

# Minimal models of canonical 3-fold singularities and their Betti numbers

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## Abstract

Let  $(X, x)$  be a germ of an isolated canonical 3-fold singularity. Fix a representative  $X$  of the germ which is Stein and contractible. Let  $\varphi' : Y' \rightarrow X$  be a crepant projective morphism from  $Y'$  with terminal  $\mathbb{Q}$ -factorial singularities, and  $\psi : Y \rightarrow Y'$  an analytic  $\mathbb{Q}$ -factorialisation of  $Y'$  at its singular points. The aim of this paper is to calculate the Betti numbers of  $Y$  or, equivalently, the intersection cohomology Betti numbers of  $Y'$ .

*Key words and phrases:* canonical singularities, minimal models, Betti numbers,  $\mathbb{Q}$ -factoriality.

## 1 Introduction

Let  $X$  be an algebraic 3-fold with canonical singularities. In [R2] Reid proved that there exists a crepant projective morphism  $\varphi' : Y' \rightarrow X$  from a 3-fold  $Y'$  with  $\mathbb{Q}$ -factorial terminal singularities (a minimal model of  $X$  in the sense of Mori theory). Recall that a singularity  $p \in Y'$  is algebraically  $\mathbb{Q}$ -factorial if, for any Weil divisor  $D$ , there exists an integer  $m$  such that  $mD$  can be defined by one equation near  $p$ ; in other words, the local algebraic divisor class group  $\text{Cl } \mathcal{O}_{Y', p}$  is torsion.

When passing from the surface case to 3-folds, various new phenomena occur; one of these is that a  $\mathbb{Q}$ -factorial terminal minimal model  $Y'$  of  $X$  is not unique in general. Different minimal models of a 3-fold  $X$  are closely related, and their study involves finding properties which are independent of choices. By a result of Kollár [K1], [K2], the intersection cohomology groups  $IH^k(Y', \mathbb{C})$  are among the invariants of  $X$ .

An important feature of the cohomology groups  $IH^k(Y')$  is that, by the Decomposition Theorem of Beilinson, Bernstein, Deligne and Gabber [BBD],

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<sup>0</sup>*Mathematics Subject Classification 2000:* 14J17, 14B05, 14J30, 14E30

they are direct summands in  $H^k(\tilde{Y})$ , for any resolution of singularities  $\tilde{Y}$  of  $X$ . The usual cohomology groups  $H^k(Y')$  do not have this birational invariance property.

The minimal model  $Y'$  is a  $\mathbb{Q}$ -homology manifold if and only if it is locally analytically  $\mathbb{Q}$ -factorial if and only if, for each  $p \in Y'$ , the local algebraic divisor class group  $\text{Cl } \mathcal{O}_{Y',p}$  and the local analytic divisor class group  $\text{Cl } \mathcal{O}_{Y',p}^{\text{an}}$  have the same rank. This is not the case in general, but we can slightly modify  $Y'$  to make it  $\mathbb{Q}$ -homology. This is done by taking an analytic  $\mathbb{Q}$ -factorialisation  $Y$  of  $Y'$ , which exists by [R2], [Ka], at its singular points. This is a small morphism and, therefore, it leaves the intersection cohomology groups unchanged (see [K1, Corollary 4.12] and [Ki, p. 112]). In this paper, we will occasionally refer to  $Y$  as an analytic minimal model of  $X$ . The 3-fold  $Y$  is a  $\mathbb{Q}$ -homology manifold, hence its intersection cohomology (with rational coefficients) coincides with the usual cohomology.

The topology of  $X$  and of its partial resolutions have not been studied in detail since the 1980s [R1], [R2], except in some special cases. In this paper we aim to determine the cohomology groups  $H^k(Y, \mathbb{Q})$  of an analytic minimal model of  $X$ , or, equivalently, the intersection cohomology groups of a minimal model  $Y'$  of  $X$ , in the case of an isolated singularity  $x \in X$ . The main class of singularities we are interested in is the class of hypersurface singularities  $0 \in X : (f = 0) \subset \mathbb{C}^4$ .

Let  $(X, x)$  be a germ of an isolated canonical 3-fold singularity. Fix a representative  $X$  of the germ  $(X, x)$  which is Stein and contractible, and let  $\varphi: Y \rightarrow X$  be a crepant partial resolution from  $Y$  with terminal analytically  $\mathbb{Q}$ -factorial singularities. Denote by  $b_i(Y) = \dim H^i(Y, \mathbb{Q})$  the Betti numbers of  $Y$ . Under certain strong assumptions, we show in Section 3 that  $b_3(Y) = \sum_{i=1}^{c(X)} b_3(E_i)$ , where  $E_i$  for  $1 \leq i \leq c(X)$  are any projective nonsingular representatives of the crepant valuations of  $X$  (see Proposition 3.3 and the subsequent remarks). This is the “ideal case”, on which we hope to model the general situation. Although the assumptions are too strong, there are examples satisfying them (see Section 6.1).

Section 4 contains a collection of lemmas, leading to a simplification of the Mayer–Vietoris sequence (1) in the previous section. These results are also used in Section 5, which calculates the 2nd and 4th Betti numbers of an analytic minimal model in terms of the number of crepant valuations  $c(X)$  of  $X$  and the rank  $\rho(X)$  of the analytic divisor class group  $\text{Cl } X$  of  $x \in X$ . More precisely, we show that  $b_2(Y) = \rho(X) + c(X)$  and  $b_4(Y) = c(X)$  (Theorem 5.2). Together with the formulas from [Ca1] and [Ca2] about  $c(X)$  and  $\rho(X)$ , we thus obtain explicit answers about  $b_i(Y)$ ,  $i \neq 3$ , for a

large class of isolated canonical singularities  $0 \in X : (f = 0) \subset \mathbb{C}^4$ .

