

# A SHORT COURSE ON WITTEN HELFFER-SJÖSTRAND THEORY

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## ABSTRACT.

Witten-Helffer-Sjöstrand theory is an addition to Morse theory and Hodge-de Rham theory for Riemannian manifolds and considerably improves on them by injecting some spectral theory of elliptic operators. It can serve as a general tool to prove results about comparison of numerical invariants associated to compact manifolds analytically, i.e. by using a Riemannian metric, or combinatorially, i.e. by using a triangulation. It can be also refined to provide an alternative presentation of Novikov Morse theory and improve on it in many respects. In particular it can be used in symplectic topology and in dynamics. This material represents my Notes for a three lectures course given at the Goettingen summer school on groups and geometry, June 2000.

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## 0. Introduction.

Witten Helffer Sjöstrand theory, or abbreviated, WHS -theory, consists of a number of results which considerably improve on Morse theory and De Rham -Hodge theory.

The intuition behind the WHS -theory is provided by physics and consists in regarding a compact smooth manifold equipped with a Riemannian metric and a Morse function (or closed 1-form) as an interacting system of harmonic oscillators. This intuition was first exploited by E. Witten, cf[Wi], in order to provide a short "physicist's proof" of Morse inequalities, a rather simple but very useful result in topology.

Helffer and Sjöstrand have completed Witten's picture with their results on Schrödinger operators and have considerably strengthened Witten's mathematical statements, cf [HS2]. The work of Helffer and Sjöstrand on the Witten theory can be substantially simplified by using simple observations more or less familiar to topologists, cf [BZ] and [BFKM].

The mathematics behind the WHS-theory is almost entirely based on the following two facts: the existence of a gap in the spectrum of the Witten Laplacians (a one parameter family of deformed Laplace-Beltrami operator involving a Morse function  $h$ ), detected by elementary mini-max characterization of the spectrum of selfadjoint positive operators and simple estimates involving the equations of the harmonic oscillator. The Witten Laplacians in the neighborhood of critical points in "admissible coordinates" are given by such equations. The simplification we

referred to are due to a compactification theorem for the space of trajectories and of the unstable sets of the gradient of a Morse function with respect to a "good" Riemannian metric. This theorem can be regarded as a strengthening of the basic results of elementary Morse theory.

The theory, initially considered for a Morse function, can be easily extended to a Morse closed one form, and even to the more general case of a Morse-Bott form. These are closed 1-forms which in some neighborhood of a connected component of the zero set are differential of a Morse-Bott function. So far the theory has been very useful to obtain an alternative derivation of results concerning the comparison of numerical invariants associated to compact manifolds analytically (i.e. by using a Riemannian metric,) and combinatorially cf[BZ1], [BZ2], [BFKM], [BFK1], [BFK4], [BFKM], [BH].

The theory provides an alternative (analytic) approach to the Novikov- Morse theory with considerable improvements and consequently has applications in symplectic topology and dynamics. These aspects will be developed in a forthcoming paper [BH].

This minicourse is presented as a series of three lectures. The first is a reconsideration of elementary Morse theory, with the sketch of the proof of the compactification theorem. The second discusses the "Witten deformation" of the Laplace Beltrami operator and its implications and provides a sketch of the proof of Theorem 2.1 the main result of the section. The third presents Helffer-Sjöstrand Theorem as an asymptotic improvement of the Hodge-de Rham Theorem and finally surveys some of the existing applications. Part of the material presented in these notes will be contained in a book [BFK5] in preparation, which will be written in collaboration with L. Friedlander and T. Kappeler.

## Lecture 1: Morse Theory revisited.

### a. Generalized triangulations.

Let  $M^n$  be a compact closed smooth manifold of dimension  $n$ . A *generalized triangulation* is provided by a pair  $(h, g)$ ,  $h : M \rightarrow \mathbb{R}$  a smooth function,  $g$  a Riemannian metric so that

**C1.** For any critical point  $x$  of  $h$  there exists a coordinate chart in the neighborhood of  $x$  so that in these coordinates  $h$  is quadratic and  $g$  is Euclidean.

More precisely, for any critical point  $x$  of  $h$ , ( $x \in Cr(h)$ ), there exists a coordinate chart  $\varphi : (U, x) \rightarrow (D_\varepsilon, 0)$ ,  $U$  an open neighborhood of  $x$  in  $M$ ,  $D_\varepsilon$  an open disc of

radius  $\varepsilon$  in  $\mathbb{R}^n$ ,  $\varphi$  a diffeomorphism with  $\varphi(x) = 0$ , so that :

$$(i) \ h \cdot \varphi^{-1}(x_1, x_2, \dots, x_n) = c - 1/2(x_1^2 + \dots + x_k^2) + 1/2(x_{k+1}^2 + \dots + x_n^2)$$

$$(ii) \ (\varphi^{-1})^*(g) \text{ is given by } g_{ij}(x_1, x_2, \dots, x_n) = \delta_{ij}$$

Coordinates so that (i) and (ii) hold are called *admissible*.

It follows then that any critical point  $x \in Cr(h)$  has a well defined index,  $i(x) = \text{index}(x) = k$ ,  $k$  the number of the negative squares in the expression (i), which is independent of the choice of a coordinate system (with respect to which  $h$  has the form (i)).

Consider the vector field  $-\text{grad}_g(h)$  and for any  $y \in M$ , denote by  $\gamma_y(t)$ ,  $-\infty < t < \infty$ , the unique trajectory of  $-\text{grad}_g(h)$  which satisfies the condition  $\gamma_y(0) = y$ . For  $x \in Cr(h)$  denote by  $W_x^-$  resp.  $W_x^+$  the sets

$$W_x^\pm = \{y \in M \mid \lim_{t \rightarrow \pm\infty} \gamma_y(t) = x\}.$$

In view of (i), (ii) and of the theorem of existence, unicity and smooth dependence on the initial condition for the solutions of ordinary differential equations,  $W_x^-$  resp.  $W_x^+$  is a smooth submanifold diffeomorphic to  $\mathbb{R}^k$  resp. to  $\mathbb{R}^{n-k}$ , with  $k = \text{index}(x)$ . This can be verified easily based on the fact that

$$\varphi(W_x^- \cap U_x) = \{(x_1, x_2, \dots, x_n) \in D(\varepsilon) \mid x_{k+1} = x_{k+2} = \dots = x_n = 0\},$$

and

$$\varphi(W_x^+ \cap U_x) = \{(x_1, x_2, \dots, x_n) \in D(\varepsilon) \mid x_1 = x_2 = \dots = x_k = 0\}.$$

Since  $M$  is compact and C1 holds, the set  $Cr(h)$  is finite and since  $M$  is closed (i.e. compact and without boundary),  $M = \bigcup_{x \in Cr(h)} W_x^-$ . As already observed each  $W_x^-$  is a smooth submanifold diffeomorphic to  $\mathbb{R}^k$ ,  $k = \text{index}(x)$ , i.e. an open cell.

**C2.** The vector field  $-\text{grad}_g h$  satisfies the Morse-Smale condition if for any  $x, y \in Cr(h)$ ,  $W_x^-$  and  $W_y^+$  are transversal.

C2 implies that  $\mathcal{M}(x, y) := W_x^- \cap W_y^+$  is a smooth manifold of dimension equal to  $\text{index}(x) - \text{index}(y)$ .  $\mathcal{M}(x, y)$  is equipped with the action  $\mu : \mathbb{R} \times \mathcal{M}(x, y) \rightarrow \mathcal{M}(x, y)$ , defined by  $\mu(t, z) = \gamma_z(t)$ .

If  $\text{index}(x) \leq \text{index}(y)$ , and  $x \neq y$ , in view of the transversality requested by the Morse Smale condition,  $\mathcal{M}(x, y) = \emptyset$ .

If  $x \neq y$  and  $\mathcal{M}(x, y) \neq \emptyset$ , the action  $\mu$  is free and we denote the quotient  $\mathcal{M}(x, y)/\mathbb{R}$  by  $\mathcal{T}(x, y)$ ;  $\mathcal{T}(x, y)$  is a smooth manifold of dimension  $\text{index}(x) \leq \text{index}(y) - 1$ , diffeomorphic to the submanifold  $\mathcal{M}(x, y) \cap h^{-1}(\lambda)$ , for any real number  $\lambda$  in the open interval  $(\text{index } x, \text{index } y)$ . The elements of  $\mathcal{T}(x, y)$  are the trajectories from “ $x$  to  $y$ ” and such a trajectory will usually be denoted by  $\gamma$ .

If  $x = y$ , then  $W_x^- \cap W_x^+ = x$ .

Further the condition C2 implies that the partition of  $M$  into open cells is actually a smooth cell complex. To formulate this fact precisely we recall that an  $n$ -dimensional manifold  $X$  with corners is a paracompact Hausdorff space equipped with a maximal smooth atlas with charts  $\varphi : U \rightarrow \varphi(U) \subseteq \mathbb{R}_+^n$  with  $\mathbb{R}_+^n = \{(x_1, x_2, \dots, x_n) | x_i \geq 0\}$ . The collection of points of  $X$  which correspond (by some and then by any chart) to points in  $\mathbb{R}^n$  with exactly  $k$  coordinates equal to zero is a well defined subset of  $X$  and it will be denoted by  $X_k$ . It has a structure of a smooth  $(n - k)$ -dimensional manifold.  $\partial X = X_1 \cup X_2 \cup \dots \cup X_n$  is a closed subset which is a topological manifold and  $(X, \partial X)$  is a topological manifold with boundary  $\partial X$ . A compact smooth manifold with corners,  $X$ , with interior diffeomorphic to the Euclidean space, will be called a compact smooth cell.

For any string of critical points  $x = y_0, y_1, \dots, y_k$  with

$$\text{index}(y_0) > \text{index}(y_1) > \dots > \text{index}(y_k),$$

consider the smooth manifold of dimension  $\text{index } y_0 - k$ ,

$$\mathcal{T}(y_0, y_1) \times \dots \times \mathcal{T}(y_{k-1}, y_k) \times W_{y_k}^-,$$

and the smooth map

$$i_{y_0, y_1, \dots, y_k} : \mathcal{T}(y_0, y_1) \times \dots \times \mathcal{T}(y_{k-1}, y_k) \times W_{y_k}^- \rightarrow M,$$

defined by  $i_{y_0, y_1, \dots, y_k}(\gamma_1, \dots, \gamma_k, y) := i_{y_k}(y)$ , where  $i_x : W_x^- \rightarrow M$  denotes the inclusion of  $W_x^-$  in  $M$ .

**Theorem 1.1.** *Let  $\tau = (h, g)$  be a generalized triangulation.*

1) *For any critical point  $x \in Cr(h)$  the smooth manifold  $W_x^-$  has a canonical compactification  $\hat{W}_x^-$  to a compact manifold with corners and the inclusion  $i_x$  has a smooth extension  $\hat{i}_x : \hat{W}_x^- \rightarrow M$  so that :*

(a)  $(\hat{W}_x^-)_k = \bigsqcup_{(x, y_1, \dots, y_k)} \mathcal{T}(x, y_1) \times \dots \times \mathcal{T}(y_{k-1}, y_k) \times W_{y_k}^-$ ,

(b) *the restriction of  $\hat{i}_x$  to  $\mathcal{T}(x, y_1) \times \dots \times \mathcal{T}(y_{k-1}, y_k) \times W_{y_k}^-$  is given by  $i_{x, y_1, \dots, y_k}$ .*

2) *For any two critical points  $x, y$  with  $i(x) > i(y)$  the smooth manifold  $\mathcal{T}(x, y)$  has a canonical compactification  $\hat{\mathcal{T}}(x, y)$  to a compact manifold with corners and*

$$\hat{\mathcal{T}}(x, y)_k = \bigsqcup_{(x, y_1, \dots, y_k=y)} \mathcal{T}(x, y_1) \times \dots \times \mathcal{T}(y_{k-1}, y_k).$$

The proof of Theorem 1.1 will be sketched in the last subsection of this section.

This theorem was probably well known to experts before it was formulated by Floer in the framework of infinite dimensional Morse theory cf. [F]. As formulated,

Theorem 1.1 is stated in [AB]. The proof sketched in [AB] is excessively complicated and incomplete. A considerably simpler proof will be sketched in Lecture 1 subsection d) and is contained in [BFK5] and [BH].

**Observation:**

O1: The name of generalized triangulation for  $\tau = (h, g)$  is justified by the fact that any simplicial smooth triangulation can be obtained as a generalized triangulation, cf [Po].

O2: Given a Morse function  $h$  and a Riemannian metric  $g$ , one can perform arbitrary small  $C^0$ -perturbations of  $g$ , so that the pair consisting of  $h$  and the perturbed metric is a generalized triangulation, cf [Sm] and [BFK5].

Given a generalized triangulation  $\tau = (h, g)$ , and for any critical point  $x \in Cr(h)$  an orientation  $\mathcal{O}_x$  of  $W_x^-$ , one can associate a cochain complex of vector spaces over the field  $\mathbb{K}$  of real or complex numbers,  $(C^*(M, \tau), \partial^*)$ . Denote the collection of these orientations by  $o$ . The differential  $\partial^*$  depends on the chosen orientations  $o := \{\mathcal{O}_x | x \in Cr(h)\}$ . To describe this complex we introduce the incidence numbers

$$I_q : Cr(h)_q \times Cr(h)_{q-1} \rightarrow \mathbb{Z}$$

defined as follows:

If  $\mathcal{T}(x, y) = \emptyset$ , we put  $I_q(x, y) = 0$ .

