q)^\sharp &= (-1)^{nq-1} R^{n-q} \cdot \iota_X^{n-q-1} \cdot R^{q-1}, \\ (L_X^q)^\sharp &= (-1)^{(n+1)q+1} R^{n-q} \cdot L_X^{n-q} \cdot R^q \end{aligned} \tag{3.9}$$

These operators satisfy:

$$\begin{aligned} \ll d\omega_1, \omega_2 \gg &= \ll \omega_1, \delta\omega_2 \gg, \\ \ll d(t)\omega_1, \omega_2 \gg &= \ll \omega_1, \delta(t)\omega_2 \gg, \\ \ll \iota_X\omega_1, \omega_2 \gg &= \ll \omega_1, (\iota_X)^\sharp\omega_2 \gg, \\ \ll L_X\omega_1, \omega_2 \gg &= \ll \omega_1, (L_X)^\sharp\omega_2 \gg, \end{aligned} \tag{3.10}$$

and

$$(L_X)^\sharp = (\iota_X)^\sharp \cdot \delta + \delta \cdot (\iota_X)^\sharp. \tag{3.11}$$

Note that $L_X^q + (L_X^q)^\sharp$ is a zeroth order differential operator. Let $X^\sharp$ denote the element in $\Omega^1(M)$ defined by $X^\sharp(Y) := \ll X, Y \gg$ and for $\omega \in \Omega^1(M)$ let $E_\omega^q : \Omega^q(M) \to \Omega^{q+1}(M)$, denote the exterior product by ω. Then we have

$$(\iota_X^q)^\sharp = E_{X^\sharp}^{q-1}. \tag{3.12}$$

It is easy to see that the scalar products $\ll .,. \gg$ and the operators $\delta^q, \delta^q(t), \iota_X^\sharp$ and $L_X^\sharp$ are independent of the orientation of M. Therefore they are defined (first locally and then being differential operators globally) for an arbitrary Riemannian manifold, not necessary orientable, and satisfy (3.8), (3.10)-(3.12) above.

For a Riemannian manifold (M, g) one introduces the scalar product $\Omega^q(M) \times \Omega^q(M) \to \mathbb{C}$ by

$$< \omega, \omega' > := \int_M \omega \wedge \omega' = \int_M \ll \omega, \omega' \gg dvol(g). \tag{3.13}$$

In view of (3.10), $\delta^{q+1}(t), (\iota_X^q)^\sharp$ and $(L_X^q)^\sharp$ are formal adjoints of $d^q(t)$, $\iota_X^q(t)$ and L_X^q with respect to the scalar product $< .,. >$.

For a Riemannian manifold (M, g), one introduces the second order differential operators $\Delta_q : \Omega^q(M) \to \Omega^q(M)$, the Laplace Beltrami operator, and $\Delta_q(t) : \Omega^q(M) \to \Omega^q(M)$, the Witten Laplacian for the function h, by

$$\Delta_q := \delta^{q-1} \cdot d^q + d^{q-1} \cdot \delta^q,$$

and

$$\Delta_q(t) := \delta^{q-1}(t) \cdot d^q(t) + d^{q-1}(t) \cdot \delta^q(t).$$

Note that $\Delta_q(0) = \Delta_q$. In view of (3.1) -(3.8) and (3.10) one verifies

$$\Delta_q(t) = \Delta_q + t(L_{-grad_g h} + L^\sharp_{-grad_g h}) + t^2 ||grad_g h|| Id \tag{3.14}$$

and that $L_{-grad_g h} + L^\sharp_{-grad_g h}$ is a zeroth order differential operator.

The operators $\Delta_q(t)$ are elliptic selfadjoint and positive, hence their spectra spect$\Delta_q(t)$, lie on $[0, \infty)$. Further, as

$$\ker \Delta_q(t) = \{\omega \in \Omega^q(M) | d^q(t) = 0, \delta^q(t) = 0\}$$

one can see that for all $t \geq 0$ $\ker \Delta_q(t)$ is isomorphic to $\ker \Delta_q(0)$. Hence if 0 is an eigenvalue of $\Delta_q(0)$, then it is an eigenvalue of $\Delta_q(t)$ for all t and with the same multiplicity.