The rest of this paper is concerned with calculating the remaining interesting Betti number of  $Y$ , namely  $b_3$ . In Section 6 we work out some examples involving nondegenerate hypersurface singularities. The singularities considered in Section 6.1 are quasihomogeneous. In this case, there is no subtlety related to the difference between algebraic and analytic divisor classes. The situation is quite different in Section 6.2, where the singularities are not quasihomogeneous. One difference between the two cases comes from the fact that analytic  $\mathbb{Q}$ -factoriality is not a local condition. Example 6.3 gives an instance of a globally analytically factorial 3-fold, which is not locally analytically factorial. In Section 7 we prove that

$$b_3(Y) = \sum_{i=1}^{c(X)} b_3(E_i),$$

where  $E_i$  are any nonsingular projective representatives of the crepant valuations of  $X$ . This is the conclusion of Theorem 7.2. Since each  $E_i$  is rational or ruled [R1, Corollary 2.14], this result implies that  $b_3(Y) = 2 \sum g(C_i)$ , where the summation is taken over the surfaces  $E_i$  which are ruled over  $C_i$ . In particular, for a nondegenerate hypersurface singularity  $0 \in X : (f = 0) \subset \mathbb{C}^4$ , we obtain a formula for  $b_3(Y)$  in terms of the Newton diagram  $\Gamma(f)$  of the defining equation. For a weighting  $\alpha$ , denote by  $\alpha(\mathbf{1})$  the sum of the weights, and by  $\alpha(f)$  the degree of the  $\alpha$ -tangent cone of  $X$ . Denote also by  $W(f)$  the set of weightings satisfying the equation  $\alpha(\mathbf{1}) = \alpha(f) + 1$ . Then we prove that

$$b_3(Y) = 2 \sum_{\substack{\alpha \in W(f) \\ \dim \Gamma_\alpha = 2}} \#(\Gamma_\alpha^\circ \cap N),$$

where  $\Gamma_\alpha$  is the face of  $\Gamma(f)$  corresponding to  $\alpha$  (Corollary 7.4).

## Acknowledgements

It is a great pleasure to thank my research advisor, Miles Reid, for suggesting this problem and for his constant support and guidance.

I am also grateful to Alessio Corti and Mark Gross for their help and suggestions during various stages of this project.

## 2 Preliminaries

This section contains the basic terminology and notation used throughout the paper.

A normal variety  $Z$  has *canonical singularities* if the canonical divisor  $K_Z$  is  $\mathbb{Q}$ -Cartier, and if, for any resolution  $\varphi: W \rightarrow Z$ , with exceptional divisors  $E_i$ , the rational numbers  $a_i$  defined by

$$K_W = \varphi^*(K_Z) + \sum a_i E_i$$

are nonnegative. If all the *discrepancies*  $a_i$  are positive,  $Z$  is said to have *terminal singularities*. The number of crepant valuations  $c(Z) := \#\{i : a_i = 0\}$ , measuring how far is  $Z$  from having terminal singularities, is finite and independent of  $W$ . A partial resolution  $\varphi: W \rightarrow Z$  of  $Z$  is *crepant* if  $K_W = \varphi^*(K_Z)$ .

Let  $Z$  be a normal algebraic variety. Denote by  $\text{WDiv } Z$  the group of Weil divisors, and by  $\text{CDiv } Z \subset \text{WDiv } Z$  the subgroup of Cartier divisors on  $Z$ . If  $Z$  has canonical singularities, then the Abelian group  $\text{WDiv } Z / \text{CDiv } Z$  is finitely generated [Ka]; denote by  $\sigma(Z)$  its rank. The same holds if we replace  $Z$  by a pair  $(Z, \Sigma)$  consisting of an analytic space and its compact subset; denote in this case

$$\sigma(Z) = \sigma(Z, \Sigma) = \text{rank } \varinjlim \text{WDiv } U / \varinjlim \text{CDiv } U,$$

where the limit is taken over all open neighbourhoods  $U$  of  $\Sigma$ . In particular, for a germ  $(Z, p)$  with  $p \in Z$  a point, we denote by  $\rho(Z)$  or  $\rho(p)$  the number  $\sigma(Z, p)$ . A normal algebraic (or analytic) 3-fold  $Z$  is (*globally analytically*)  $\mathbb{Q}$ -factorial if  $\sigma(Z) = 0$ . This condition is local in the Zariski topology, but not in the analytic topology. A variety is *locally analytically  $\mathbb{Q}$ -factorial* if the analytic germ  $(Z, p)$  is  $\mathbb{Q}$ -factorial, for any point  $p \in Z$ .

The aim of this paper is to compare the cohomology of 3-fold  $Z$  with isolated canonical singularities with the cohomology of its crepant partial resolutions. We illustrate the computations involved in several examples, which are hypersurface singularities, assumed to be nondegenerate with respect to their Newton polyhedra. Recall now the relevant toric terminology.

Let  $M$  be the free Abelian group  $\mathbb{Z}^n$ ,  $N = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$  its dual,  $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$  and  $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$  the vector spaces obtained by extending scalars to  $\mathbb{R}$ . We identify  $m \in M$  with the monomial  $x^m = \prod_{i=1}^n x_i^{m_i}$ . A weighting  $\alpha = (\alpha_1, \dots, \alpha_n) \in N$  is *primitive* if the  $\alpha_i$  have no common factor. Denote by  $\sigma \subset N_{\mathbb{R}}$  the positive quadrant, and by  $\sigma^{\vee}$  the dual quadrant. It is convenient to write  $m \in f$  if  $m$  appears in the polynomial  $f = \sum_{m \in M} a_m x^m$  with  $a_m \neq 0$ . For each  $\alpha \in \sigma \cap N$ , define then  $\alpha(m) = \sum_{i=1}^n \alpha_i m_i$  and  $\alpha(f) = \min_{m \in f} \alpha(m)$ .