If  $\mathcal{T}(x, y) \neq \emptyset$ , for any  $\gamma \in \mathcal{T}(x, y)$ , the set  $\gamma \times W_y^-$  is an open subset of the boundary  $\partial \hat{W}_x^-$  and the orientation  $\mathcal{O}_x$  induces an orientation on it. If this is the same as the orientation  $\mathcal{O}_y$ , we set  $\varepsilon(\gamma) = +1$ , otherwise we set  $\varepsilon(\gamma) = -1$ . Define  $I_q(x, y)$  by

$$I_q(x, y) = \sum_{\gamma \in \mathcal{T}(x, y)} \varepsilon(\gamma).$$

In the case  $M$  is an oriented manifold, the orientation of  $M$  and the orientation  $\mathcal{O}_x$  on  $W_x^-$  induce an orientation  $\mathcal{O}_x^+$  on the stable manifold  $W_x^+$ .

For any  $c$  in the open interval  $(h(y), h(x))$ ,  $h^{-1}(c)$  carries a canonical orientation induced from the orientation of  $M$ . One can check that  $I_q(x, y)$  is the intersection number of  $W_x^- \cap h^{-1}(c)$  with  $W_y^+ \cap h^{-1}(c)$  inside  $h^{-1}(c)$  and is also the incidence number of the open cells  $W_x^-$  and  $W_y^-$  in the  $CW$ - complex structure provided by  $\tau$ .

Denote by  $(C^*(M, \tau), \partial_{(\tau, o)}^*)$  the cochain complex of  $\mathbb{K}$ - Euclidean vector spaces defined by

$$(1) C^q(M, \tau) := Maps(Cr_q(h), \mathbb{K})$$

$$(2) \partial_{(\tau, o)}^{q-1} : C^{q-1}(M, \tau) \rightarrow C^q(M, \tau), (\partial^{q-1} f)(x) = \sum_{y \in Cr_{q-1}(h)} I_q(x, y) f(y),$$

where  $x \in Cr_q(h)$ .

(3) Since  $C^q(M, \tau)$  is equipped with a canonical base provided by the maps  $E_x$  defined by  $E_x(y) = \delta_{x, y}$ ,  $x, y \in Cr_q(h)$ , it carries a natural scalar product which makes  $E_x$ ,  $x \in Cr_q(h)$ , orthonormal.

**Proposition 1.2.** *For any  $q$ ,  $\partial^q \cdot \partial^{q-1} = 0$ .*

A geometric proof of this Proposition follows from Theorem 1.1. The reader can also derive it by observing that  $(C^*(M, \tau), \partial^*)$  as defined is nothing but the cochain complex associated to the  $CW$ -complex structure provided by  $\tau$  via Theorem 1.1.

### b. Morse Bott generalized triangulations.

The concept of generalized triangulation and Theorem 1.1 above can be extended to pairs  $(h, g)$  with  $h$  a Morse Bott function; i.e  $Cr(h)$  consists of a disjoint union of compact connected smooth submanifolds  $\Sigma$  and the Hessian of  $h$  at any  $x \in \Sigma$  is nondegenerated in the normal directions of  $\Sigma$ . More precisely a *MB-generalized triangulation* is a pair  $\tau = (h, g)$  which satisfies C'1 and C'2 below:

**C'1:**  $Cr(h)$  is a disjoint union of closed connected submanifolds  $\Sigma$ , and for any  $\Sigma$  there exist *admissible* coordinates in some neighborhood of  $\Sigma$ . An admissible coordinate chart around  $\Sigma$  is provided by:

1) two orthogonal vector bundles  $\nu_{\pm}$  over  $\Sigma$  equipped with scalar product preserving connections (parallel transports)  $\nabla_{\pm}$  so that  $\nu_+ \oplus \nu_-$  is isomorphic to the normal bundle of  $\Sigma$ ,

2) a closed tubular neighborhood of  $\Sigma$ ,  $\varphi : (U, \Sigma) \rightarrow (D_{\varepsilon}(\nu_- \oplus \nu_+), \Sigma)$ ,  $U$  closed neighborhood of  $\Sigma$ , so that :

(i) :  $h \cdot \varphi^{-1}(v_1, v_2) = c - 1/2\|v_1\|^2 + 1/2\|v_2\|^2$  where  $v_{\pm} \in E(\nu_{\pm})$ ;

(ii) :  $(\varphi^{-1})^*(g)$  is the metric induced from the restriction of  $g$  on  $\Sigma$ , the scalar products and the connections in  $\nu_{\pm}$ .

The rank of  $\nu_-$  will be called the index of  $\Sigma$  and denoted by  $i(\Sigma) = \text{index}(\Sigma)$ . As before for any  $x \in Cr(h)$  consider  $W_x^{\pm}$  and introduce  $W_{\Sigma}^{\pm} = \cup_{x \in \Sigma} W_x^{\pm}$ .

**C'2:**(the Morse Smale condition) For any two critical manifolds  $\Sigma, \Sigma'$  and  $x \in \Sigma$ ,  $W_x^-$  and  $W_{\Sigma'}^+$  are transversal.

As before C'2 implies that  $\mathcal{M}(\Sigma, \Sigma') := W_{\Sigma}^- \cap W_{\Sigma'}^+$  is a smooth manifold of dimension equal to  $i(\Sigma) - i(\Sigma') - \dim \Sigma$ , and that the evaluation maps  $u : \mathcal{M}(\Sigma, \Sigma') \rightarrow \Sigma$  is a smooth bundle with fiber  $W_x^- \cap W_{\Sigma'}^+$ , a smooth manifold of dimension  $i(\Sigma) - i(\Sigma')$ .

$\mathcal{M}(\Sigma, \Sigma')$  is equipped with the free action  $\mu : \mathbb{R} \times \mathcal{M}(\Sigma, \Sigma') \rightarrow \mathcal{M}(\Sigma, \Sigma')$  defined by  $\mu(t, z) = \gamma_z(t)$  and we denote the quotient space  $\mathcal{M}(\Sigma, \Sigma')/\mathbb{R}$  by  $\mathcal{T}(\Sigma, \Sigma')$ .  $\mathcal{T}(\Sigma, \Sigma')$  is a smooth manifold of dimension  $i(\Sigma) - i(\Sigma') + \dim \Sigma - 1$ , diffeomorphic to the submanifold  $\mathcal{M}(\Sigma, \Sigma') \cap h^{-1}(\lambda)$ , for any real number  $\lambda$  in the open interval  $(h(\Sigma), h(\Sigma'))$ . In addition, one has the evaluation maps,  $u_{\Sigma, \Sigma'} : \mathcal{T}(\Sigma, \Sigma') \rightarrow \Sigma$  which is a smooth bundle with fiber  $W_x^- \cap W_{\Sigma'}^+/\mathbb{R}$ , a smooth manifold of dimension  $i(\Sigma) - i(\Sigma') - 1$ , and  $l_{\Sigma, \Sigma'} : \mathcal{T}(\Sigma, \Sigma') \rightarrow \Sigma'$  a smooth map. The maps  $u_{\dots}$  and  $l_{\dots}$  induce by pull- back constructions the smooth bundles

$$u_{(\Sigma_0, \Sigma_1, \dots, \Sigma_k)} : \mathcal{T}(\Sigma_0, \Sigma_1) \times_{\Sigma_1} \cdots \times_{\Sigma_{k-1}} \mathcal{T}(\Sigma_{k-1}, \Sigma_k) \rightarrow \Sigma_0,$$

the smooth maps

$$l_{(\Sigma_0, \Sigma_1, \dots, \Sigma_k)} : \mathcal{T}(\Sigma_0, \Sigma_1) \times_{\Sigma_1} \cdots \times_{\Sigma_{k-1}} \mathcal{T}(\Sigma_{k-1}, \Sigma_k) \rightarrow \Sigma_k$$

and

$$i_{\Sigma_0, \Sigma_1, \dots, \Sigma_k} : \mathcal{T}(\Sigma_0, \Sigma_1) \times_{\Sigma_1} \cdots \times_{\Sigma_{k-1}} \mathcal{T}(\Sigma_{k-1}, \Sigma_k) \times_{\Sigma_k} W_{\Sigma_k}^- \rightarrow M,$$

defined by  $i_{\Sigma_0, \Sigma_1, \dots, \Sigma_k}(\gamma_1, \dots, \gamma_k, y) := i_{\Sigma_k}(y)$ , for  $\gamma_i \in \mathcal{T}(\Sigma_{i-1}, \Sigma_i)$  and  $y \in W_{\Sigma_k}^-$ ,

The analogue of Theorem 1.1 is Theorem 1.1' below. The proof of Theorem 1.1 as given below, subsection d), and is formulated in such way that the extension to the Morse Bott case is straightforward.

**Theorem 1.1'.**

Let  $\tau = (h, g)$  be a MB generalized triangulation.

1) For any critical manifold  $\Sigma \subset Cr(h)$ , the smooth manifold  $W_{\Sigma}^-$  has a canonical compactification to a compact manifold with corners  $\hat{W}_{\Sigma}^-$ , and the smooth bundle  $\pi_{\Sigma}^- : W_{\Sigma}^- \rightarrow \Sigma$  resp. the smooth inclusion  $i_{\Sigma} : W_{\Sigma}^- \rightarrow M$  have extensions  $\hat{\pi}_{\Sigma}^- : \hat{W}_{\Sigma}^- \rightarrow \Sigma$ , a smooth bundle whose fibers are compact manifolds with corners, resp.  $\hat{i}_{\Sigma} : \hat{W}_{\Sigma}^- \rightarrow M$  a smooth map, so that

$$(a): (\hat{W}_{\Sigma}^-)_k = \bigsqcup_{(\Sigma, \Sigma_1, \dots, \Sigma_k)} \mathcal{T}(\Sigma, \Sigma_1) \times_{\Sigma_1} \cdots \times_{\Sigma_{k-1}} \mathcal{T}(\Sigma_{k-1}, \Sigma_k) \times_{\Sigma_k} W_{\Sigma_k}^-,$$

(b): the restriction of  $\hat{i}_{\Sigma}$  to  $\mathcal{T}(\Sigma, \Sigma_1) \times_{\Sigma_1} \cdots \times_{\Sigma_{k-1}} \mathcal{T}(\Sigma_{k-1}, \Sigma_k) \times_{\Sigma_k} W_{\Sigma_k}^-$  is given by  $i_{\Sigma, \Sigma_1, \dots, \Sigma_k}$ .

2) For any two critical manifolds  $\Sigma, \Sigma'$  with  $i(\Sigma) > i(\Sigma')$  the smooth manifold  $\mathcal{T}(\Sigma, \Sigma')$  has a canonical compactification to a compact manifold with corners  $\hat{\mathcal{T}}(\Sigma, \Sigma')$  and the smooth maps  $u : \mathcal{T}(\Sigma, \Sigma') \rightarrow \Sigma$  and  $l : \mathcal{T}(\Sigma, \Sigma') \rightarrow \Sigma'$  have smooth extensions  $\hat{u} : \hat{\mathcal{T}}(\Sigma, \Sigma') \rightarrow \Sigma$  and  $\hat{l} : \hat{\mathcal{T}}(\Sigma, \Sigma') \rightarrow \Sigma'$  with  $\hat{u}$  a smooth bundle whose fibers are compact manifolds with corners. Precisely

$$(\hat{\mathcal{T}}(\Sigma, \Sigma'))_k = \bigsqcup_{(\Sigma, \Sigma_1, \dots, \Sigma_k)} \mathcal{T}(\Sigma, \Sigma_1) \times_{\Sigma_1} \cdots \times_{\Sigma_{k-1}} \mathcal{T}(\Sigma_{k-1}, \Sigma_k).$$

For a critical manifold  $\Sigma$  choose an orientation of  $\nu_-$  if this bundle is orientable and an orientation of the orientable double cover of  $\nu_-$  if not. Such an object will be denoted by  $\mathcal{O}_{\Sigma}$  and the collection of all  $\mathcal{O}_{\Sigma}$  will be denoted by  $o \equiv \{\mathcal{O}_{\Sigma} | \Sigma \subset Cr(h)\}$ .

Choosing the collection  $o$  in addition to the Morse-Bott generalized triangulation  $\tau$  one can provide as an analogue of the geometric complex  $(C^*(M, \tau), \partial_{(\tau, o)}^*)$  the complex  $(C^*, D^*)$  defined by

$$C^r =: \bigoplus_{\{(k, \Sigma) | k+i(\Sigma)=r\}} \Omega^k(\Sigma, o(\nu_-))$$

and  $D^r : C^r \rightarrow C^{r+1}$  given by the matrix  $|||\partial_{\Sigma, \Sigma'}^k|||$  whose entries

$\partial^k(\Sigma, \Sigma') : \Omega^k(\Sigma', o(\nu_-)) \rightarrow \Omega^{r-i(\Sigma)+i(\Sigma')+1}(\Sigma, o(\nu_-))$  are given by<sup>1</sup>

$$\partial^k(\Sigma, \Sigma') = \left\{ \begin{array}{l} d^k : \Omega^k(\Sigma, o(\nu_-)) \rightarrow \Omega^{k+1}(\Sigma, o(\nu_-)) \text{ if } \Sigma = \Sigma' \\ (-1)^k (\hat{u}_{\Sigma, \Sigma'})_* \cdot (\hat{l}_{\Sigma, \Sigma'})^* \text{ otherwise} \end{array} \right\}$$

Here  $\Omega^*(\Sigma, o(\nu_-))$  denotes the differential forms on  $\Sigma$  with coefficients in the orientation bundle of  $\nu_-$  and  $(\dots)_*$  denotes the integration along the fiber of  $\hat{u}_{\Sigma, \Sigma'}$ .