A very important fact in the proof of Theorems 3.1 and 3.2 below is is that $\Delta_q(t) - \Delta_q$ is a zeroth order operator for any t.

The following result is essentially due to E.Witten, and provides the first main result of the WHS-theory.

Theorem 3.1. *Suppose that $\tau = (g, h)$ is a generalized triangulation of the closed Riemannian manifold M. There exist the constants C_1, C_2, C_3 and T_0 depending on τ, so that for any $t > T_0$, $spect\Delta_q(t) \subset [0, C_1e^{-C_2t}] \cup [C_3t, \infty)$ and the number of the eigenvalues of $\Delta_q(t)$ in the interval $[0, C_1e^{-C_2t}]$ counted with their multiplicity is equal to the number of critical points of index q.*

The above theorem states the existence of a gap in the spectrum of $\Delta_q(t)$, namely the open interval $(C_1e^{-C_2t}, C_3t)$, which widens to $(0, \infty)$ when $t \to \infty$.

Clearly C_1, C_2, C_3 and T_0 determine a constant T, so that $1 \in (C_1e^{-C_2t}, C_3t)$ and for $t \geq T$,

$$spect\Delta_q(t) \cap [0, C_1e^{-C_2t}] = spect\Delta_q(t) \cap [0, 1]$$

and

$$spect\Delta_q(t) \cap [C_3t, \infty) = spect\Delta_q(t) \cap [1, \infty).$$

For $t > T$ we denote by $\Omega^q(M)(t)_{sm}$ the finite dimensional subspace of dimension m_q, the number of critical points of index q, generated by the $q-$eigenforms of $\Delta_q(t)$ corresponding to the eigenvalues of $\Delta_q(t)$ smaller than 1. The elliptic theory implies that these eigenvectors, a priori elements in the L_2-completion of $\Omega^q(M)$, are actually in $\Omega^q(M)$. Note that $d(t)(\Omega^q(M)(t)_{sm}) \subset \Omega^{q+1}(M)(t)_{sm}$, so that $(\Omega^*(M)(t)_{sm}, d^*(t))$ is a finite dimensional cochain subcomplex of $(\Omega^*, d^*(t))$ and $e^{th}(\Omega^*(M)(t)_{sm}, d^*(t))$ is a finite dimensional subcomplex of $(\Omega^*(M), d^*)$.

For $t > T$, consider the composition of morphisms of cochain complexes denoted by $l^*(t)$,

$$(\Omega^*(M)_{sm}, \overline{d}^*(t)) \xrightarrow{S^*(t)} (\Omega^*(M)_{sm}, d^*(t)) \xrightarrow{e^{th}} (\Omega^*(M), d^*) \xrightarrow{Int^*} (C^*(M, \tau), \partial^*),$$

with $S^q(t) = (\frac{\pi}{t})^{\frac{n-2q}{4}} e^{tq} Id$, and $\overline{d}^q(t) := (\frac{\pi}{t})^{1/2} e^{-t} d^q(t)$. $S^*(t)$ is an isomorphism of cochain complexes referred to as the "rescaling isomorphism". The following theorem due to Helffer-Sjöstrand, cf [HS2], provides the second main result of the WHS-theory.

Theorem 3.2. *(Helffer-Sjöstrand) Given M a closed manifold and $\tau = (g, h)$ a generalized triangulation, there exists $T_1 > 0$, depending on τ, so that for $t > T_1$ $1 \notin spect\Delta_q(t)$ and $l^*(t)$ is an isomorphism of cochain complexes.*

Moreover, for $t > T_1$ there exists a family of isometries $\mathcal{J}^q(t) : C^q(M, \tau) \to \Omega^q(M)(t)_{sm}$ of finite dimensional vector spaces so that $l^q(t)\mathcal{J}^q(t) = Id + O(1/t)$. It is understood that $C^q(M, \tau)$ is equipped with the canonical scalar product defined in section 1, before Theorem 1.2, and $\Omega^q(M)(t)_{sm}$ with the scalar product $< .,. >$ defined by (3.13).