Let  $0 \in X : (f = 0) \subset \mathbb{C}^n$  be an isolated hypersurface singularity. The *Newton polyhedron* of  $f$ ,  $\Gamma_+(f)$ , is the convex hull in  $M_{\mathbb{R}}$  of the set

$\bigcup_{m \in f} (m + \sigma^\vee)$ ; the union of all its compact facets,  $\Gamma(f)$ , is called the *Newton diagram* of  $f$ . For any face  $\Gamma \prec \Gamma(f)$  denote  $f_\Gamma = \sum_{m \in \Gamma \cap M} a_m x^m$ . The polynomial  $f$  is *nondegenerate* with respect to its Newton polyhedron if, for each face  $\Gamma$  of  $\Gamma(f)$ , the hypersurface defined by  $f_\Gamma = 0$  is nonsingular on  $(\mathbb{C}^*)^n$ .

### 3 A “model case”

Let  $\psi: Z' \rightarrow Z$  be a proper birational morphism between algebraic varieties or analytic spaces, such that for  $C \subset Z$  and  $E = \psi^{-1}(C)$  we have  $Z' \setminus E \cong Z \setminus C$ . There is then a long exact sequence in reduced cohomology

$$\begin{aligned} \cdots \rightarrow H^k(Z, \mathbb{Z}) \rightarrow H^k(Z', \mathbb{Z}) \oplus H^k(C, \mathbb{Z}) \rightarrow H^k(E, \mathbb{Z}) \rightarrow \\ H^{k+1}(Z, \mathbb{Z}) \rightarrow \cdots \end{aligned} \quad (1)$$

The following result from [K1] relates topological and geometric properties of a 3-fold with canonical singularities.

**Proposition 3.1** *Let  $Z$  be a 3-fold with at most isolated canonical singularities. Then  $Z$  is a rational homology manifold if and only if its singular points are analytically  $\mathbb{Q}$ -factorial.*

As a consequence of the previous result and of the fact that (1) is a sequence of mixed Hodge structures if  $Z$  and  $Z'$  are algebraic varieties ([D, 8.3.9 and 8.3.10]), we obtain the following:

**Proposition 3.2** *Let  $\psi: Z' \rightarrow Z$  be as above. Suppose that  $Z$  and  $Z'$  are algebraic 3-folds and  $Z$  has at most isolated canonical analytically  $\mathbb{Q}$ -factorial singularities. Then the exact sequence (1) breaks into short exact sequences*

$$0 \rightarrow H^k(Z, \mathbb{Q}) \rightarrow H^k(Z', \mathbb{Q}) \oplus H^k(C, \mathbb{Q}) \rightarrow H^k(E, \mathbb{Q}) \rightarrow 0.$$

**Proof** Indeed,  $Z$  is a  $\mathbb{Q}$ -homology manifold by Proposition 3.1 and therefore  $W_{k-1}H^k(Z, \mathbb{Q}) = 0$  ([D, Théorème 8.2.4]). Since  $E$  is projective, we also have  $W_k H^k(E, \mathbb{Q}) = H^k(E, \mathbb{Q})$  again by [D, Théorème 8.2.4]. This shows that the morphisms

$$H^k(E, \mathbb{Q}) \rightarrow H^{k+1}(Z, \mathbb{Q}),$$

which are morphisms of mixed Hodge structures, are all zero, hence the conclusion.  $\square$

Let  $Z$  be an algebraic 3-fold with canonical singularities. Suppose that we have a chain of crepant projective morphisms

$$W = Z_c \xrightarrow{\varphi_c} Z_{c-1} \longrightarrow \cdots \longrightarrow Z_1 \xrightarrow{\varphi_1} Z$$

from  $W$  with terminal analytically  $\mathbb{Q}$ -factorial singularities, where  $c = c(Z)$  is the number of crepant valuations of  $Z$ , and  $\varphi_i$  are divisorial contractions, contracting the divisor  $E_i$  corresponding to the crepant valuation  $v_i$ .

The next result is a “model theorem” for the 3rd Betti number of  $W$ ; under strong hypotheses, we obtain a formula expressing  $b_3(W)$  in terms of the 3rd Betti numbers of arbitrary nonsingular representatives of the crepant valuations. Although these assumptions are too strong, there are examples satisfying them (see Section 6.1).

**Proposition 3.3** *With the above assumptions and notation, suppose that  $Z$  and its partial resolutions  $Z_i$  have isolated locally analytically  $\mathbb{Q}$ -factorial singularities, and the  $\varphi_i$ -exceptional divisors  $E_i$  are normal for  $i = 1, \dots, c$ . Then*

$$b_3(W) = b_3(Z) + \sum_{i=1}^c b_3(\tilde{E}_i), \quad (2)$$

where  $\tilde{E}_i$  is any nonsingular representative of the valuation  $v_i$ .

**Proof** Apply Proposition 3.2 to each morphism  $\varphi_i$  for  $1 \leq i \leq c$ .  $\square$

**Remark 3.4** The above argument shows that  $b_3(W) = b_3(Z) + \sum_{i=1}^c b_3(E_i)$ . The assumption about the normality of  $E_i$  is only made to emphasise the fact that we are looking for a similar birational invariant statement in the general case (see Theorem 7.2).

**Remark 3.5** Let  $Z$  and  $W$  be as above, and assume that  $\text{Sing } Z = \{x\}$ . Let  $E$  be the exceptional locus of  $\varphi$ , and  $\varphi: (Y, E) \rightarrow (X, x)$  a minimal model of  $X$ . Fix a representative  $X$  of the germ  $(X, x)$  which is Stein and contractible. Then the conclusion of Proposition 3.3 is equivalent to

$$b_3(Y) = \sum_{i=1}^c b_3(\tilde{E}_i). \quad (3)$$

Indeed, part of the Mayer–Vietoris sequence of the cover  $W = (W \setminus E) \cup Y$  is

$$H^2(Y \setminus E) \longrightarrow H^3(W) \longrightarrow H^3(W \setminus E) \oplus H^3(Y) \longrightarrow H^3(Y \setminus E),$$

where all the cohomology groups have rational coefficients. The singularity  $x \in X$  is  $\mathbb{Q}$ -factorial, and so  $H^2(Y \setminus E) = H^3(Y \setminus E) = 0$ . Thus  $b_3(W) = b_3(W \setminus E) + b_3(Y)$ . A similar argument shows that  $b_3(Z) = b_3(Z \setminus x)$ , and this proves the claim.