The orientation bundle of  $\nu_-$  has a canonical flat connection. When  $\nu_-$  is orientable then this bundle is trivial as bundle with connection and  $\Omega^*$  identifies to the ordinary differential forms.

A Morse Bott function  $h$  is a smooth function for which the critical set consists of a disjoint union of connected manifolds  $\Sigma$ , so that the Hessian of  $h$  at each critical point of  $\Sigma$  is nondegenerated in the normal direction.

**Observation:**

O.2': Given a pair  $(h, g)$  with  $h$  a Morse Bott function and  $g$  is a Riemannian metric one can provide arbitrary small  $C^0$  perturbation  $g'$  of  $g$  so that the pair  $(h, g')$  satisfies  $C'$ . If  $(h, g)$  satisfies  $C'1$  one can choose  $g'$  arbitrary closed to  $g$  in  $C^0$ -topology so that  $g = g'$  away from a given neighborhood of the critical point set,  $g' = g$  in some (smaller) neighborhood of the critical point set and  $(h, g')$  is an MB generalized triangulation.

**c. G-Generalized triangulations.**

Of particular interest is the case of a smooth  $G$ -manifold  $(M, \mu : G \times M \rightarrow M)$  where  $G$  is a compact Lie group and  $\mu$  a smooth action. In this case we consider pairs  $(h, g)$  with  $h$  a  $G$ -invariant smooth function and  $g$  a  $G$ -invariant Riemannian metric. Then  $Cr(h)$  consists of a union of  $G$ -orbits. The  $G$ -version of conditions **C1** and **C2** are obvious to formulate. We say that the pair  $(h, g)$  with both  $h$  and  $g$   $G$ -invariant is a  $G$ -generalized triangulation resp. normal  $G$ -generalized triangulation if G-C1 (resp. normal G-C1) and G-C2 hold.

**G-C1:**  $Cr(h)$  is a finite union of orbits denoted by  $\Sigma$  and for any critical orbit  $\Sigma$  we require the existence of an admissible chart. More precisely, such a chart around  $\Sigma$  is provided by the following data:

1: A closed subgroup  $H \subset G$ , two orthogonal representations  $\rho_{\pm} : H \rightarrow O(V_{\pm})$  and a scalar product on the Lie algebra  $\mathfrak{g}$  of  $G$  which is invariant with respect to the adjoint representation restricted to  $H$ .

$\rho_{\pm}$  induce orthogonal bundles  $\nu_{\pm} : E(\nu_{\pm}) \rightarrow \Sigma$ . The total space of these  $G$ -bundles are  $E(\nu_{\pm}) = G \times_H V_{\pm}$ . The scalar product on  $\mathfrak{g}$  and the the scalar product

<sup>1</sup>this formula is implicit in (2.2) in view of the fact that  $Int^* : (\Omega^*, d^*) \rightarrow (C^*(M, \tau), \partial^*)$  is supposed to be a surjective morphism of cochain complexes

on  $V = V_- \oplus V_+$  induces a  $G$ -invariant Riemannian metric on  $G$  and on  $G \times V$ . The metric on  $G \times V$  descends to a  $G$ -invariant Riemannian metric on  $E(\nu_- \oplus \nu_+)$ .

2) A positive number  $\epsilon$ , a constant  $c \in \mathbb{R}$  and a  $G$ -equivariant diffeomorphism  $\varphi : (U, \Sigma) \rightarrow D_\epsilon(\rho_+ \oplus \rho_-)$  where  $U$  is a closed  $G$ -tubular neighborhood of  $\Sigma$  in  $M$ , and  $D_\epsilon$  denotes the disc of radius  $\epsilon$  in the underlying Euclidean space of the representation  $\rho_+ \oplus \rho_-$ , so that

$$(i) : h \cdot \varphi^{-1}((g, v_1, v_2)) = c - 1/2\|v_1\|^2 + 1/2\|v_2\|^2 \text{ where } v_\pm \in E(\nu_\pm)$$

$$(ii) : (\varphi^{-1})^*(g) \text{ is the Riemannian metric on } E(\nu_- \oplus \nu_+) \text{ described above.}$$

We call the admissible chart "normal" if in addition  $\rho_-$  is trivial. The condition **normal G-C1** requires the admissible charts to be normal.

**G-C2:** This condition is the same as C'2.

**Observation:**

O1": Given a pair  $(h, g)$  one can perform an arbitrary small  $C^0$  perturbation  $(h', g')$  so that  $(h', g')$  satisfies normal G-C1. This was proven in [M].

O2": Given  $(h, g)$  a pair which satisfies normal G-C1 then one can perform an arbitrary small  $C^0$  perturbation on the metric  $g$  and obtain the  $G$ -invariant metric  $g'$  (away from the critical set) so that  $(h, g')$  satisfies G-C2. This result is proven in [B].

Clearly, a  $G$ -generalized triangulation is a  $MB$ -generalized triangulation, hence Theorem 1.1' above can be restated in this case as Theorem 1.1" with the additional specifications that in the statement of Theorem 1.1' all compact manifolds with corners are  $G$ -manifolds and all maps are  $G$ -equivariant.

Note that a  $G$ -generalized triangulation provides via Theorem 1.1" a structure of a smooth  $G$ -handle body and a normal  $G$ -generalized triangulation provides a structure of smooth  $G$ -CW complex for  $M$ . The smooth triangulability of compact smooth manifolds with corners if combined with the existence of normal  $G$ -generalized triangulation lead to the existence of a smooth  $G$ -triangulation in the sense of [I] and then to the existence of smooth triangulation of the orbit spaces of a smooth  $G$ -manifold when  $G$  is compact. There is no proof for this result in literature. The best known result so far, is the existence of a  $C^0$  triangulation of the orbit space established by Verona. [V].

**d): Proof of Theorem 1.1.**

*Some notations*

We begin by introducing some notations:

Let  $c_0 < c_1 \cdots < c_N$  be the collection of all critical values ( $c_0$  the absolute minimum,  $c_N$  the absolute maximum) and fix  $\epsilon > 0$  small enough so that  $c_i - \epsilon > c_{i-1} + \epsilon$  for all  $i \geq 1$ . Denote by:

$$Cr(i) := Cr(h) \cap h^{-1}(c_i),$$

$$\begin{aligned}
M_i &:= h^{-1}(c_i), \\
M_i^\pm &:= h^{-1}(c_i \pm \epsilon) \\
M(i) &:= h^{-1}(c_{i-1}, c_{i+1})
\end{aligned}$$

For any  $x \in Cr(i)$  denote by:

$$\begin{aligned}
S_x^\pm &:= W_x^\pm \cap M_i^\pm \\
\mathbf{S}_x &:= S_x^+ \times S_x^- \\
W_x^\pm(i) &:= W_x^\pm \cap M(i) \\
\mathbf{S}W_x(i) &:= S_x^+ \times W_x^-(i).
\end{aligned}$$

It will be convenient to write

$$\begin{aligned}
S_i^\pm &:= \bigcup_{x \in Cr(i)} S_x^\pm \\
\mathbf{S}_i &:= \bigcup_{x \in Cr(i)} S_x \subset M_i^- \times M_i^+ \\
W^\pm(i) &:= \bigcup_{x \in Cr(i)} W_x^\pm(i) \\
\mathbf{S}W(i) &:= \bigcup_{x \in Cr(i)} S_x^+ \times W_x^-(i)
\end{aligned}$$

Observe that:

- 1)  $\mathbf{S}_i \subset M_i^+ \times M_i^-$ ,  $\mathbf{SW}(i) \subset M_i^+ \times M(i)$
- 2)  $M_i^\pm$  is a smooth manifold of dimension  $n - 1$ , ( $n = \dim M$ ) and  $M(i)$  is a smooth manifold of dimension  $n$ , actually an open set in  $M$ .  $M_i$  is not a manifold, however,  $\mathring{M}_i := M_i \setminus Cr(i)$ ,  $\mathring{M}_i^\pm := M_i^\pm \setminus S_i^\pm$  are smooth manifolds (submanifolds of  $M$ ) of dimension  $n - 1$ .

*The flow  $\Phi_t$  and few induced maps*

Let  $\Phi_t$  be the flow associated to the vector field  $-\text{grad}_g h / \|\text{grad}_g h\|$  on  $M \setminus Cr(h)$  and consider:

a) the diffeomorphisms

$$\begin{aligned} \psi_i &: M_i^- \rightarrow M_{i-1}^+ \\ \varphi_i^\pm &: M_i^\pm \rightarrow \mathring{M}_i \end{aligned}$$

obtained by the restriction of  $\Phi_{(c_i - c_{i-1} - 2\epsilon)}$  and  $\Phi_{\pm\epsilon}$ ,

b) the submersion  $\varphi(i) : M(i) \setminus (W^-(i) \cup W^+(i)) \rightarrow \mathring{M}_i$  defined by  $\varphi(i)(x) := \Phi_{h(x) - c_i}(x)$ .

Observe that  $\varphi_i^\pm$  and  $\varphi(i)$  extend in a unique way to continuous maps

$$\varphi_i^\pm : M_i^\pm \rightarrow M, \quad \varphi(i) : M(i) \rightarrow M_i.$$

*Two manifolds with boundary*

*The manifold  $P_i$  :* Define

$$P_i := \{(x, y) \in M_i^- \times M_i^+ \mid \varphi_i^-(x) = \varphi_i^+(y)\},$$

and denote by  $p_i^\pm : P_i \rightarrow M_i^\pm$  the canonical projections. One can verify that  $P_i$  is a compact smooth  $(n-1)$  dimensional manifold with boundary, (smooth submanifold of  $M_i^- \times M_i^+$ ) whose boundary  $\partial P_i$  can be identified to  $\mathbf{S}_i \subset M_i^- \times M_i^+$ . Precisely

OP1:  $p_i^\pm : P_i \setminus \partial P_i \rightarrow \overset{\circ}{M}_i^\pm$  are diffeomorphisms,

OP2: the restriction of  $p_i^+ \times p_i^-$  to  $\partial P_i$  is a diffeomorphism onto  $\mathbf{S}_i$ . (Each  $p_i^\pm$  restricted to  $\partial P_i$  is the projection onto  $S_i^\pm$ .)

*The manifold  $Q(i)$  :* Define

$$Q(i) = \{(x, y) \in M_i^+ \times M(i) \mid \varphi_i^+(x) = \varphi(i)(y)\}$$

or equivalently,  $Q(i)$  consists of pairs of points  $(x, y)$ ,  $x \in M_i^+$ ,  $y \in M(i)$  which lie on the same (possibly broken) trajectory and denote by  $l_i : Q(i) \rightarrow M_i^+$  resp.  $r_i : Q(i) \rightarrow M(i)$  the canonical projections.

One can verify that  $Q(i)$  is a smooth  $n$ -dimensional manifold with boundary, (smooth submanifold of  $M_i^+ \times M(i)$ ) whose boundary  $\partial Q(i)$  is diffeomorphic to  $\mathbf{SW}(i) \subset M_i^+ \times M(i)$ . More precisely

OQ1:  $l_i : Q(i) \setminus \partial Q(i) \rightarrow \overset{\circ}{M}_i^+$  is a smooth bundle with fiber an open segment and  $r_i : Q(i) \setminus \partial Q(i) \rightarrow M(i) \setminus W^-(i)$  a diffeomorphism,

OQ2: the restriction of  $l \times r$  to  $\partial Q(i)$  is a diffeomorphism onto  $\mathbf{SW}(i)$ . ( $l$  resp.  $r$  restricted to  $\partial Q(i)$  identifies with the projection onto  $S_i^+$  resp.  $W(i)$ ).

Since  $P_i$  and  $Q(i)$  are smooth manifolds with boundaries,

$$\mathcal{P}_{r,r-k} := P_r \times P_{r-1} \cdots P_{r-k}$$

and

$$\mathcal{P}_r(r-k) := P_r \times \cdots P_{r-k+1} \times Q(r-k)$$

are smooth manifolds with corners.

Our arguments for the proof of Theorem 1.1 will be based on the following method for recognizing a smooth manifold with corners. If  $\mathcal{P}$  is a smooth manifold with corners,  $\mathcal{O}, \mathcal{S}$  smooth manifolds,  $p : \mathcal{P} \rightarrow \mathcal{O}$  and  $s : \mathcal{S} \rightarrow \mathcal{O}$  smooth maps so that  $p$  and  $s$  are transversal ( $p$  is transversal to  $s$  if its restriction to each  $k$ -corner of  $\mathcal{P}$  is transversal to  $s$ ,) then  $p^{-1}(s(\mathcal{S}))$  is a smooth submanifold with corners of  $\mathcal{P}$ .

**Proof of Theorem 1.1** First we prove part (2). We want to verify that  $\hat{T}(x, y)$  (cf the definition in the statement of Theorem 1.1) is a smooth manifold with corners. Let  $x \in Cr(r+1)$  and  $y \in Cr(r-k-1)$ ,  $k \geq -2$ . If  $k = -2$  the statement is empty, if  $k = -1$  there is nothing to check, so we suppose  $k \geq 0$ .