Theorem 3.2 provides inside $(\Omega^*(M), d^*(t))$, (a reparametrization of $(\Omega^*(M), d^*)$ induced by the multiplication operator $e^{th}S^q(t) : \Omega^q(M) \to \Omega^q(M))$ the finite dimensional subcomplex $(\Omega^*(M)_{sm}, \overline{d}^*(t))$ which, after rescaling, is asymptotically isometric to $(C^*(M,\tau), \partial^*)$.

Recall that De Rham Hodge theory provides a canonical and unique representation of each cohomology class of $(C^*(M,\tau), \partial^*)$ by harmonic $q-$forms with respect to g. Theorem 3.2 provides, asymptotically, a canonical and unique representation of the full complex $(C^*(M,\tau), \partial^*)$ and its base E_x inside $(\Omega^*(M), d^*)$.

4. Ideas of the proof of Theorems 3.1 and 3.2.

The proof of Theorems 3.1 and 3.2 is based on a mini-max criterion for detecting a gap in the spectrum of a positive selfadjoint operator in a Hilbert space H, Lemma 4.1 below, and on the explicit formula for $\Delta_q(t)$ in admissible coordinates in a neighborhood of the critical points.

Lemma 4.1. *Let $A : H \to H$ be a densely defined (not necessary bounded) self adjoint positive operator in a Hilbert space $(H, <,>)$ and a, b two real numbers so that $0 < a < b < \infty$. Suppose that there exists two closed subspaces H_1 and H_2 of H with $H_1 \cap H_2 = 0$ and $H_1 + H_2 = H$ such that:*
(1) $< Ax_1, x_2 > \leq\ a||x_1||^2$ for $x_1 \in H_1$,
(2) $< Ax_1, x_2 > \geq\ b||x_2||^2$ for $x_1 \in H_2$.
Then $spect A \bigcap (a, b) = \emptyset$.

The proof of this Lemma is elementary and is left as an exercise for the reader.

Consider $x \in Cr(h)$ and choose admissible coordinates $(x_1, x_2, ..., x_n)$ in the neighborhood of x. Since with respect to these coordinates

$$h(x_1, x_2, ..., x_n) = k - 1/2(x_1^2 + \cdots + x_k^2) + 1/2(x_{k+1}^2 + \cdots + x_n^2)$$

and $g_{ij}(x_1, x_2, ..., x_n) = \delta_{ij}$, by (3.14) the operator $\Delta_q(t)$ has the form:

$$\Delta_{q,k}(t) = \Delta_q + tM_{q,k} + t^2(x_1^2 + \cdots + x_n^2)Id \tag{4.1}$$

with

$$\Delta_q(\sum_I a_I(x_1, x_2, ..., x_n)dx_I) = -(\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} a_I(x_1, x_2, ..., x_n))dx_I,$$

and $M_{q,k}$ the linear operator determined by

$$M_{q,k}(\sum_I a_I(x_1, x_2, ..., x_n)dx_I) = \sum_I \epsilon_I^{q,k} a_I(x_1, x_2, ..., x_n)dx_I. \tag{4.2}$$

Here $I = (i_1, i_2 \cdots i_q)$, $1 \leq i_1 < i_2 \cdots < i_q \leq n$, $d_I = dx_{i_1} \wedge \cdots \wedge d_{i_q}$ and

$$\epsilon_I^{q,k} = -n + 2k - 2q + 4\sharp\{j | k + 1 \leq i_j \leq n\},$$

where $\sharp A$ denotes the cardinality of the set A. Note that $\epsilon_I^{q,k} \geq -n$ and is $= -n$ iff $q = k$.