## 4 Some lemmas

**Lemma 4.1** *Let  $(Z, E) \rightarrow (X, x)$  be a birational morphism. Take a common resolution of  $Z$  and  $X$ ,  $\varphi: Z' \rightarrow Z$ ,  $\psi: Z' \rightarrow X$  and suppose that  $R^1\varphi_*\mathcal{O}_{Z'} = R^1\psi_*\mathcal{O}_{Z'} = 0$ . Then  $H^1(Z, \mathbb{Z}) = 0$ .*

**Proof** The exponential exact sequence  $0 \rightarrow \mathbb{Z}_{Z'} \rightarrow \mathcal{O}_{Z'} \rightarrow \mathcal{O}_{Z'}^* \rightarrow 0$  gives

$$0 \rightarrow R^1\varphi_*\mathbb{Z}_{Z'} \rightarrow R^1\varphi_*\mathcal{O}_{Z'} \rightarrow \cdots$$

and this yields  $R^1\varphi_*\mathbb{Z}_{Z'} = 0$ . From the Leray spectral sequence of the morphism  $\varphi$  we obtain

$$0 \rightarrow H^1(Z, \mathbb{Z}) \rightarrow H^1(Z', \mathbb{Z}) \rightarrow H^0(Z, R^1\varphi_*\mathbb{Z}_{Z'}) = 0,$$

hence the isomorphism  $H^1(Z, \mathbb{Z}) \cong H^1(Z', \mathbb{Z})$ .

The above argument applied to the morphism  $\psi$  shows that

$$H^1(Z', \mathbb{Z}) \cong H^1(X, \mathbb{Z}),$$

and, since  $H^1(X, \mathbb{Z}) = 0$ , this proves the lemma.  $\square$

**Remark 4.2** In [SB] it is shown that if  $(X, x)$  is an isolated canonical 3-fold singularity and  $Y \rightarrow X$  a crepant morphism from  $Y$  with terminal singularities, then  $Y$  is, in fact, simply connected.

From now on fix a 3-dimensional canonical singularity  $(X, x)$ ; we take  $X$  to be a representative of the germ which is Stein and topologically contractible.

**Lemma 4.3** *Let  $Z$  be a partial crepant resolutions of  $X$ . Then  $\text{Pic } Z \cong H^2(Z, \mathbb{Z})$ .*

**Proof** Take a common resolution  $Z'$  of  $X$  and  $Z$ ,  $\varphi: Z' \rightarrow Z$ ,  $\psi: Z' \rightarrow X$ . Since  $Z$  has canonical singularities, it follows from the Leray spectral sequence associated to  $\varphi$  that, for each  $i \geq 0$ ,

$$H^i(Z', \mathcal{O}_{Z'}) = H^i(Z, \varphi_* \mathcal{O}_{Z'}) = H^i(Z, \mathcal{O}_Z).$$

The same applies to  $\psi$ , giving, for each  $i > 0$ ,

$$H^i(Z', \mathcal{O}_{Z'}) = H^i(X, \psi_* \mathcal{O}_{Z'}) = H^i(X, \mathcal{O}_X) = 0.$$

The conclusion now follows from the exponential sequence

$$H^1(Z, \mathcal{O}_Z) \rightarrow \text{Pic } Z \rightarrow H^2(Z, \mathbb{Z}) \rightarrow H^2(Z, \mathcal{O}_Z).$$

□

**Remark 4.4** If  $X$  is algebraic, then the same proof shows that

$$\text{Pic}^{\text{alg}} Z \cong H^2(Z, \mathbb{Z}),$$

where  $\text{Pic}^{\text{alg}} Z$  is the algebraic Picard group of  $Z$ .

**Lemma 4.5** *Let  $\varphi: Z' \rightarrow Z$  be a morphism between partial crepant resolutions of  $X$ . Then the natural maps*

$$\begin{aligned} \varphi^2: H^2(Z, \mathbb{Z}) &\rightarrow H^2(Z', \mathbb{Z}) \\ \varphi^4: H^4(Z, \mathbb{Z}) &\rightarrow H^4(Z', \mathbb{Z}) \end{aligned}$$

*are injective.*

**Proof** By Lemma 4.3, the first injection is equivalent to  $\varphi^*: \text{Pic } Z \hookrightarrow \text{Pic } Z'$ . But this is clearly injective since

$$\varphi_*(\varphi^* \mathcal{L}) = \mathcal{L} \quad \text{for any } \mathcal{L} \in \text{Pic } Z.$$

The second injection is also clear since, for any partial resolution  $Z$  of  $X$ , with exceptional set  $E$ , we have

$$H^4(Z, \mathbb{Z}) \cong H^4(E, \mathbb{Z}) \cong \bigoplus H^4(E_i, \mathbb{Z}),$$

where the  $E_i$  are the prime divisor components of  $E$ .

□



Let  $\psi: Z' \rightarrow Z$  be a morphism such that  $Z' \setminus E \cong Z \setminus C$  as in Section 3. Suppose that  $Z$  and  $Z'$  are partial crepant resolutions of  $X$ . Then, by Lemma 4.5, the exact sequence (1) becomes

$$\begin{aligned} 0 \rightarrow H^2(Z, \mathbb{Z}) \rightarrow H^2(Z', \mathbb{Z}) \oplus H^2(C, \mathbb{Z}) \rightarrow H^2(E, \mathbb{Z}) \\ \rightarrow H^3(Z, \mathbb{Z}) \rightarrow H^3(Z', \mathbb{Z}) \rightarrow H^3(E, \mathbb{Z}) \rightarrow 0. \end{aligned}$$

In particular, we have

$$b_3(Z') = b_3(Z) + b_3(E) + b_2(Z') - b_2(Z) + b_2(C) - b_2(E). \quad (4)$$

In some examples, this information is enough to calculate the 3rd Betti number of a minimal model (see Section 6).