We consider  $\mathcal{P} = \mathcal{P}_{r,k}$  as defined above,  $\mathcal{O} = \prod_r^{r-k}(M_i^+ \times M_i^-)$ , and  $\mathcal{S} = S_x^- \times M_r^- \cdots \times M_{(r-k+1)}^- \times S_y^+$ . In order to define the maps  $p$  and  $s$  we consider

$\omega_i : M_i^- \rightarrow M_i^- \times M_{i+1}^+$  given by  $\omega_i(x) = (x, \psi_i(x))$ , and  
 $\tilde{p}_i : P_i \rightarrow M_i^+ \times M_i^-$  given by  $\tilde{p}_i(y) = (p_i^+(y), p_i^-(y))$ .

We also denote by  $\alpha : S_x^- \rightarrow M_r^+$  and  $\beta : S_y^+ \rightarrow M_{k-r}^-$  the restriction of  $\psi_{r+1}$  resp. of  $\psi_{r-k}^{-1}$  to  $S_x^-$  resp.  $S_y^+$ . Take  $s = \alpha \times \omega_r \cdots \omega_{r-k+1} \times \beta$  and  $p := \prod_{i=r}^{r-k} \tilde{\pi}_i$ .

The verification of the transversality follows easily from OP1, OP2 and the Morse Smale condition C2. It is easy to see that  $p^{-1}(s(\mathcal{S}))$  is compact and identifies to  $\hat{T}(x, y)$ ; the verification of this fact is left to the reader.

To prove part (1) we first consider the map  $\hat{i}_x : X = \hat{W}_x^- \rightarrow M$  defined by (a) and (b) in Theorem 1.1 (1). Let  $X := \hat{W}_x^-$  and for any positive integer  $k$  we denote by  $X(k) := \hat{i}_x^{-1}(M(k))$ . The proof will be given in two steps. First we will topologize  $X(k)$  and put on it a structure of a smooth manifold with corners, so that the restriction of  $\hat{i}_x$  to  $X(k)$  is a smooth map. Second we check that  $X(k)$  and  $X(k')$  induce on the intersection  $X(k) \cap X(k')$  the same topology and the same smooth structure. These facts imply that  $X$  has a canonical structure of smooth manifold with corners and  $\hat{i}_x$  is a smooth map. The compactness of  $X$  follows by observing that the image  $\hat{i}_x(X)$  is compact and the preimage of any point is compact.

To accomplish the first step we proceed in exactly the same way as in the proof of part (2). Suppose  $x \in Cr(r+1)$ . Consider

$$\mathcal{P} := \mathcal{P}_r(k),$$

$$\mathcal{O} := (M_r^+ \times M_r^-) \times \cdots \times (M_{r-k+1}^+ \times M_{k+r-1}^-) \times M_{r-k}^+, \text{ and}$$

$$\mathcal{S} := S_x^- \times M_r^- \cdots \times M_{r-k+1}^-.$$

Take

$$p := \tilde{p}_r \times \cdots \times \tilde{p}_{r-k+1} \times l_{r-k} \text{ and}$$

$$s = \alpha \times \omega_r \cdots \times \omega_{r-k+1}.$$

The verification of the transversality follows from OP1, OP2, OQ1, OQ2 and the Morse Smale condition C2 above as explained in the Appendix. It is easy to see that  $p^{-1}(s(\mathcal{S}))$  identifies to  $X(r-k)$ .

The second step is more or less straightforward, so it will be left again to the reader. q.e.d.

## Appendix : The verification of transversality of the maps $p$ and $s$ .

Consider the diagrams

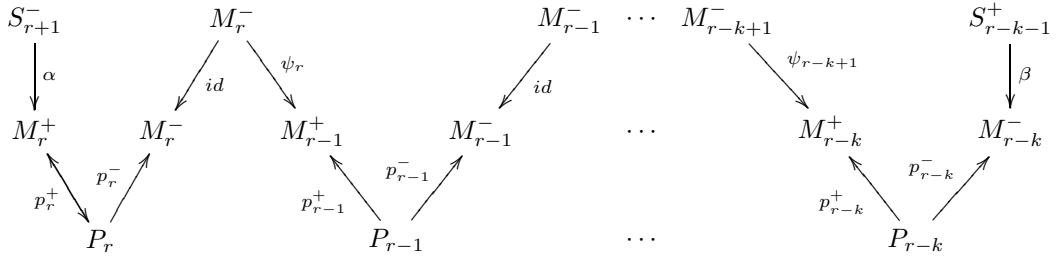


Diagram 1

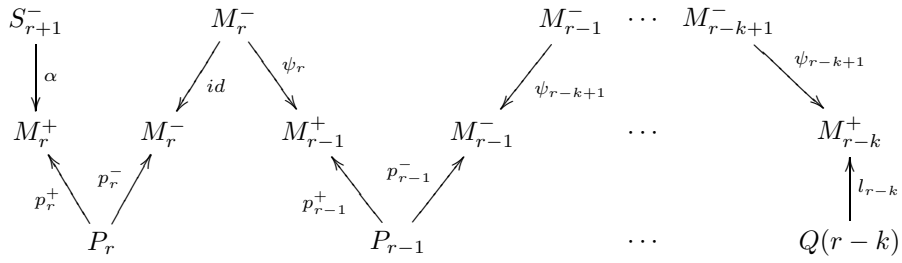


Diagram 1'

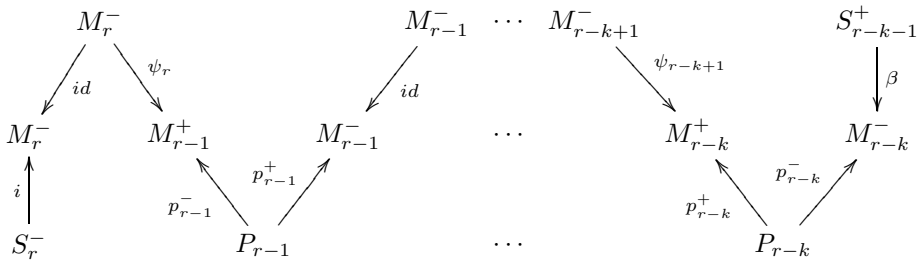


Diagram 2

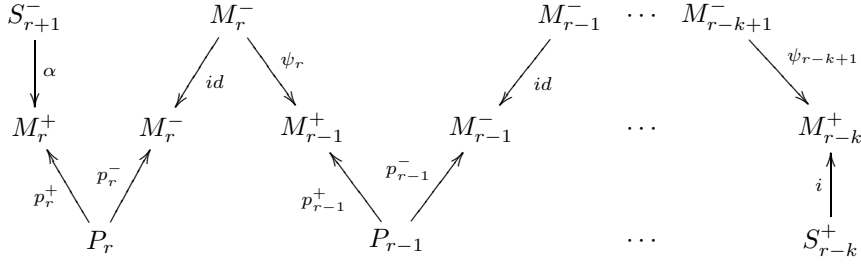


Diagram 3

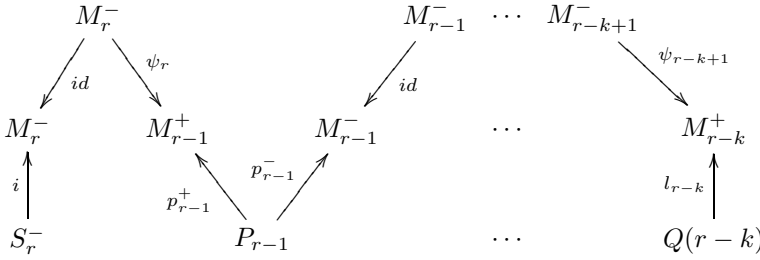


Diagram 4

For each of these diagrams denote by  $\mathcal{P}$  resp.  $\mathcal{O}$  resp.  $\mathcal{R}$ , the product of the manifolds on the third resp. second resp. first row and let  $p : \mathcal{P} \rightarrow \mathcal{O}$  resp.  $s : \mathcal{R} \rightarrow \mathcal{O}$  denote the product of the maps from the third to the second row resp. the first to the second row. Clearly  $\mathcal{P}$  is a smooth manifold with corners. Denote by  $\overset{\circ}{\mathcal{P}}$  the interior of  $\mathcal{P}$ , i.e.  $\mathcal{P} \setminus \partial\mathcal{P}$ , and by  $\overset{\circ}{p} : \overset{\circ}{\mathcal{P}} \rightarrow \mathcal{O}$  the restriction of  $p$  to  $\overset{\circ}{\mathcal{P}}$ .

We refer to the statement "  $\overset{\circ}{p}$  transversal to  $s$ " with  $p$  and  $s$  obtained from the diagram 1, 1', 2, 2', 3, 4 as:  $T_{r,k}^1$ ,  $T_{r,k}^{1'}$ ,  $T_{r,k}^2$ ,  $T_{r,k}^{2'}$ ,  $T_{r,k}^3$ ,  $T_{r,k}^4$ .

In view of the observation OP2 and OQ2 and of the fact that "  $m_i : \mathcal{M}_i \rightarrow \mathcal{O}_i$  and  $s_i : \mathcal{R}_i \rightarrow \mathcal{O}_i$ ,  $i = 1, 2$ ,  $\mathcal{M}_i, \mathcal{O}_i, \mathcal{R}_i$  smooth manifolds imply the transversality of  $m_1 \times m_2 : \mathcal{M}_1 \times \mathcal{M}_2 \rightarrow \mathcal{O}_1 \times \mathcal{O}_2$  and  $s_1 \times s_2 : \mathcal{R}_1 \times \mathcal{R}_2 \rightarrow \mathcal{O}_1 \times \mathcal{O}_2$ ", it is easy to see that the transversality of  $p$  and  $s$  obtained from the diagram 1 resp. 1' can be derived from the validity of the statements  $T_{r,k}^1$ ,  $T_{r,k}^2$ ,  $T_{r,k}^3$  resp.  $T_{r,k}^{1'}$ ,  $T_{r,k}^3$ ,  $T_{r,k}^4$  for various  $r, k$ .

In view of the fact that all arrows except  $\alpha, \beta, l_{r-k}$  and the inclusions ( cf Diagrams 2,3,4) are open embeddings, the properties  $T_{r,k}^1, \dots, T_{r,k}^4$  follow from the transversality of  $W_{r+1}^-$  and  $W_{r-k-1}^+$ . This finishes the verification of the transversality statement needed for the proof of Theorem 1.1. q.e.d

## Lecture 2: Witten deformation and the spectral properties of the Witten Laplacian.

### a. De Rham theory and integration.

Let  $M$  be a closed smooth manifold and  $\tau = (h, g)$  be a generalized triangulation or more general a Morse-Bott generalized triangulation and  $o$  a system of orientations. Denote by  $(\Omega^*(M), d^*)$  the De Rham complex of  $M$ . This is a cochain complex whose component  $\Omega^q(M)$  is the (Fréchet) space of smooth differential forms of degree  $q$  and whose differential  $d^q : \Omega^q(M) \rightarrow \Omega^{q+1}(M)$  is given by the exterior differential  $d$ . Recall that Stokes' theorem for manifolds with corners can be formulated as follows:

**Theorem.** *Let  $P$  be a compact  $r$ -dimensional oriented smooth manifold with corners and  $f : P \rightarrow M$  a smooth map. Denote by  $\partial f : P_1 \rightarrow M$  the restriction of  $f$  to the smooth oriented manifold  $P_1$  ( $P_1$  defined as in section 1a). If  $\omega \in \Omega^{r-1}(M)$  is a smooth form then  $\int_{P_1} (\partial f)^*(\omega)$  is convergent and*

$$\int_P f^*(d\omega) = \int_{P_1} (\partial f)^*(\omega).$$

Define the linear maps  $Int^q : \Omega^q(M) \rightarrow C^q(M, \tau)$  as follows.

a) In the case  $\tau$  is a generalized triangulation and  $C^q(M, \tau) := \text{Maps}(Cr_q(h), \mathbb{K})$ ,  $Int^q$  is defined by

$$(2.1) \quad Int^q(\omega)(x) := \int_{\hat{W}_x^-} \omega,$$

where  $\mathbb{K}$  is the field of real or complex numbers.

b) In the case  $\tau$  is an Morse-Bott generalized triangulation and

$$C^q(M, \tau) := \bigoplus_{k+\dim(\Sigma)=q} \Omega^k(\Sigma, o(\nu_-)),$$

$Int^q$  is given by integrating the pull back by  $\hat{i}_\Sigma : \hat{W}_\Sigma^- \rightarrow M$  of a form  $\omega \in \Omega^q(M)$  along the fiber of  $\hat{\pi}_\Sigma : \hat{W}_\Sigma^- \rightarrow \Sigma$ , i.e.

$$(2.2) \quad Int^q(\omega) = \bigoplus_{k+\dim(\Sigma)=q} (\hat{\pi}_\Sigma^-)_* \hat{i}_\Sigma^* \omega \in \Omega^{q-i(\Sigma)}(\Sigma, o(\nu_-))$$

It is a consequence of Theorem 1.1 (1.1') that the collection of the linear maps  $Int^q$  defines a morphism

$$Int^* : (\Omega^*(M), d^*) \rightarrow (C^*(M, \tau), \partial^*)$$

of cochain complexes which has the following property (de Rham):

**Theorem.** *Int\* induces an isomorphism in cohomology.*

c) Suppose that we are in the case of a  $G$ -manifold and  $\tau$  is a  $G$ -generalized triangulation as described in section 1. Fix an irreducible representation  $\xi : G \rightarrow O(V_\xi)$  and denote by  $(\Omega_\xi^*, d_\xi^*)$  resp.  $(C_\xi^*, \partial_\xi^*)$  the subcomplex of  $(\Omega^*, d^*)$  resp.  $(C^*, \partial^*)$  defined by the property that  $\Omega_\xi^*$  resp.  $C_\xi^*$  is the largest  $G$ -invariant subspace of  $\Omega^*$  resp.  $C^*$  which contains no other irreducible representation but  $\xi$ . Equivalently for any irreducible representation  $\xi'$ ,  $\xi' \neq \xi$ ,  $\Omega_\xi^* \otimes_G V_{\xi'} = 0$  resp.  $C_\xi^* \otimes_G V_{\xi'} = 0$ . Since  $G$  is compact,  $(\Omega^*, d^*) = \hat{\bigoplus}_\xi (\Omega_\xi^*, d_\xi^*)$  resp.  $(C^*, \partial^*) = \hat{\bigoplus}_\xi (C_\xi^*, \partial_\xi^*)$ . Since the integration map is  $G$ -equivariant,  $Int(\Omega_\xi^*) \subset C_\xi^*$ .