Let $\mathcal{S}^q(\mathbb{R}^n)$ denote the space of smooth $q-$forms $\omega = \sum_I a_I(x_1, x_2, ..., x_n)dx_I$ with $a_I(x_1, x_2, ..., x_n)$ rapidly decaying functions. The operator $\Delta_{q,k}(t)$ acting on $\mathcal{S}^q(\mathbb{R}^n)$ is globally elliptic (in the sense of [Sh1] or [Hö]), selfadjoint and positive. This operator is the harmonic oscillator in n variables acting on $q-$forms and its properties can be derived from the harmonic oscillator in one variable $-\frac{d^2}{dx^2} + a + bx^2$ acting on functions. In particular the following result holds.

Proposition 4.2. *(1) $\Delta_{q,k}(t)$, regarded as an unbounded densely defined operator on the L_2-completion of $\mathcal{S}^q(\mathbb{R}^n)$, is selfadjoint, positive and its spectrum is contained in $2t\mathbb{Z}_{\geq 0}$ (i.e positive integer multiple of $2t$).*

(2) $\ker \Delta_{q,k}(t) = 0$ if $k \neq q$ and dim $\ker \Delta_{q,q}(t) = 1$.

(3) $\omega_{q,t} = (t/\pi)^{n/2} e^{-t\sum_i x_i^2/2} dx_1 \wedge \cdots \wedge dx_q$ is the generator of $\ker \Delta_{q,q}(t)$ with the L_2-norm 1.

For details consult [BFKM] page 805.

Choose a smooth function $\gamma_\eta(u)$, $\eta \in (0, \infty)$, $u \in \mathbb{R}$, which satisfies :

$$\gamma_\eta(u) = \begin{Bmatrix} 1 \text{ if } u \leq \eta/2 \\ 0 \text{ if } u > \eta \end{Bmatrix}. \tag{4.3}$$

Introduce $\tilde{\omega}^\eta_{q,t} \in \Omega^q_c(\mathbb{R}^n)$defined by
$\tilde{\omega}^\eta_{q,t}(x) = \beta_q^{-1}(t)\ \gamma_\eta(|x|)\omega_{q,t}(x)$ with $|x| = \sqrt{\sum_i x_i^2}$ and

$$\beta_q(t) = (t/\pi)^{n/4} \left(\int_{\mathbb{R}^n} \gamma_\eta^2(|x|) e^{-t\sum_i x_i^2} dx_1 \cdots dx_n\right)^{1/2}. \tag{4.4}$$

The smooth form $\tilde{\omega}^\eta_{q,t}$ has the support in the disc of radius η, agrees with $\omega_{q,t}$ on the disc of radius $\eta/2$ and satisfies

$$< \tilde{\omega}^\eta_q(t), \tilde{\omega}^\eta_q(t) >= 1 \tag{4.5}$$

with respect to the scalar product $< .,. >$ on $\mathcal{S}^q(\mathbb{R}^n)$ induced by the Euclidean metric. The following proposition can be obtained by elementary calculations in coordinates in view of the explicit formula of $\Delta_{q,k}(t)$ cf [BFKM], Appendix 2.

Proposition 4.3. *For a fixed $r \in \mathbb{N}_{\geq 0}$ there exists $C, C', C'', T_0, \epsilon_0$ so that $t > T_0$ and $\epsilon < \epsilon_0$ imply*

(1) $|\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \Delta_{q,q}(t) \tilde{\omega}^\epsilon_{q,t}(x)| \leq Ce^{-C't}$ for any $x \in \mathbb{R}^n$ and multiindex $\alpha = (\alpha_1, \cdots, \alpha_n)$, with $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq r$.

(2) $< \Delta_{q,k}(t)\tilde{\omega}^\epsilon_{q,t}, \tilde{\omega}^\epsilon_{q,t} > \geq\ 2t|q-k|$

(3) If $\omega \perp \tilde{\omega}^\epsilon_{q,t}$ with respect to the scalar product $< .,. >$ then

$$< \Delta_{q,q}\omega, \omega > \geq\ C''t||\omega||^2.$$

For the proof of Theorems 3.1 and 3.2 we choose $\epsilon_0 > 0$ so that for each $y \in Cr(h)$ there exists an admissible coordinate chart $\varphi_y : (U_y, y) \to (D_{2\epsilon}, 0)$ so that $U_y \cap U_z = \emptyset$ for $y \neq z$, $y, z \in Cr(h)$.