## 5 The 2nd and 4th Betti numbers of a minimal model

There are several differences between the algebraic and analytic category related to divisor class groups. For algebraic varieties, there is a surjection

$$\mathrm{Cl} Z \rightarrow \mathrm{Cl}(Z \setminus \Sigma) \rightarrow 0$$

given by restricting divisors on  $Z$  to  $Z \setminus \Sigma$ , for any proper closed subset  $\Sigma$  of  $Z$ . If  $(Z, \Sigma)$  is a pair consisting of an analytic space and its compact subset, this is no longer true in general.

The following result from [BS] gives an instance when this difference between the algebraic and the analytic case does not occur.

**Proposition 5.1** *Let  $\psi: Z' \rightarrow Z$  be a proper birational morphism between algebraic varieties (resp. analytic spaces), and  $C \subset Z$  an algebraic (resp. analytic) set of codimension at least 2 such that  $Z' \setminus \psi^{-1}(C) \cong Z \setminus C$ . Let  $K$  be the subgroup of  $\mathrm{WDiv} Z'$  with support contained in  $\psi^{-1}(C)$ . There is then a diagram with exact rows*

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & \mathrm{WDiv} Z' & \xrightarrow{\psi_*} & \mathrm{WDiv} Z \longrightarrow 0 \\ & & \cong \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K & \longrightarrow & \mathrm{Cl} Z' & \xrightarrow{\psi_*} & \mathrm{Cl} Z \longrightarrow 0 \end{array}$$

Moreover, if  $Z$  is algebraic or compact analytic, then  $K$  is a finitely generated free group.

In [Ca1] and [Ca2] we obtained formulas for the number of crepant divisors  $c(X)$  and the divisor class number  $\rho(X)$  for a large class of isolated canonical singularities  $0 \in X : (f = 0) \subset \mathbb{C}^4$ . Together with the following result, these formulas often calculate the 2nd and 4th Betti numbers of an (analytic) minimal model of  $X$ .

**Theorem 5.2** *Let  $x \in X$  be an isolated canonical 3-fold singularity and let  $\varphi: Y \rightarrow X$  be a crepant partial resolution of  $X$  from  $Y$  with terminal analytically  $\mathbb{Q}$ -factorial singularities. Then the Betti numbers of  $Y$  other than  $b_3(Y)$  are given by:*

- (1)  $b_0(Y) = 1$ ;
- (2)  $b_1(Y) = b_5(Y) = b_6(Y) = 0$ ;
- (3)  $b_4(Y) = c(X)$ ;
- (4)  $b_2(Y) = \rho(X) + c(X)$ .

**Proof** The exceptional set  $E = \varphi^{-1}(x)$  is connected, since  $X$  is irreducible. The retraction  $X \searrow x$  coming from the cone structure of  $X$  can be lifted via  $\varphi$  to a retraction  $Y \searrow E$ . This, together with Lemma 4.1, gives (1) and (2).

If  $E_i$  are the prime divisor components of  $E$  for  $1 \leq i \leq c(X)$ , then

$$H^4(Y, \mathbb{Z}) \cong H^4(E, \mathbb{Z}) \cong \bigoplus_{i=1}^{c(X)} H^4(E_i, \mathbb{Z}),$$

which shows (3).

Finally, to prove (4), look at the short exact sequence

$$0 \rightarrow K \rightarrow \text{Cl} Y \rightarrow \text{Cl} X \rightarrow 0,$$

of (analytic) divisor class groups from Proposition 5.1. The result follows once we show that the kernel  $K$  is equal to  $\bigoplus_{i=1}^{c(X)} \mathbb{Z}[E_i]$ , that is, we have an exact sequence

$$0 \rightarrow \bigoplus_{i=1}^{c(X)} \mathbb{Z}[E_i] \rightarrow \text{Cl} Y \rightarrow \text{Cl} X \rightarrow 0.$$

To see this, it suffices to prove that the components  $E_i$  are linearly independent in  $\text{Cl} Y$ . Since  $\text{Cl} Y$  is left unchanged under a small modification,

we can assume that  $X$  and  $Y$  are algebraic. Since  $x \in X$  is an isolated singularity, we can assume that  $X$  and  $Y$  are projective. Take then a general hyperplane section  $H$  on  $Y$ ; the curves  $C_i = E_i \cap H$  are exceptional on the nonsingular surface  $H$ , hence the matrix

$$(C_i \cdot C_j)$$

is negative definite. □

**Remark 5.3** The above argument about the components  $E_i$  being linearly independent in  $\text{Cl} Y$  also shows that, given any crepant partial resolution  $\psi: Z \rightarrow X$ , the prime divisor components  $E'_i$  of  $\psi^{-1}(x)$  are linearly independent in  $\text{Cl} Z$  as well.

To see this, notice that, by Proposition 5.1,  $\text{Cl} Z$  does not change under a  $\mathbb{Q}$ -factorialisation; thus we can assume that  $Z$  is  $\mathbb{Q}$ -factorial. Regard now an exceptional divisor  $D$  on  $Z$  as an element of  $\text{Pic} Z$ . Then, if  $\varphi: Y \rightarrow Z$  is a minimal model of  $Z$  and  $D'$  denotes the proper transform of  $D$  on  $Y$ , the equality

$$\varphi^*(D) = D' + \sum a_i E_i,$$

where the  $E_i$  are now  $\varphi$ -exceptional, reduces the problem to  $Y$ , where it is already solved.

## 6 Examples

All the examples we consider in this section are isolated canonical nondegenerate hypersurface singularities  $0 \in X : (f = 0) \subset \mathbb{C}^4$ . We start with  $X$  and then make explicit blowups leading to an analytic minimal model  $Y$ ; at each step  $Z' \rightarrow Z$ , we use formula (4) to relate the cohomologies of  $Z$  and  $Z'$ .