Denote by

$$Int_\xi^* : (\Omega_\xi^*, d_\xi^*) \rightarrow (C_\xi^*, \partial_\xi^*)$$

the restriction of  $Int^*$  to the components  $(\Omega_\xi^*, d_\xi^*)$  corresponding to  $\xi$ .

We have the following refinement of de Rham Theorem.

**Theorem.** *Int\*\_xi induces an isomorphism in cohomology.*

Theorem 3.1-1" below (Lecture 3) will imply these theorems.

## b) Witten deformation and Witten Laplacians.

Let  $M$  be a closed manifold and  $\alpha \in \Omega^1(M)$  a closed 1-form, i.e.  $d\alpha = 0$ . For  $t > 0$  consider the complex  $(\Omega^*(M), d^*(t))$  with differential

$$(2.3) \quad d^q(t)(\omega) = d\omega + t\alpha \wedge \omega.$$

If  $\alpha = dh$  with  $h : M \rightarrow \mathbb{R}$  a smooth function,  $d^q(t)(\omega) = e^{-th} d e^{th}(\omega)$  and  $d^*(t)$  is the unique differential in  $\Omega^*(M)$  which makes the multiplication by the smooth function  $e^{th}$  an isomorphism of cochain complexes

$$e^{th} : (\Omega^*(M), d^*(t)) \rightarrow (\Omega^*(M), d^*).$$

Recall that for any vector field  $X$  on  $M$  one defines a zero order differential operator,  $\iota_X^* : \Omega^*(M) \rightarrow \Omega^{*-1}(M)$ , by

$$(2.4) \quad (\iota_X^q \omega)(X_1, X_2, \dots, X_{q-1}) := \omega(X, X_1, \dots, X_{q-1})$$

and a first order differential operator  $L_X^* : \Omega^*(M) \rightarrow \Omega^*(M)$ , the Lie derivative in the direction  $X$ , by

$$(2.5) \quad L_X^q := d^{q-1} \cdot \iota_X^q + \iota_X^{q+1} \cdot d^q.$$

They satisfy the following identities:

$$(2.6) \quad \iota_X(\omega_1 \wedge \omega_2) = \iota_X(\omega_1) \wedge \omega_2 + (-1)^{|\omega_1|} \omega_1 \wedge \iota_X(\omega_2).$$

where  $|\omega_1|$  denotes the degree of  $\omega_1$  and

$$(2.7) \quad L_X(\omega_1 \wedge \omega_2) = L_X(\omega_1) \wedge \omega_2 + \omega_1 \wedge L_X(\omega_2).$$

Given a Riemannian metric  $g$  on the oriented manifold  $M$  we have the zero order operator  $R^q : \Omega^q(M) \rightarrow \Omega^{n-q}(M)$ , known as the Hodge-star operator which, with respect to an oriented orthonormal frame  $e_1, e_2, \dots, e_n$  in the cotangent space at  $x$ , is given by

$$(2.8) \quad R_x^q(e_{i_1} \wedge \dots \wedge e_{i_q}) = \epsilon(i_1, \dots, i_q) e_1 \wedge \dots \wedge \hat{e}_{i_1} \wedge \dots \wedge \hat{e}_{i_q} \wedge \dots \wedge e_n,$$

where  $1 \leq i_1 < i_2 < \dots < i_q \leq n$ , and  $\epsilon(i_1, i_2, \dots, i_q)$  denotes the sign of the permutation of  $(1, \dots, n)$  given by

$$(i_1, \dots, i_q, 1, 2, \dots, \hat{i}_1, \dots, \hat{i}_2, \dots, \hat{i}_2, \dots, \hat{i}_q, \dots, n).$$

Here a “hat” above a symbol means the deletion of this symbol.

The operators  $R^q$  satisfy

$$(2.9) \quad R^q \cdot R^{n-q} = (-1)^{q(n-q)} Id.$$

With the help of the operators  $R^q$  for an oriented Riemannian manifold of dimension  $n$ , one defines the fiberwise scalar product  $\ll , \gg : \Omega^q(M) \times \Omega^q(M) \rightarrow \Omega^0(M)$  and the formal adjoints of  $d^q, d^q(t), \iota_X^q, L_X^q$ , by the formulas

$$(2.10) \quad \ll \omega_1, \omega_2 \gg = (R^n)^{-1}(\omega_1 \wedge R^q(\omega_2)),$$

$$(2.11) \quad \begin{aligned} \delta^{q+1} &= (-1)^{nq+1} R^{n-q} \cdot d^{n-q-1} \cdot R^{q+1} : \Omega^{q+1}(M) \rightarrow \Omega^q(M), \\ \delta^{q+1}(t) &= (-1)^{nq+1} R^{n-q} \cdot d^{n-q-1}(t) \cdot R^{q+1} : \Omega^{q+1}(M) \rightarrow \Omega^q(M), \\ (\iota_X^q)^\sharp &= (-1)^{nq-1} R^{n-q} \cdot \iota_X^{n-q-1} \cdot R^{q-1} : \Omega^{q-1}(M) \rightarrow \Omega^q(M), \\ (L_X^q)^\sharp &= (-1)^{(n+1)q+1} R^{n-q} \cdot L_X^{n-q} \cdot R^q : \Omega^q(M) \rightarrow \Omega^q(M) \end{aligned}$$

These operators satisfy

$$(2.12) \quad \ll \iota_X \omega_1, \omega_2 \gg = \ll \omega_1, (\iota_X)^\sharp \omega_2 \gg$$

and

$$(2.13) \quad (L_X)^\sharp = (\iota_X)^\sharp \cdot \delta + \delta \cdot (\iota_X)^\sharp.$$

Note that  $L_X^q + (L_X^q)^\sharp$  is a zeroth order differential operator. Let  $X^\sharp$  denote the element in  $\Omega^1(M)$  defined by  $X^\sharp(Y) := \ll X, Y \gg$  and for  $\alpha \in \Omega^1(M)$  let  $(E_\alpha)^q : \Omega^q(M) \rightarrow \Omega^{q+1}(M)$ , denote the exterior product by  $\alpha$ . Then we have

$$(2.14) \quad (\iota_X^q)^\sharp = (E_{X^\sharp})^{q-1}.$$

It is easy to see that the scalar products  $\ll, \gg$  and hence the operators  $\delta^q, \delta^q(t), \iota_X^\sharp$  and  $L_X^\sharp$  are independent of the orientation of  $M$ . Therefore they are defined (first locally and then, being differential operators, globally) for an arbitrary Riemannian manifold, not necessarily orientable, and satisfy (2.10), (2.12)-(2.14) above.

For a Riemannian manifold  $(M, g)$  one introduces the scalar product  $\Omega^q(M) \times \Omega^q(M) \rightarrow \mathbb{C}$  by

$$(2.15) \quad \langle \omega, \omega' \rangle := \int_M \omega \wedge \omega' = \int_M \ll \omega, \omega' \gg \, dvol(g).$$

In view of (2.12),  $\delta^{q+1}(t), (\iota_X^q)^\sharp$  and  $(L_X^q)^\sharp$  are formal adjoints of  $d^q(t), \iota_X^q(t)$  and  $L_X^q$  with respect to the scalar product  $\langle \cdot, \cdot \rangle$ .

For a Riemannian manifold  $(M, g)$ , one introduces the second order differential operators  $\Delta_q : \Omega^q(M) \rightarrow \Omega^q(M)$ , the Laplace Beltrami operator, and  $\Delta_q(t) : \Omega^q(M) \rightarrow \Omega^q(M)$ , the Witten Laplacian (for the 1-form  $\alpha$ ) by

$$\Delta_q := \delta^{q+1} \cdot d^q + d^{q-1} \cdot \delta^q,$$

and

$$\Delta_q(t) := \delta^{q+1}(t) \cdot d^q(t) + d^{q-1}(t) \cdot \delta^q(t).$$

Note that  $\Delta_q(0) = \Delta_q$ . In view of (2.3) -(2.10) and (2.12) one verifies

$$(2.16) \quad \Delta_q(t) = \Delta_q + t(L_{-grad_g \alpha} + L_{-grad_g \alpha}^\sharp) + t^2 \|\alpha\|^2 Id$$

where  $grad_g \alpha$  is the unique vector field defined by  $(grad_g \alpha)^\sharp = \alpha$ . One verifies that  $L_{-grad_g \alpha} + L_{-grad_g \alpha}^\sharp$  is a zeroth order differential operator.

The operators  $\Delta_q(t)$  are elliptic, essentially selfadjoint, and positive, hence their spectra,  $\text{spect} \Delta_q(t)$ , are contained in  $[0, \infty)$ . Further

$$\ker \Delta_q(t) = \{\omega \in \Omega^q(M) | d^q(t) = 0, \delta^q(t) = 0\}$$

If  $\alpha = dh$  and if 0 is an eigenvalue of  $\Delta_q(0)$ , then 0 is an eigenvalue of  $\Delta_q(t)$  for all  $t$  and with the same multiplicity: this because  $d^q(t) = e^{-th} \cdot d^q \cdot e^{th}$  and  $\delta^q(t) = e^{-th} \cdot \delta^q \cdot e^{th}$ .

### c) Spectral gap theorems.

If  $\alpha$  is a closed 1-form we write  $Cr(\alpha)$  for the set of zeros of  $\alpha$ . This notation is justified because in a neighborhood of any connected component of  $Cr(\alpha)$ ,  $\alpha = dh$  for some smooth function  $h$ , unique up to an additive constant.

The pair  $(\alpha, g)$  is called a Morse pair, resp. Morse-Bott pair, resp. G-Morse pair, resp. normal G-Morse pair if  $(h, g)$  satisfies C1, resp. C'1, resp. G-C1, resp. normal G-C1.

In this subsection we will study the spectrum of  $\Delta_q(t)$  for the Witten deformation associated with  $(\alpha, g)$  being a Morse pair, resp. a Morse Bott pair or a G-Morse pair.

In the case of a Morse-Bott or a G-Morse pair we consider an additional complex, namely the complex of the critical sets

$$(C^*, d^*) := \bigoplus_{\Sigma} (\Omega^{*-i(\Sigma)}(\Sigma, o(\nu_-)), d^{*-i(\Sigma)}).$$

Note that  $(C^*, d^*) \neq (C^*, \partial^*)$ . In the case of a Morse pair this complex is trivial as it is concentrated in degree zero. The metric  $g$  induces a Riemannian metric on  $Cr(\alpha) = \cup \Sigma$ , hence the complex  $(C^*, d^*)$  gives rise to Laplacians

$$\Delta'_q := \bigoplus_{\Sigma} \Delta_{q-i(\Sigma)}(\Sigma).$$

In the case of a Morse-Bott pair  $\Delta_q(\Sigma)$  denotes the Laplacian on  $q$ -forms with coefficients the orientation bundle in  $o(\nu_-)$ , of  $\nu_-$ . In the case of a G-Morse pair,  $\Delta_q(\Sigma)$  denotes the Laplacian acting on  $\Omega^q(G/H_{\Sigma}, \det(\rho_-))$  where  $G/H_{\Sigma}$  is equipped with the Riemannian metric induced by a scalar product on  $\mathfrak{g}$  which invariant with respect to the adjoint action of  $H_{\Sigma}$ . Here  $H_{\Sigma}$  is the isotropy group of a point  $x \in \Sigma$ , which is independent of  $x$  up to conjugacy. If the  $G$ -triangulation is normal then  $\rho_-$  is trivial and  $\Omega^q(G/H_{\Sigma}, \det(\rho_-)) = \Omega^q(G/H_{\Sigma})$ .

Note that  $\Delta(G/H_{\Sigma}) = \hat{\oplus} \Delta_q^{\xi}(G/H_{\Sigma})$  and  $\Delta_q^{\xi}(G/H_{\Sigma})$  is an endomorphism in the finite dimensional vector space  $\Omega_{\xi}^q(G/H_{\Sigma}, o(\rho_-))$ , which can be explicitly calculated by elementary representation theory.

The following result is essentially due to E. Witten.

**Theorem 2.1.** *Suppose that  $(\alpha, g)$  is a Morse pair. Then there exist constants  $C_1, C_2, C_3$  and  $T_0$  depending on  $(\alpha, g)$  so that for any  $t > T_0$*

$$1): \text{spect} \Delta_q(t) \cap (C_1 e^{-C_2 t}, C_3 t) = \emptyset$$

and

2) the number of eigenvalues of  $\Delta_q(t)$  in the interval  $[0, C_1 e^{-C_2 t}]$  counted with their multiplicity is equal to the number of zeros of  $\alpha$  of index  $q$ .