Choose once for all such an admissible coordinate chart for any $y \in Cr_q(h)$. Introduce the smooth forms $\overline{\omega}_{y,t} \in \Omega^q(M)$ defined by

$$\overline{\omega}_{y,t}|_{M\setminus\varphi_y^{-1}(D_{2\epsilon})} = 0,\ \overline{\omega}_{y,t}|_{\varphi_y^{-1}(D_{2\epsilon})} = \varphi_y^*(\tilde{\omega}^{\epsilon}_{q,t}). \tag{4.6}$$

The forms $\overline{\omega}_{y,t} \in \Omega^q(M),\ y \in Cr_q(h)$ are orthonormal. Indeed if $y \neq z,\ y, z \in Cr_q(h)$, $\overline{\omega}_{y,t}$ and $\overline{\omega}_{z,t}$ have disjoint support, hence are orthogonal, and because the support of $\overline{\omega}_{y,t}$ is contained in an admissible chart, $< \overline{\omega}_{y,t}, \overline{\omega}_{y,t} >= 1$ by (4.5).

For $t > T_0$, with T_0, given by Proposition 4.3, we define $J^q(t) : C^q(X,\tau) \rightarrow \Omega^q(M)$ to be the linear map determined by

$$J^q(t)(E_y) = \overline{\omega}_y(t),$$

where $E_y \in C^q(X,\tau)$ is given by $E_y(z) = \delta_{yz}$ for $y, z \in Cr(h)_q$. $J_q(t)$ is an isometry, thus in particular injective.

Proof of Theorems 3.1 and 3.2: (sketch). Take H to be the L_2-completion of $\Omega^q(M)$ with respect to the scalar product $< .,. >$, $H_1 := J^q(t)(C^q(M,\tau))$ and $H_2 = H_1^{\perp}$. Let T_0, C, C', C'' be given by Proposition 4.3 and define

$$C_1 := \inf_{z\in M'} ||grad_g h(z)||,$$

with $M' = M \setminus \bigcup_{y\in Cr_q(h)} \varphi_y^{-1}(D_\epsilon)$,

$$C_2 = \sup_{x\in M} ||(L_{-grad_g h} + L^\sharp_{-grad_g h})(z)||;$$

here $||grad_g h(z)||$ resp. $||(L_{-grad_g h} + L^\sharp_{-grad_g h})(z)||$ denotes the norm of the vector $grad_g h(z) \in T_z(M)$ resp. of the linear map $(L_{-grad_g h} + L^\sharp_{-grad_g h})(z) : \Lambda^q(T_z(M)) \rightarrow \Lambda^q(T_z(M))$ with respect to the scalar product induced in $T_z(M)$ and $\Lambda^q(T_z(M))$ by $g(z)$. Recall that if X is a vector field then $L_X + L^\sharp_X$ is a zeroth order differential operator, hence an endomorphism of the bundle $\Lambda^q(T^*M) \rightarrow M$.

We can use the constants $T_0, C, C', C'', C_1, C_2$ to construct C''' and ϵ_1 so that for $t > T_0$ and $\epsilon < \epsilon_1$, we have $< \Delta_q(t)\omega, \omega >\geq C_3 t < \omega, \omega >$ for any $\omega \in H_2$ (cf. [BFKM], page 808-810).

Now one can apply Lemma 4.1 whose hypotheses are satisfied for $a = Ce^{-C't}, b = C'''t$ and $t > T_0$. This concludes the first part of Theorem 3.1.