### 6.1 Normal exceptional divisors

The singularities  $(X, 0)$  in this section are quasihomogeneous; their various partial resolutions have only isolated quasihomogeneous singularities, as well. Therefore, in these cases, there is no subtlety related to the difference between algebraic and analytic divisor classes. For  $(X, 0)$  factorial, these examples are also instances where the strong assumptions of Proposition 3.3 are satisfied.

**Example 6.1** Let  $0 \in X : (f = 0) \subset \mathbb{C}^4$ , where  $f = x^2 + y^3 + z^6 + t^n$  with  $n \geq 6$ , and  $Y$  a minimal model of  $X$ . We want to show that

$$b_3(Y) = \begin{cases} 2([n/6] - 1) & \text{if } n \equiv 0 \pmod{6} \\ 2[n/6] & \text{otherwise.} \end{cases} \quad (5)$$

The number of crepant divisors and the Picard number of  $X$  are given by  $c(X) = [n/6]$  and

$$\rho(X) = \begin{cases} 0 & \text{if } n \equiv \pm 1 \pmod{6} \\ 2 & \text{if } n \equiv \pm 2 \pmod{6} \\ 4 & \text{if } n \equiv 3 \pmod{6} \\ 8 & \text{if } n \equiv 0 \pmod{6}. \end{cases}$$

Let  $\alpha = (3, 2, 1, 1)$  and  $X_1 = X(\alpha)$  the  $\alpha$ -blowup of  $X$ . This is a crepant morphism and, on one affine piece,

$$X_1 : x^2 + y^3 + z^6 + t^{n-6} = 0.$$

Blowing up again, we obtain a chain of crepant morphisms

$$X_c \rightarrow X_{c-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X,$$

where  $X_{i+1} = X_i(\alpha)$ , and  $X_c$  has at most terminal singularities,  $c = c(X)$ . Let  $E_i$  be the exceptional divisor of the  $i$ th morphism. Then

$$b_2(E_i) = \begin{cases} 9 & \text{if } i = c \text{ and } n \equiv 0 \pmod{6} \\ 1 & \text{otherwise,} \end{cases} \quad (6)$$

$$b_3(E_i) = \begin{cases} 0 & \text{if } i = c \text{ and } n \equiv 0 \pmod{6} \\ 2 & \text{otherwise.} \end{cases} \quad (7)$$

Apart from the case  $n \equiv 0 \pmod{6}$ , we have  $\sigma(X_c) = \rho(X)$ . Thus

$$b_2(X_c) = \text{rank Pic } X_c = c(X). \quad (8)$$

In the remaining case  $n \equiv 0 \pmod{6}$  the variety  $X_c$  is  $\mathbb{Q}$ -factorial (which is the same as locally  $\mathbb{Q}$ -factorial by the observation before this example), and

$$b_2(X_c) = \text{rank Pic } X_c = \text{rank Cl } X_c = c(X) + \rho(X). \quad (9)$$

From (4), (6), (7), (8) and (9) it follows that

$$b_3(X_c) = \sum_{i=1}^c b_3(E_i) = \begin{cases} 2(c-1) & \text{if } n \equiv 0 \pmod{6} \\ 2c & \text{otherwise.} \end{cases}$$

A minimal model of  $X$  is obtained from  $X_c$  by a  $\mathbb{Q}$ -factorialisation. For  $n \equiv 0, 1$  or  $5 \pmod{6}$   $X_c$  is already  $\mathbb{Q}$ -factorial, while in the remaining cases it is not. To prove the claim, we are left with showing that  $b_3$  does not change under a  $\mathbb{Q}$ -factorialisation. But this is clear since, if  $C$  denotes the exceptional curve,  $b_2(Y) = \rho(X) + c(X)$ ,  $b_2(X_c) = c(X)$ ,  $b_2(C) = \rho(X)$  and, again by (4),

$$b_3(Y) = b_3(X_c) + b_2(Y) - b_2(X_c) - b_2(C).$$

**Example 6.2 ([G])** Let  $0 \in X : (f = 0) \subset \mathbb{C}^4$  be the singularity defined by  $f = x^3 + y^4 + z^4 + t^4$ , and  $Y$  a minimal model of  $X$ . Then  $b_3(Y) = 12$ .

To see this, let  $X_1 = \text{Bl}_0 X \rightarrow X$  be the blowup of  $X$ . This is a crepant morphism and the blown up variety  $X_1$  is singular along the quartic curve of  $A_2$ -singularities

$$C : (y^4 + z^4 + t^4 = 0) \subset \mathbb{P}^2$$

of genus  $g(C) = 3$ ; the corresponding exceptional divisor  $E_1 : (x^3 = 0) \subset \mathbb{P}^3$  is a triple plane. Blowing up  $X_1$  along its singular curve  $C$  leads to a nonsingular variety  $Y$ . Again the morphism  $Y = \text{Bl}_C X_1 \rightarrow X_1$  is crepant, and its exceptional locus  $E$  is a union of two  $\mathbb{P}^1$ -bundles,  $E_2$  and  $E_3$ , over  $C$ . The singularity  $0 \in X$  is factorial thus, by Theorem 5.2,  $b_2(Y) = 3$ . Since  $b_2(X_1) = 1$ ,  $b_2(C) = 1$  and  $b_2(E) = 3$ , it follows by (4) that

$$b_3(Y) = b_3(X_1) + b_3(E) = 4g(C) = 12.$$

## 6.2 Nonnormal exceptional divisors

In the next example the singularity  $(X, 0)$  is factorial and its blowup  $Y'$  is globally factorial, but not locally factorial near its singular point  $p \in Y'$ . If  $Y$  is a  $\mathbb{Q}$ -factorialisation of  $Y'$  at  $p$ , then the global  $\mathbb{Q}$ -factoriality of  $Y'$  implies that  $b_2(Y) = b_2(Y')$ , while  $b_3$  changes under this morphism precisely by  $\rho(p)$ .