The above theorem states the existence of a gap in the spectrum of  $\Delta_q(t)$ , namely the open interval  $(C_1 e^{-C_2 t}, C_3 t)$ , which widens to  $(0, \infty)$  when  $t \rightarrow \infty$ .

Clearly  $C_1, C_2, C_3$  and  $T_0$  determine a constant  $T > T_0$ , so that for  $t \geq T$ ,  $1 \in (C_1 e^{-C_2 t}, C_3 t)$  and therefore

$$\text{spect} \Delta_q(t) \cap [0, C_1 e^{-C_2 t}] = \text{spect} \Delta_q(t) \cap [0, 1]$$

and

$$\text{spect} \Delta_q(t) \cap [C_3 t, \infty) = \text{spect} \Delta_q(t) \cap [1, \infty).$$

For  $t > T$  we denote by  $\Omega^q(M)_{sm}(t)$  the finite dimensional subspace of  $\Omega^q(M)$  generated by the  $q$ -eigenforms of  $\Delta_q(t)$  corresponding to the eigenvalues of  $\Delta_q(t)$  smaller than 1. Note that  $\Omega^q(M)_{sm}(t)$  is of dimension  $m_q$  where  $m_q$  is the number of critical points of index  $q$  of the closed 1-form  $\alpha$ .

The theory of elliptic operators implies that these eigenforms which are a priori elements in the  $L_2$ -completion of  $\Omega^q(M)$ , are actually smooth, i.e. in  $\Omega^q(M)$ . Note that  $d(t)(\Omega^q(M)_{sm}(t)) \subset \Omega^{q+1}(M)_{sm}(t)$ , so that  $(\Omega^*(M)_{sm}(t), d^*(t))$  is a finite dimensional cochain subcomplex of  $(\Omega^*(M), d^*(t))$  and  $(e^{th}(\Omega^*(M)_{sm}(t)), d^*)$  is a finite dimensional subcomplex of  $(\Omega^*(M), d^*)$ . Clearly the  $L_2$ -orthogonal complement of  $\Omega^*(M)_{sm}(t)$  in  $\Omega^*(M)$  is also a closed Frechet subcomplex  $(\Omega^*(M)_{la}(t), d^*(t))$  of  $(\Omega^*(M), d^*(t))$  and we have the following decomposition

$$(2.17) \quad (\Omega^*(M), d^*(t)) = (\Omega^*(M)_{sm}(t), d^*(t)) \oplus (\Omega^*(M)_{la}(t), d^*(t))$$

with  $(\Omega^*(M)_{la}(t), d^*(t))$  acyclic.

Let us consider now the case of a Morse-Bott pair. Denote by  $\lambda'_{q,1} \leq \lambda'_{q,2} \leq \dots \leq \lambda'_{q,r} \leq \dots$  be the spectrum of  $\Delta'_q$ , and by  $\lambda'_q$  the first nonzero eigenvalue of  $\Delta'_q$ . The following result is due to Helffer [H] (cf also [P]).

**Theorem 2.1'.** *Suppose that  $(\alpha, g)$  is a Morse-Bott pair and  $r \geq 1$ , an integer. Then there exist positive constants  $C_1, C_2, T_0$ , so that for any  $t > 0$  and  $t \geq T$  and  $1 \leq q \leq n$*

$$|\lambda_{q,r}(t) - \lambda'_{q,r}| < C_1 t^{-C_2} \text{ where } \lambda_{q,r}(t) \text{ is the } r\text{-th eigenvalue of } \Delta_q(t).$$

In particular one can find  $T \geq T_0$  so that for  $\lambda' = \inf\{\lambda'_q\}$  and  $t > T$

- 1)  $\text{Spect} \Delta_q(t) \cap (C_1 t^{-C_2}, -C_1 t^{-C_2} + \lambda') = \emptyset$
- 2)  $\lambda'/2 \in (C_1 t^{-C_2}, -C_1 t^{-C_2} + \lambda')$ .

Therefore one can again produce a decomposition of the form

$$(2.17') \quad (\Omega^*(M), d^*(t)) = (\Omega^*(M)_{sm}(t), d^*(t)) \oplus (\Omega^*(M)_{la}(t), d^*(t))$$

where  $(\Omega^*(M)_{sm}(t), d^*(t))$  is a finite dimensional cochain subcomplex of  $(\Omega^*, d^*(t))$  with  $(\Omega^q(M)_{sm}(t))$  given by the span of the eigenforms corresponding to eigenvalues

of  $\Delta_q(t)$  smaller than  $\lambda'$  and  $(\Omega^*(M)(t)_{la})$  its orthogonal complement in  $\Omega^*(M)$ . Clearly  $(\Omega^*(M)_{la}(t), d^*(t))$  is acyclic.

In [H] one finds  $C_2 \geq 5/2$  and in [P]  $C_2 > 1/2$ . Under additional hypothesis better estimate that the one stated in Theorem 2.1' can be obtained. As an example we mention the case of a  $G$ -Morse pair considered below.

Suppose  $\mu : G \times M \rightarrow M$  is a smooth  $G$ -manifold,  $G$  being a compact Lie group,  $(\alpha, g)$  a  $G$ -Morse pair and  $\xi$  an irreducible representation of  $G$ . Since  $G$  is compact and the Riemannian metric  $g$  is  $G$ -invariant  $\Delta'$  and  $\Delta_q(t)$  decompose orthogonally as  $\Delta'_q = \sum \Delta'^{\xi}_q$  and  $\Delta_q(t) = \sum \Delta_q^{\xi}(t)$

Let  $\lambda_{q,1}^{\xi}, \dots, \lambda_{q,N}^{\xi}(t)$  be the eigenvalues of  $(\Delta'^{\xi}_q)$  and  $\lambda_{q,1}^{\xi}(t) \leq \dots$  the ones of the Witten Laplacian  $\Delta_q(t)^{\xi}$ .

**Theorem 2.1''.** *There exist the positive constants  $C_1, C_2, C_3$  and  $T_0$  depending on  $M, \alpha, g$  and  $\xi$  so that for any  $t > T_0$*

- 1) *Spect* $\Delta_q(t)^{\xi} \subset \bigcup_{i=1, \dots, N} (-C_1 e^{-tC_2} + \lambda_{q,i}^{\xi}, C_1 e^{-tC_2} + \lambda_{q,i}^{\xi}) \cup [C_3 t, \infty)$  and
- 2) *the number of the eigenvalues of  $\Delta_q(t)^{\xi}$  in the interval  $(-C_1 e^{-tC_2} + \lambda_{q,i}^{\xi}, C_1 e^{-tC_2} + \lambda_{q,i}^{\xi})$  equals the multiplicity of  $\lambda_{q,i}^{\xi}$*

In particular one can canonically decompose  $(\Omega^*(M)_{\xi}, d^*(t)_{\xi})$  in an orthogonal sum

$$(2.17'') \quad (\Omega_{\xi}^*(M), d^*(t)_{\xi}) = ((\Omega_{\xi}^*(M))_{sm}(t), d^*(t)_{\xi}) \oplus ((\Omega_{\xi}^*(M))_{la}(t), d^*(t)_{\xi})$$

where  $((\Omega_{\xi}^q(M)(t))_{sm}(t))$  is the span of the eigenforms corresponding to the eigenvalues  $\lambda_{q,i}^{\xi}$  and  $((\Omega_{\xi}^*(M))_{la}(t))$  is the orthogonal complement. Note that  $((\Omega_{\xi}^*(M))_{la}(t), d^*(t))$  is acyclic.

The proof of Theorem 2.1 will be given in subsection e) (cf [BZ1] and [BFKM]). The proof of Theorem 2.1' and 2.1'' can be found in literature in [H] and [BFK5].

#### d) Applications.

*Morse inequalities:*

Let  $\alpha$  be a closed 1-form and denote by  $[\alpha]$  its cohomology class, which can be interpreted as a 1-dimensional representation of the fundamental group of  $M$ . Let  $\beta_i(M, [\alpha]) := \dim H^i(M; [\alpha])$ . It is not hard to show that the integer valued function  $\beta(M, [t\alpha])$  is constant in  $t$  for  $t$  large enough so that  $\hat{\beta}_i(M, [\alpha]) := \lim_{t \rightarrow \infty} \beta_i(M, [t\alpha])$  is well defined.

As an immediate consequence of the decompositions discussed in section c) we have the following result:

**Theorem 2.2.** *Suppose that  $\alpha$  is a Morse one form and let  $C_i := \sharp(Cr_i(\alpha))$ . Then for any integer  $N$ , we have*

$$(-1)^N \sum_{i=0}^N (-1)^i C_i \geq (-1)^N \sum_{i=0}^N (-1)^i \beta_i(M, [t\alpha])$$

for  $t \geq T_0$ . If  $\alpha$  is a Morse Bott one form and  $Cr(\alpha)$  is the union of the closed connected submanifolds  $\Sigma$ , then the above formula holds with  $C_i$  given by  $C_i = \sum_{\Sigma} \dim H^{i-i(\Sigma)}(\Sigma, o(\nu_-))$ .

Clearly the inequalities remain true with  $\beta_i(M, [t\alpha])$  replaced by  $\hat{\beta}_i(M, [\alpha])$ . The above result is known as the Morse inequalities when  $\alpha = dh$  and as the Novikov-Morse inequalities when  $\alpha$  is a closed 1-form. It has a number of pleasant consequences in symplectic topology which will be discussed below.

*Symplectic vector fields:*

Let  $(M^{2n}, \omega)$  be a symplectic manifold. The nondegenerated 2- form  $\omega$  establishes a bijective correspondence  $X \rightarrow \alpha_X$  between the set of smooth vector fields  $X$  on  $M$  and the smooth 1-forms  $\alpha$  with  $\alpha_X$  defined by the formula  $\alpha_X(Y) = \omega(X, Y)$  for any vector field  $Y$ .

Given a vector field  $X$  consider  $\text{Zeros}(X) := \{x \in M | X(x) = 0\}$ . We say that  $x^0 \in \text{Zeros}(X)$  is nondegenerated if in one (and then any) coordinate system  $\{x_1, \dots, x_{2n}\}$  around  $x^0 \in M$  the vector field  $X = \sum_{i=1, \dots, 2n} a_i(x_1, \dots, x_{2n}) \partial / \partial x_i$  satisfies  $\det(\partial a_i / \partial x_j(x_1^0, \dots, x_{2n}^0)) \neq 0$  where  $(x_1^0, \dots, x_{2n}^0)$  are the coordinates of  $x^0$ . In this case we set

$$(2.18) \quad I(x^0) := \text{sign} \det(\partial a_i / \partial x_j(x_1^0, \dots, x_{2n}^0)).$$

The famous Hopf Theorem states:

**Theorem.** *If  $M$  is closed and all zeros of  $X$  are nondegenerated, then*

$$\sum_{x \in \text{Zeros}(X)} I(x) = \chi(M)$$

where  $\chi(M)$  denotes the Euler Poincaré characteristic of  $M$ . In particular

$$\sharp(\text{Zeros}(X)) \geq |\chi(M)|.$$

This is the best general result about such vector fields. The estimate is sharp.

The vector field  $X$  is called symplectic if the 1-parameter (local) group of diffeomorphisms induced by  $X$  preserves the form  $\omega$ , equivalently if  $L_X(\omega) = 0$  or,

equivalently if  $d(\alpha_X) = 0$ . Clearly,  $\text{Zeros}(X) = Cr(\alpha_X)$  and if all zeros of  $X$  are nondegenerated,  $\alpha_X$  is a Morse form and one can easily verify that  $I(x) = (-1)^{i(x)}$  where  $i(x)$  denotes the index of the critical point  $x$  of the form  $\alpha_X$ . Theorem 2.2 provides a considerable improvement of the Hopf Theorem, in particular it says that for a symplectic vector field with all zeroes nondegenerated

$$(2.19) \quad \sharp(\text{Zeros}(X)) \geq \sum_i \hat{\beta}_i(M, [\alpha]).$$

In fact, as shown in [BH2], it is possible to prove the existence of zeros of a symplectic vector field in certain cases by "torsion methods" even when  $\hat{\beta}_i(M, [\alpha]) = 0$  for any  $i$ .

One can also apply the above theory to the study of some Lagrangian intersections. A precise situation is the case of the symplectic manifold  $T^*M$ , the cotangent bundle of a smooth manifold  $M$ , equipped with the canonical symplectic structure.

We are interested in the intersection of the zero section, the canonical Lagrangian in  $T^*M$ , with the image of a Lagrangian immersion  $i : \Sigma \rightarrow T^*M$ . In case that  $\Sigma$  has a generating function  $h : E \rightarrow R$ , where  $E$  is the total space of a smooth vector bundle on  $M$ , and all critical points of  $h$  are nondegenerated (cf [MS] for the definition of a generating function), the count of the intersection points of  $i(\Sigma)$  and  $M$  reduces to the count of zeroes of the closed form  $i^*(\sigma)$  on  $\Sigma$ . As all zeros are nondegenerated and Theorem 2.2 applies. Here  $\sigma$  is the canonical 1-form on  $T^*M$  whose differential  $d(\sigma)$  defines the canonical symplectic structure on  $T^*M$ .

#### e) Sketch of the proof of Theorems 2.1.

The proof of Theorems 2.1 stated in the next section c) is based on a mini-max criterion for detecting a gap in the spectrum of a positive selfadjoint operator in a Hilbert space  $H$  (cf Lemma 2.3 below) and uses the explicit formula for  $\Delta_q(t)$  in admissible coordinates in a neighborhood of the set of critical points.