Let $Q_q(t),\ t > T_0$ denote the orthogonal projection in H on the span of the eigenvectors corresponding the eigenvalues smaller than 1. In view of the ellipticity of $\Delta_q(t)$ all these eigenvectors are smooth $q-$forms. An additional important estimate is given by the following Proposition:

Proposition 4.4. *For $r \in \mathbb{N}_{\geq 0}$ one can find $\epsilon_0 > 0$ and C_3, C_4 so that for $t > T_0$ as constructed above, and any $\epsilon < \epsilon_0$ one has, for any $v \in C^q(M, \tau)$*

$$\sup_{x \in M} ||(Q_q(t) J^q(t) - J^q(t))(v)|| \leq C_3 e^{-C_4 t} ||v||,$$

$(Q_q(t) J^q(t) - J^q(t))(v) \in \Omega^q(M)$, with similar estimates for the C^p-norm of $(Q_q(t) J^q(t) - J^q(t)$, with $p \leq r$.

The proof of this Proposition is contained in [BZ1], page 128 and [BFKM] page 811. Its proof requires (3.14), Proposition 4.3 and general estimates coming from the ellipticity of $\Delta_q(t)$.

Proposition 4.4 implies that for t large enough, say $t > t_0$, $\mathcal{I}^q(t) := Q_q(t) J^q(t)$ is bijective, which finishes the proof of Theorem 3.1.

For $t \geq T_0$, as constructed in Proposition 4.4, let $\mathcal{J}^q(t)$ be the isometry defined by

$$\mathcal{J}^q(t) := \mathcal{I}^q(t)(\mathcal{I}^q(t)^\sharp \mathcal{I}^q(t))^{-1/2} \tag{4.7}$$

and denote by $U_{y,t} := \mathcal{J}^q(t)(E_y) \in \Omega^q(M), y \in Cr(h)_q$. Proposition 4.4 implies that there exists t_0 and C so that for $t > t_0$ and $y \in Cr(h)_q$ one has:

$$\sup_{z \in M \setminus \varphi_y^{-1}(D_\epsilon)} ||U_{y,t}(z)|| \leq C e^{-\epsilon t}, \tag{4.8}$$

$$||U_{y,t}(z) - \overline{\omega}_{y,t}(z)|| \leq C \frac{1}{t}, \text{ for } z \in W_y^- \cup \varphi_y^{-1}(D_\epsilon). \tag{4.9}$$

To check Theorem 3.2 it suffices to show that

$$|\int_{W_{x'}^-} U_{x,t} e^{th} - (\frac{t}{\pi})^{\frac{n-2q}{4}} e^{tq} \delta_{xx'}| \leq C'' \ \frac{1}{t}$$

for some $C'' > 0$ and any $x, x' \in Cr(h)_q$.

If $x \neq x'$ this follows from (1). If $x = x'$ from (4.8) and (4.9). $\square$

5. Extensions and a survey of applications.

1. One can relax the definition of the generalized triangulation by dropping from $C1$ the constraint on g to be Euclidean in the neighborhood of the critical points. This will keep Theorems 3.1 and 3.2 valid as stated; however almost all calculations will be longer since the explicit formulae for $\Delta_q(t)$ and its spectrum when regarded on $\mathcal{S}^*(\mathbb{R}^n)$ will be more complicated.

2.One can drop the hypothesis that the Morse function h is self indexing. In this case Theorem 3.1 remains true as stated but in Theorem 3.2, $l^q(t)$ should be replaced by

$$(\Omega^*(M)_{sm}, d^*(t)) \xrightarrow{e^{th}} (\Omega^*(M), d^*) \xrightarrow{Int^*} (C^*(M,\tau), \partial^*),$$

and $\mathcal{J}^q(t)$ by $\mathcal{J}^q(t)\cdot\Sigma^q(t)$ with $\Sigma^q(t) : (C^q(M,\tau), \partial^*) \to (C^q(M,\tau), \partial^*(t))$, and $\partial^*(t) = \Sigma^{q+1}(t)\cdot\partial^q\cdot(\Sigma^q(t))^{-1}$ the morphism of cochain complexes defined by

$$\Sigma^q(t)(E_x) = (\frac{\pi}{t})^{\frac{n-2q}{4}} e^{h(x)} E_x,$$

$x \in Cr_q(h)$ cf. [BFK3].