**Example 6.3** The canonical singularity  $0 \in X : (f = 0) \subset \mathbb{C}^4$  given by  $x^3 + x^2z + y^2z + z^4 + t^5$  is (analytically) factorial. Its blowup  $\varphi' : Y' \rightarrow X$  has one singular point  $p$  and the divisor class number of  $Y'$  at this point is

1. The exceptional divisor  $E = (\varphi')^{-1}(0)$  is a projective cone over the nodal curve  $(x^3 + x^2z + y^2z = 0) \subset \mathbb{P}^2$ ; thus  $b_2(Y') = b_3(Y') = 1$ . Let  $\psi : Y \rightarrow Y'$  be an analytic  $\mathbb{Q}$ -factorialisation of  $Y'$  at  $p$ . Since  $Y$  is an analytic minimal model of  $X$ , its Betti numbers are  $b_2(Y) = b_4(Y) = 1$ . If  $\tilde{\Sigma} = \psi^{-1}(p)$ , it follows by (4) that

$$b_3(Y) = b_3(Y') + b_2(Y) - b_2(Y') - b_2(\tilde{\Sigma}) = 0.$$

Note that  $b_3(Y)$  does not coincide to  $b_3(E)$ , but to  $b_3$  of the normalisation of  $E$ . The neighbourhood  $Y'$  of  $E$  gives an example of a variety which is globally analytically  $\mathbb{Q}$ -factorial, but is not locally analytically  $\mathbb{Q}$ -factorial.

The next example is based on an idea from [C]. We start with a singularity  $(X, 0)$  which is factorial iff  $\text{hcf}(n, 4) = 1$ . Its blowup  $Y'$  has 3 planes which are not  $\mathbb{Q}$ -Cartier. Under a  $\mathbb{Q}$ -factorialisation of  $Y'$  at  $\text{Sing } Y'$ , both the 2nd and the 3rd Betti numbers change,  $b_2$  by  $\sigma(Y')$ , and  $b_3$  by  $\sum_{p \in \text{Sing } Y'} \rho(p) - \sigma(Y')$ .

**Example 6.4** The polynomial  $f = xyz + x^n + y^n + z^n + t^4$ , for  $n \geq 4$ , defines an isolated canonical singularity  $0 \in X$ . Its local divisor class number and number of crepant divisors are

$$\rho(X) = 3(\text{hcf}(n, 4) - 1) \quad \text{and} \quad c(X) = 3.$$

The blowup of  $X$  at 0 is a crepant morphism from  $Y' = \text{Bl}_0 X$  with terminal singularities. Its exceptional divisor  $E$  is the projective cone over the curve  $C : (xyz = 0) \subset \mathbb{P}^2$ , therefore  $b_2(Y') = b_3(Y') = 1$  and  $b_4(Y') = 3$ . Since the 3 components of  $E$  are not  $\mathbb{Q}$ -Cartier, we also have

$$\sigma(Y') = \rho(X) + 2 = 3 \text{hcf}(n, 4) - 1.$$

For  $n = 4$ , the singular locus  $\Sigma = \text{Sing } Y'$  consists of 12 ordinary double points, 4 on each line of intersection between 2 components of  $E$ . Let  $\psi : Y \rightarrow Y'$  be a  $\mathbb{Q}$ -factorialisation of  $Y'$  at  $\Sigma$ , obtained, for instance, by blowing up the divisors which are not  $\mathbb{Q}$ -Cartier in some chosen order. Then  $Y$  is an analytic minimal model of  $X$  and  $\tilde{\Sigma} = \varphi^{-1}(\Sigma)$  is a union of 12 lines. Therefore

$$b_2(Y) = 12, \quad b_4(Y) = 3 \quad \text{and} \quad b_3(Y) = 0.$$

For  $n \geq 5$  the blown up variety  $Y'$  has 3 singular points on  $E$ . These are the vertices  $p = (1 : 0 : 0 : 0)$ ,  $q = (0 : 1 : 0 : 0)$  and  $r = (0 : 0 : 1 : 0)$  of the triangle  $C$ . The (local analytic) divisor class numbers of  $Y'$  at these points

are  $\rho(p) = \rho(q) = \rho(r) = \text{hcf}(n, 4)$ . For instance,  $p$  belongs to the first affine piece of  $Y'$ , which is given by

$$yz + x^{n-3}(1 + y^n + z^n) + xt^4 = 0,$$

and, near  $p$ , this equation is equivalent to  $yz + x(x^{n-4} + t^4) = 0$ . With the same notation as in the case  $n = 4$ , it follows that  $b_2(\tilde{\Sigma}) = 3 \text{hcf}(n, 4)$ , hence

$$b_2(Y) = 3 \text{hcf}(n, 4), \quad b_4(Y) = 3 \quad \text{and} \quad b_3(Y) = 0.$$

Notice that  $b_3(Y) = b_3(\tilde{E})$ , where  $\tilde{E}$  is the normalisation, or a resolution of  $E$ .

## 7 The 3rd Betti number of a minimal model

It seems clear from the above examples that the obstructions to obtaining a formula like (3), giving the third Betti number of an analytic minimal model  $b_3(Y)$  purely in terms of the third Betti numbers of the crepant divisors, consist both of problems related to factoriality of the intermediate  $X_i$  and the nonnormality of the exceptional divisors  $E_i$  (and the change in their  $b_3(E_i)$  on resolving).