**Lemma 2.3.** *Let  $A : H \rightarrow H$  be a densely defined (not necessary bounded) self adjoint positive operator in a Hilbert space  $(H, \langle, \rangle)$  and  $a, b$  two real numbers so that  $0 < a < b < \infty$ . Suppose that there exist two closed subspaces  $H_1$  and  $H_2$  of  $H$  with  $H_1 \cap H_2 = 0$  and  $H_1 + H_2 = H$  such that*

$$(1) \langle Ax_1, x_1 \rangle \leq a \|x_1\|^2 \text{ for any } x_1 \in H_1,$$

$$(2) \langle Ax_2, x_2 \rangle \geq b \|x_2\|^2 \text{ for any } x_2 \in H_2.$$

*Then  $\text{spect}A \cap (a, b) = \emptyset$ .*

The proof of this Lemma is elementary (cf [BFK3] Lemma 1.2) and might be a good exercise for the reader.

Consider  $x \in Cr(\alpha)$  and choose admissible coordinates  $(x_1, x_2, \dots, x_n)$  in a neigh-

borhood of  $x$ . With respect to these coordinates  $\alpha = dh$ ,

$$h(x_1, x_2, \dots, x_n) = -1/2(x_1^2 + \dots + x_k^2) + 1/2(x_{k+1}^2 + \dots + x_n^2)$$

and  $g_{ij}(x_1, x_2, \dots, x_n) = \delta_{ij}$ , and hence by (2.16) the operator  $\Delta_q(t)$  has the form

$$(2.20) \quad \Delta_{q,k}(t) = \Delta_q + tM_{q,k} + t^2(x_1^2 + \dots + x_n^2)Id$$

with

$$\Delta_q(\sum_I a_I(x_1, x_2, \dots, x_n)dx_I) = \sum_I (-\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} a_I(x_1, x_2, \dots, x_n))dx_I,$$

and  $M_{q,k}$  is the linear operator determined by

$$(2.21) \quad M_{q,k}(\sum_I a_I(x_1, x_2, \dots, x_n)dx_I) = \sum_I \epsilon_I^{q,k} a_I(x_1, x_2, \dots, x_n)dx_I.$$

Here  $I = (i_1, i_2, \dots, i_q)$ ,  $1 \leq i_1 < i_2 < \dots < i_q \leq n$ ,  $dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_q}$  and

$$\epsilon_I^{q,k} = -n + 2k - 2q + 4\sharp\{j | k+1 \leq i_j \leq n\},$$

where  $\sharp A$  denotes the cardinality of the set  $A$ . Note that  $\epsilon_I^{q,k} \geq -n$  and is  $= -n$  iff  $q = k$ .

Let  $\mathcal{S}^q(\mathbb{R}^n)$  denote the space of smooth  $q$ -forms  $\omega = \sum_I a_I(x_1, x_2, \dots, x_n)dx_I$  with  $a_I(x_1, x_2, \dots, x_n)$  rapidly decaying functions. The operator  $\Delta_{q,k}(t)$  acting on  $\mathcal{S}^q(\mathbb{R}^n)$  is globally elliptic (in the sense of [Sh1] or [Hö]), selfadjoint and positive. This operator is the harmonic oscillator in  $n$  variables acting on  $q$ -forms and its properties can be derived from the harmonic oscillator in one variable  $-\frac{d^2}{dx^2} + a + bx^2$  acting on functions. In particular the following result holds.

**Proposition 2.4.** (1)  $\Delta_{q,k}(t)$ , regarded as an unbounded densely defined operator on the  $L_2$ -completion of  $\mathcal{S}^q(\mathbb{R}^n)$ , is selfadjoint, positive and its spectrum is contained in  $2t\mathbb{Z}_{\geq 0}$  (i.e positive integer multiples of  $2t$ ).

(2)  $\ker \Delta_{q,k}(t) = 0$  if  $k \neq q$  and  $\dim \ker \Delta_{q,q}(t) = 1$ .

(3)  $\omega_{q,t} = (t/\pi)^{n/4} e^{-t\sum_i x_i^2/2} dx_1 \wedge \dots \wedge dx_q$  is the generator of  $\ker \Delta_{q,q}(t)$  with  $L_2$ -norm 1.

For a proof consult [BFKM] page 805.

Choose a smooth function  $\gamma_\eta(u)$ ,  $\eta \in (0, \infty)$ ,  $u \in \mathbb{R}$ , which satisfies

$$(2.23) \quad \gamma_\eta(u) = \begin{cases} 1 & \text{if } u \leq \eta/2 \\ 0 & \text{if } u > \eta \end{cases}.$$

Introduce  $\tilde{\omega}_{q,t}^\eta \in \Omega_c^q(\mathbb{R}^n)$  defined by

$$\tilde{\omega}_{q,t}^\eta(x) = \beta_q(t)^{-1} \gamma_\eta(|x|) \omega_{q,t}(x)$$

with  $|x| = \sqrt{\sum_i x_i^2}$  and

$$(2.24) \quad \beta_q(t) = (t/\pi)^{n/4} \left( \int_{\mathbb{R}^n} \gamma_\eta^2(|x|) e^{-t \sum_i x_i^2} dx_1 \cdots dx_n \right)^{1/2}.$$

The smooth form  $\tilde{\omega}_{q,t}^\eta$  has its support in the ball  $\{|x| \leq \eta\}$ , agrees with  $\omega_{q,t}$  on the ball  $\{|x| \leq \eta/2\}$  and satisfies

$$(2.25) \quad \langle \tilde{\omega}_q^\eta(t), \tilde{\omega}_q^\eta(t) \rangle = 1$$

with respect to the scalar product  $\langle \cdot, \cdot \rangle$  on  $\mathcal{S}^q(\mathbb{R}^n)$ , induced by the Euclidean metric. The following proposition can be obtained by elementary calculations in coordinates in view of the explicit formula of  $\Delta_{q,k}(t)$  (cf [BFKM], Appendix 2).

**Proposition 2.5.** *For a fixed  $r \in \mathbb{N}_{\geq 0}$  there exist positive constants  $C, C', C'', T_0$ , and  $\epsilon_0$  so that  $t > T_0$  and  $\epsilon < \epsilon_0$  imply*

(1)  $\left| \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \Delta_{q,q}(t) \tilde{\omega}_{q,t}^\epsilon(x) \right| \leq C e^{-C't}$  for any  $x \in \mathbb{R}^n$  and multiindex  $\alpha = (\alpha_1, \dots, \alpha_n)$ , with  $|\alpha| = \alpha_1 + \dots + \alpha_n \leq r$ .

(2)  $\langle \Delta_{q,k}(t) \tilde{\omega}_{q,t}^\epsilon, \tilde{\omega}_{q,t}^\epsilon \rangle \geq 2t|q - k|$

(3) If  $\omega \perp \tilde{\omega}_{q,t}^\epsilon$  with respect to the scalar product  $\langle \cdot, \cdot \rangle$  then

$$\langle \Delta_{q,q} \omega, \omega \rangle \geq C'' t \|\omega\|^2.$$

For the proof of Theorem 2.1 (and of Theorem 3.1 in the next lecture) we set the following notations. We choose  $\epsilon > 0$  so that for each  $y \in Cr(h)$  there exists an admissible coordinate chart  $\varphi_y : (U_y, y) \rightarrow (D_{2\epsilon}, 0)$  so that  $U_y \cap U_z = \emptyset$  for  $y \neq z$ ,  $y, z \in Cr(h)$ .

Choose once and for all such an admissible coordinate chart for each  $y \in Cr_q(h)$ . Introduce the smooth forms  $\bar{\omega}_{y,t} \in \Omega^q(M)$  defined by

$$(2.26) \quad \bar{\omega}_{y,t}|_{M \setminus \varphi_y^{-1}(D_{2\epsilon})} := 0, \quad \bar{\omega}_{y,t}|_{\varphi_y^{-1}(D_{2\epsilon})} := \varphi_y^*(\tilde{\omega}_{q,t}^\epsilon).$$

For any given  $t > 0$  the forms  $\bar{\omega}_{y,t} \in \Omega^q(M)$ ,  $y \in Cr_q(h)$ , are orthonormal. Indeed, if  $y, z \in Cr_q(h)$ ,  $y \neq z$   $\bar{\omega}_{y,t}$  and  $\bar{\omega}_{z,t}$  have disjoint support, hence are orthogonal, and because the support of  $\bar{\omega}_{y,t}$  is contained in an admissible chart,  $\langle \bar{\omega}_{y,t}, \bar{\omega}_{y,t} \rangle = 1$  by (2.25).

For  $t > T_0$ , with  $T_0$  given by Proposition 2.5, we introduce  $J^q(t) : C^q(X, \tau) \rightarrow \Omega^q(M)$  to be the linear map defined by

$$J^q(t)(E_y) = \bar{\omega}_{y,t},$$

where  $E_y \in C^q(X, \tau)$  is given by  $E_y(z) = \delta_{yz}$  for  $y, z \in Cr(h)_q$ .  $J_q(t)$  is an isometry, thus in particular injective.

**Proof of Theorems 2.1:** (sketch). Take  $H$  to be the  $L_2$ -completion of  $\Omega^q(M)$  with respect to the scalar product  $\langle \cdot, \cdot \rangle$ ,  $H_1 := J^q(t)(C^q(M, \tau))$  and  $H_2 = H_1^\perp$ . Let  $T_0, C, C', C''$  be given by Proposition 2.5 and define

$$C_1 := \inf_{z \in M'} \|grad_g \alpha(z)\|,$$

with  $M' = M \setminus \bigcup_{y \in Cr_q(\alpha)} \varphi_y^{-1}(D_\epsilon)$ , and

$$C_2 = \sup_{x \in M} \|(L_{-grad_g \alpha} + L_{-grad_g \alpha}^\sharp)(z)\|.$$

Here  $\|grad_g \alpha(z)\|$  resp.  $\|(L_{-grad_g \alpha} + L_{-grad_g \alpha}^\sharp)(z)\|$  denotes the norm of the vector  $grad_g \alpha(z) \in T_z(M)$  resp. of the linear map  $(L_{-grad_g \alpha} + L_{-grad_g \alpha}^\sharp)(z) : \Lambda^q(T_z(M)) \rightarrow \Lambda^q(T_z(M))$  with respect to the scalar product induced in  $T_z(M)$  and  $\Lambda^q(T_z(M))$  by  $g(z)$ . Recall that if  $X$  is a vector field then  $L_X + L_X^\sharp$  is a zeroth order differential operator, hence an endomorphism of the bundle  $\Lambda^q(T^*M) \rightarrow M$ .

We can use the constants  $T_0, C, C', C'', C_1, C_2$  to construct  $C'''$  and  $\epsilon_1$  so that for  $t > T_0$  and  $\epsilon < \epsilon_1$ , we have  $\langle \Delta_q(t)\omega, \omega \rangle \geq C_3 t \langle \omega, \omega \rangle$  for any  $\omega \in H_2$  (cf. [BFKM], page 808-810).

Now one can apply Lemma 2.3 whose hypotheses are satisfied for  $a = Ce^{-C't}$ ,  $b = C'''t$  and  $t > T_0$ . This concludes the first part of Theorem 2.1.

Let  $Q_q(t)$ ,  $t > T_0$  denote the orthogonal projection in  $H$  on the span of the eigenvectors corresponding the eigenvalues smaller than 1. In view of the ellipticity of  $\Delta_q(t)$  all these eigenvectors are smooth  $q$ -forms. An additional important estimate is given by the following Proposition:

**Proposition 2.6.** *For  $r \in \mathbb{N}_{\geq 0}$  one can find  $\epsilon_0 > 0$  and  $C_3, C_4$  so that for  $t > T_0$  as constructed above, and any  $\epsilon < \epsilon_0$  one has, for any  $v \in C^q(M, \tau)$*

$$(Q_q(t)J^q(t) - J^q(t))(v) \in \Omega^q(M),$$

and for  $0 \leq p \leq r$ ,

$$\|(Q_q(t)J^q(t) - J^q(t))(v)\|_{C^p} \leq C_3 e^{-C_4 t} \|v\|,$$

where  $\|\cdot\|_{C^p}$  denotes the  $C^p$ -norm.

The proof of this Proposition is contained in [BZ1], page 128 and [BFKM] page 811. Its proof requires (2.16), Proposition 2.5 and general estimates coming from the ellipticity of  $\Delta_q(t)$ .

Proposition 2.6 implies that for  $t$  large enough, say  $t > t_0$ ,  $\mathcal{I}^q(t) := Q_q(t)J^q(t)$  is bijective, which finishes the proof of Theorem 2.1.

### Lecture 3: Helffer Sjöstrand Theorem, an asymptotic improvement of the Hodge-de Rham theorem

In this section we will formulate the result of Helffer Sjöstrand for a generalized triangulation and its analogue for a  $G$ -generalized triangulation.

#### a. Hodge-de Rham theorem and its (asymptotic) generalization.

First let us recall the classical Hodge-de Rham theorem.

Consider a closed Riemannian manifold  $(M, g)$  and a simplicial smooth triangulation  $\tau$ . The Riemannian metric  $g$  provides a scalar product in the Frechet space of smooth forms and then the Laplace Beltrami operators  $\Delta_i : \Omega^i(M) \rightarrow \Omega^i(M)$ . Let us denote by  $\mathcal{H}^i := \ker \Delta_i$ . The simplicial triangulation  $\tau$  provides the (finite dimensional) cohomology vector spaces  $H_\tau^*(M)$ .