3. One can twist both complexes $(C^*(M,\tau),\partial^*)$ and (Ω^*, d^*) by a finite dimensional representation of the fundamental group, $\rho : \pi_1(M) \to GL(V)$. In this case an additional data is necessary: a Hermitian structure μ on the flat bundle ξ_ρ induced by ρ. The "canonical" scalar product on $(C^*(M,\tau,\rho),\partial^*)$ will be obtained by using the critical points (the cells of the generalized triangulation and the hermitian scalar product provided by μ in the fibers of ξ_ρ above the critical points. The De-Rham complex in this case is replaced by $(\Omega^*(M,\rho), d^*_\rho)$ of differential forms with coefficients in ξ_ρ and the differential is provided by the flat connection in ξ_ρ. The scalar product will require in addition of the Riemannian metric g the Hermitian structure μ (cf. [BFK] or [BFK2]). Under the hypotheses that the Hermitian structure is parallel in small neighborhoods of the critical points, the proofs of Theorems 3.1 and 3.2 remain the same. An easy continuity argument permits to reduce the case of an arbitrary Hermitian structure to the previous one by taking a C^0 approximation of a given Hermitian structure by Hermitian structures which are parallel near the critical points. Since the Witten Laplacians do not involve derivatives of the Hermitian structure such a reduction is possible. If the representation is a unitary representation in a finite dimensional Euclidean space one has a canonical Hermitian structure in ξ_ρ (parallel with respect to the flat canonical connection in ξ_ρ.) This extension was used in the new proofs of the Cheeger- Muller theorem and its extension concerning the comparison of the analytic and the Reidemeister torsion. cf. [BZ], [BFK].

4.One can further extend the WHS-theory to the case where ρ is a special type of an infinite dimensional representation, a representation of the fundamental group in an $\mathcal{A}-$ Hilbert module of finite type. This extension was done in [BFKM] for ρ unitary and in [BFK4] for ρ arbitrary. In this case the Laplacian $\Delta_q(t)$ do not have discrete spectrum and it seems quite remarkable that Theorems 3.1 and 3.2 remain true. It is even more surprising that exactly the same arguments as presented above can be adapted to prove them. A particularly interesting situation is the case of the left regular representation of a countable group Γ on the Hilbert space $L_2(\Gamma)$ when regarded as an $\mathcal{N}(\Gamma)$ right Hilbert module of the von Neumann algebra $\mathcal{N}(\Gamma)$, cf.[BFKM] for definitions. One can prove that Farber extended L_2-cohomology of M, a compact smooth manifold with infinite fundamental group defined analytically (i.e. using differential forms and a Riemannian metric) and combinatorially (i.e using a triangulation) are isomorphic and therefore the classical L_2-Betti numbers and Novikov-Shoubin invariants defined analytically and combinatorially are the same. For the last fact see see [BFKM].

This WHS-theory was a fundamental tool in the proof of the equality of the L_2-analytic and the L_2-Reidemeister torsion presented [BFKM].

5. One can further extend Theorems 3.1 and 3.2 to bordisms $(M, \partial_- M, \partial_+ M)$, and ρ a representation of $\Gamma = \pi_1(M)$ on an $\mathcal{A}-$Hilbert module of finite type. In this case one has first to extend the concept of generalized triangulation to such bordisms. This will involve a pair (h, g) which in addition to the requirements C1-C3 is supposed to satisfy the following assumptions: g is product like near $\partial M = \partial_- M \cup \partial_+ M$,

$h : M \to [a, b]$ with $h^{-1}(a) = \partial_- M$, $h^{-1}(b) = \partial_+ M$, a, b regular values, and h linear on the geodesics normal to ∂M near ∂W. This extension was done in [BFK2] and was used to prove gluing formulae for analytic torsion and to extend the results of [BFKM] to manifolds with boundary.

5. One can actually extend WHS-theory to the case where h is generalized Morse function, i.e. the critical points are either nondegenerated or birth-death. This extension is much more subtle and very important. Beginning work in this direction was done by Hon Kit Wai in his OSU dissertation.

6. References.

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