#### Theorem. (Hodge-de Rham )

Given a Riemannian manifold  $(M, g)$  equipped with a smooth triangulation  $\tau$  one can produce:

1) a canonical orthogonal decomposition  $\Omega^*(M) = \mathcal{H}^* \oplus \Omega_2^*(M)$ , with  $\mathcal{H}^* = \text{Ker} \Delta_*$  a finite dimensional graded vector space and  $\Omega_2^*(M) = d(\Omega^{*-1}(M)) \oplus d^\sharp(\Omega^{*+1}(M))$

2) a canonical linear isomorphism (whose inverse is induced from integration of forms on simplexes)  $J : H_\tau^*(M, \mathbb{R}) \rightarrow \mathcal{H}^*$ .

The above theorem provides a canonical realization of the cohomology, calculated with the help of the smooth triangulation  $\tau$ , as differential forms, harmonic with respect to the given Riemannian metric  $g$ .

One can improve the above result by realizing the full geometric complex defined by the triangulation as a subcomplex of differential forms but the "canonicity" statement remains true only asymptotically. More precisely one can show:

**Theorem.** Given a Riemannian manifold  $(M, g)$  and a smooth triangulation  $\tau$  for  $t \in \mathbb{R}$  large enough one can produce

1) a smooth one-parameter family of orthogonal decompositions

$$(\Omega^*(M), d) = (\Omega^*(M)_0(t), d) \oplus (\Omega^*(M)_1(t), d)$$

with  $(\Omega^*(M)_0(t), d)$  a finite dimensional complex, which is  $O(1/t)$  canonical<sup>2</sup>, and

2) a smooth family of isomorphisms  $\mathcal{I}^*(t) : C^*(M, \tau) \rightarrow \Omega(M)_0^*(t)$  so that the composition  $\mathcal{I}^*(t) \cdot S^*(t)$ , where  $S^*(t)$  is the scaling isomorphism

$$(3.1) \quad S^*(t) : (C^*(M, \tau), \delta_{\tau, o}^*) \rightarrow (C^*(M, \tau), \delta_{\tau, o}^*(t))$$

defined by  $S^q(t)(E_x) = (\frac{\pi}{t})^{(n-2q)/4} e^{-th(x)} E_x, x \in Cr_q(h)$ , is of the form  $I^* + O(1/t)$  with  $I^*$  an isometry. Moreover  $\mathcal{I}^*(t)$  is  $O(1/t)$  canonical<sup>3</sup>.

In view of a result of Pozniak which claims that any smooth triangulation can be realized as a generalized triangulation (cf section 1 a), O.1) the above theorem is a straightforward reformulation of Theorem 3.1 below, proven by Helffer and Sjöstrand.

### b. Helffer Sjöstrand theorem (Theorem 3.1).

We consider only the case of a generalized triangulation and of a G- generalized triangulation. We pick up orientations  $o$  as indicated in section 1 and consider the scaling (3.1)  $S^q(t) : (C^*(M, \tau), \delta_{\tau, o}^*) \rightarrow (C^*(M, \tau), \delta_{\tau, o}^*(t))$  and, for  $t$  large enough, the compositions  $L(t)$  and  $L(t)_\xi$  defined by the following diagram

$$\begin{array}{ccccc} (\Omega^*(M), d^*(t)) & \xrightarrow{e^{th}} & (\Omega^*(M), d^*) & \xrightarrow{Int^*} & (C^*(M, \tau), \delta_{\tau, o}^*) \\ \uparrow in & & & & \downarrow S^*(t) \\ (\Omega_{sm}^*(M)(t), d^*(t)) & & & & (C^*(M, \tau), \delta_{\tau, o}^*(t)) \end{array}$$

and

$$\begin{array}{ccccc} (\Omega^*(M)_\xi, (d^*(t))_\xi) & \xrightarrow{e^{th}} & (\Omega^*(M)_\xi, d_\xi^*) & \xrightarrow{Int^*} & (C^*(M, \tau)_\xi, \delta_\xi^*) \\ \uparrow in & & & & \downarrow S^*(t) \\ ((\Omega_{sm}^*(M)(t))_\xi, d_\xi^*(t)_\xi) & & & & (C^*(M, \tau)_\xi, \delta_\xi^*(t)) \end{array}$$

The following theorems are reformulations of a theorem due to Helffer- Sjöstrand, cf [HS2].

<sup>2</sup>i.e. for any two such possible decompositions the finite dimensional subspaces  $\Omega^*(M)_0(t)$  are at an  $O(1/t)$  distance with respect to the scalar product induced by the metric  $g$ .

<sup>3</sup>i.e. for any two such possible  $\mathcal{I}(t)$ 's,  $\mathcal{I}_1(t)$  and  $\mathcal{I}_2(t)$ ,  $\|\mathcal{I}_1(t) - \mathcal{I}_2(t)\| = O(1/t)$

**Theorem 3.1.** (*Helfffer-Sjöstrand*). *Given  $M$  a closed manifold and  $\tau = (g, h)$  a generalized triangulation, there exists  $T > 0$ , depending on  $\tau$ , so that for  $t > T$ ,  $L^*(t)$  is an isomorphism of cochain complexes. Moreover, there exists a family of isometries  $R^q(t) : C^q(M, \tau) \rightarrow \Omega^q(M)_{sm}(t)$  of finite dimensional vector spaces so that  $L^q(t) \cdot R^q(t) = Id + O(1/t)$ .*

and

**Theorem 3.1''.** *Given a closed  $G$ -manifold  $M$ , with  $G$  being a compact Lie group and  $\tau = (g, h)$  a  $G$ -generalized triangulation and an irreducible representation  $\xi$ , there exists  $T > 0$ , depending on  $\tau$  and  $\xi$ , so that for  $t > T$ ,  $L^*(t)_\xi$  is an isomorphism of cochain complexes. Moreover, there exists a family of isometries  $R^q(t)_\xi : C^q(M, \tau)_\xi \rightarrow (\Omega^q(M)_{sm}(t))_\xi$  of finite dimensional vector spaces so that  $L^q(t)_\xi \cdot R^q(t)_\xi = Id + O(1/t)$ .*

It is understood that  $C^q(M, \tau)$  is equipped with the canonical scalar product defined in section 1, and  $\Omega^q(M)_{sm}(t)$  with the scalar product  $\langle \cdot, \cdot \rangle$  defined by (2.15).

One can also prove an analogue of Theorem 3.1 for MB-generalized triangulations. This requires a finite dimensional cell complex instead of  $(C^*, \partial^*)$ . Such complex is described in [BH] where an analogue of Theorem 3.1 for a Morse-Bott form is established.

**Sketch for the proof of Theorems 3.1:** The proof is a continuation of the proof of Theorem 2.1 in the previous lecture and we use the same notation. Therefore we invite the reader to review the section e) of Lecture 2 .

Let  $T_0$  be provided by Proposition 2.6. For  $t \geq T_0$ , let  $R^q(t)$  be the isometry defined by

$$(3.2) \quad R^q(t) := J^q(t)(J^q(t)^\sharp J^q(t))^{-1/2}$$

and introduce  $U_{y,t} := R^q(t)(E_y) \in \Omega^q(M)$  for any  $y \in Cr_q(h) = Cr_q(dh)$ . Proposition 2.6 implies that there exists  $\epsilon > 0$ ,  $t_0$  and  $C$  so that for any  $t > t_0$  and any  $y \in Cr(h)_q$  one has

$$(3.3) \quad \sup_{z \in M \setminus \varphi_y^{-1}(D_\epsilon)} \|U_{y,t}(z)\| \leq C e^{-\epsilon t}$$

and

$$(3.4) \quad \|U_{y,t}(z) - \bar{w}_{y,t}(z)\| \leq C \frac{1}{t}, \text{ for any } z \in W_y^- \cap \varphi_y^{-1}(D_\epsilon).$$

To check Theorem 3.1 it suffices to show that

$$\left| \int_{W_{x'}^-} U_{x,t} e^{th} - \left(\frac{t}{\pi}\right)^{\frac{n-2q}{4}} e^{-th(x)} \delta_{xx'} \right| \leq C'' \frac{1}{t}$$

for some  $C'' > 0$  and any  $x, x' \in Cr(h)_q$ .

If  $x \neq x'$  this follows from (3.3). If  $x = x'$  from (3.3) and (3.4).  $\square$

## 5. Extensions and a survey of other applications.

1. One can relax the definition of admissible coordinates in C1, and C1" by dropping the requirements on the metric. Theorems 2.1, 2.1" and 3.1 and 3.1" remain true as stated; however almost all calculations will be longer since the explicit formulae for  $\Delta_q(t)$  and its spectrum when regarded on  $\mathcal{S}^*(\mathbb{R}^n)$  will be more complicated.

2. One can provide an analogue of Theorems 2.1 and 3.1 in the case of a closed one form. This will be elaborated in a forthcoming paper [BH2].

3. One can twist both complexes  $(\Omega^*, d^*)$  and  $(C^*(M, \tau), \partial^*)$  resp  $(C^*(M, \tau), D^*)$  by a finite dimensional representation of the fundamental group,  $\rho : \pi_1(M) \rightarrow GL(V)$ . In this case additional data is necessary: a Hermitian structure  $\mu$  on the flat bundle  $\xi_\rho$  induced by  $\rho$ . The "canonical" scalar product on  $(C^*(M, \tau, \rho), \partial^*)$ , in the case of a generalized triangulation will be obtained by using the critical points (the cells of the generalized triangulation) and the Hermitian scalar product provided by  $\mu$  in the fibers of  $\xi_\rho$  above the critical points. The de-Rham complex in this case is replaced by  $(\Omega^*(M, \rho), d_\rho^*)$  the de-Rham complex of differential forms with coefficients in  $\xi_\rho$  whose differential is given by the covariant differentiation w.r. to the canonical flat connection in  $\xi_\rho$ . To have a scalar product on the spaces of smooth forms, in addition to the Riemannian metric on  $M$  one needs a Hermitian structure  $\mu$  (cf. [BFK1] or [BFK4]) in  $\xi_\rho$ . The statements of Theorems 2.1 and 3.1 remain the same. Under the hypotheses that the Hermitian structure is parallel in small neighborhoods of the critical points, the proofs remain the same. An easy continuity argument permits to reduce the case of an arbitrary Hermitian structure to the previous one, by taking  $C^0$  approximation of a given Hermitian structure by Hermitian structures which are parallel near the critical points. Since the Witten Laplacians do not involve derivatives of the Hermitian structure such a reduction is possible. If the representation is a unitary representation on a finite dimensional Euclidean space one has a canonical Hermitian structure in  $\xi_\rho$  which is parallel with respect to the flat canonical connection in  $\xi_\rho$ . This extension was used in the new proofs of the Cheeger- Muller theorem and its extension about the comparison of the analytic and the Reidemeister torsion, cf. [BZ], [BFK1], [BFKM], [BFK4].

4. One can further extend the WHS-theory to the case where  $\rho$  is a special type of an infinite dimensional representation, a representation of the fundamental group in an  $\mathcal{A}$ - Hilbert module of finite type. This extension was done in [BFKM] for  $\rho$  unitary and in [BFK4] for  $\rho$  arbitrary. In this case the Laplacian  $\Delta_q(t)$  do not have

discrete spectrum and it seems quite remarkable that Theorems 2.1 and 3.1 remain true. It is even more surprising that exactly the same arguments as presented above can be adapted to prove them. A particularly interesting situation is the case of the left regular representation of a countable group  $\Gamma$  on the Hilbert space  $L_2(\Gamma)$  when regarded as an  $\mathcal{N}(\Gamma)$  right Hilbert module of the von Neumann algebra  $\mathcal{N}(\Gamma)$ , cf.[BFKM] for definitions. One can prove that Farber extended  $L_2$ -cohomology of  $M$ , a compact smooth manifold with infinite fundamental group defined analytically (i.e. using differential forms and a Riemannian metric) and combinatorially (i.e. using a triangulation) are isomorphic and therefore the classical  $L_2$ -Betti numbers and Novikov-Shubin invariants defined analytically and combinatorially are the same. For this last fact see [BFKM] section 5.3.

The WHS-theory was a fundamental tool in the proof of the equality of the  $L_2$ -analytic and the  $L_2$ -Reidemeister torsion presented [BFKM].

5. One can further extend Theorems 1.1, 1.1', 2.2, 2.2', 3.3, 3.3', to bordisms  $(M, \partial_- M, \partial_+ M)$ , and  $\rho$  a representation of  $\Gamma = \pi_1(M)$  on an  $\mathcal{A}$ -Hilbert module of finite type. In this case one has first to extend the concept of generalized triangulation to such bordisms. This will involve a pair  $(h, g)$  which in addition to the requirements C1-C3 is supposed to satisfy the following assumptions:  $g$  is product like near  $\partial M = \partial_- M \cup \partial_+ M$ , the function  $h : M \rightarrow [a, b]$  satisfies  $h^{-1}(a) = \partial_- M$ ,  $h^{-1}(b) = \partial_+ M$ ,  $a, b$  regular values, and is linear on the geodesics normal to  $\partial M$  near  $\partial M$ . In case  $h$  is replaced by a closed 1-form the requirement is that this form vanishes on  $\partial M$ . This extension was partly done in [BFK2] and was used to prove gluing formulae for analytic torsion and to extend the results of [BFKM] to manifolds with boundary.

5. One can actually extend the WHS-theory to the case where  $h$  is a generalized Morse function, i.e. the critical points are either nondegenerated or birth-death. This extension is much more subtle and very important. Beginning work in this direction was done by Hon Kit Wai in his OSU dissertation.

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