In this section we prove formula (3) for an arbitrary 3-dimensional isolated canonical singularity  $x \in X$ . We will need the following:

**Lemma 7.1** *Let  $W$  be a 3-fold with canonical analytically  $\mathbb{Q}$ -factorial singularities, and let  $C \subset W$  be a closed subset of codimension at least 2. Then  $H_C^3(W, \mathbb{Q}) = 0$ .*

**Proof** The local cohomology sheaves  $\mathcal{H}_C^0(W, \mathbb{Z})$  and  $\mathcal{H}_C^1(W, \mathbb{Z})$  are both zero. Since  $W$  has canonical singularities and  $\text{codim}_W C \geq 2$ ,  $\mathcal{H}_C^2(W, \mathbb{Z})$  is also zero [K2, Prop. 2.1.7]. We want to prove that  $\mathcal{H}_C^3(W, \mathbb{Q})$  vanishes. By the spectral sequence in [H, Prop. 1.4], this will imply the vanishing of the corresponding local cohomology group. The proof is similar to that of [K2, Prop. 2.1.7]. Assume that  $W$  is local. In the exact sequence

$$H^1(W \setminus C, \mathcal{O}_{W \setminus C}^*) \rightarrow H^2(W \setminus C, \mathbb{Z}) \rightarrow H^2(W \setminus C, \mathcal{O}_{W \setminus C})$$

the last morphism is zero [F, Lemma 6.2]. Since  $W$  is  $\mathbb{Q}$ -factorial,  $\text{Pic}(W \setminus C)$  is torsion, and so  $H^2(W \setminus C, \mathbb{Q}) = 0$ . The conclusion follows now from the exact sequence

$$H^2(W \setminus C, \mathbb{Q}) \rightarrow H_C^3(W, \mathbb{Q}) \rightarrow H^3(W, \mathbb{Q}).$$

□

**Theorem 7.2** *Let  $x \in X$  an isolated canonical 3-fold singularity,  $c = c(X)$  the number of crepant valuations of  $X$ , and let  $Y$  be a crepant partial resolution of  $X$  from  $Y$  with terminal analytically  $\mathbb{Q}$ -factorial singularities. Then*

$$b_3(Y) = \sum_{i=1}^c b_3(\tilde{E}_i), \quad (10)$$

where  $\tilde{E}_i$  are any nonsingular projective representatives of the crepant valuations of  $X$ .

**Proof** Let  $\varphi' : Y' \rightarrow X$  be a crepant projective morphism from  $Y'$  with terminal (algebraically)  $\mathbb{Q}$ -factorial singularities, and let  $\psi : Y \rightarrow Y'$  be an analytic  $\mathbb{Q}$ -factorialisation of  $Y'$ .

Choose a projective variety  $Z$  that contains  $X$  as an open set and has  $x$  as its only singular point. Denote by  $W'$  and  $W$  the partial resolutions of  $Z$  obtained by replacing  $X$  by  $Y'$  and  $Y$ , respectively, as in the diagram

$$\begin{array}{ccccc} E & \hookrightarrow & Y & \hookrightarrow & W \\ \downarrow & & \downarrow & & \psi \downarrow \\ E' & \hookrightarrow & Y' & \hookrightarrow & W' \\ \downarrow & & \downarrow & & \varphi' \downarrow \\ x & \in & X & \hookrightarrow & Z \end{array}$$

We have the following commutative diagram, where the rows are parts of the Mayer–Vietoris sequences of the covers  $W = (W \setminus E) \cup Y$  and  $W' = (W' \setminus E') \cup Y'$

$$\begin{array}{ccccc} H^3(W, \mathbb{Q}) & \longrightarrow & H^3(W \setminus E, \mathbb{Q}) \oplus H^3(Y, \mathbb{Q}) & \longrightarrow & H^3(Y \setminus E, \mathbb{Q}) \\ \psi^* \uparrow & & \psi^* \uparrow & & \cong \uparrow \\ H^3(W', \mathbb{Q}) & \longrightarrow & H^3(W' \setminus E', \mathbb{Q}) \oplus H^3(Y', \mathbb{Q}) & \longrightarrow & H^3(Y' \setminus E', \mathbb{Q}) \end{array}$$

All the cohomology groups appearing in this diagram carry mixed Hodge structures. To simplify the notation, we will omit the coefficients for the remainder of the proof. By [S], the intersection cohomology groups of a complex projective variety carry pure Hodge structures, and so  $IH^3(W')$  is pure of weight 3. Since  $\psi$  is a small morphism, it leaves the intersection cohomology groups unchanged (see [K1, Corollary 4.12] and [Ki, p. 112]). Moreover,  $W$  is a  $\mathbb{Q}$ -homology manifold, hence its intersection cohomology coincides with the usual cohomology. Thus

$$IH^3(W') = IH^3(W) = H^3(W).$$



This shows that  $H^3(W)$  is pure of weight 3. Since  $Z \setminus x$  is a nonsingular algebraic variety,  $H^3(W \setminus E) = H^3(Z \setminus x)$  has weights at least 3 ([D, Théorème 8.2.4]). Let  $L$  be the link of  $x$ . By the Semipurity Theorem [St, Corollary 1.12],  $H^3(Y \setminus E) = H^3(X \setminus x) = H^3(L)$  has weights at least 4. Finally,  $Y$  retracts to  $E$  and  $H^3(E)$  has weights at most 3, again by [D, Théorème 8.2.4].

Since the map  $\psi$  is small, the vertical morphisms  $\psi^*$  are surjective. The bottom row of the diagram is an exact sequence of mixed Hodge structures, and so the top row has the same property once we show that the vertical morphism  $\psi^* : H^3(W') \rightarrow H^3(W)$  is a morphism of mixed Hodge structures. To see this, let  $\Sigma$  be the set of singular points of  $W'$ , and let  $C = \psi^{-1}(\Sigma)$ . Denote by  $i : W' \setminus \Sigma \rightarrow W'$  and  $j : W \setminus C \rightarrow W$  the inclusions, and by  $i^*$  and  $j^*$  the corresponding restrictions. It is known that  $IH^3(W') = \text{Im}(i^*)$  (see e.g. [Ki, p. 48]). The morphism  $j^*$  is injective by the previous lemma, and we clearly have  $\text{Im}(i^*) = \text{Im}(j^*)$ . This shows that the diagram above is a diagram of mixed Hodge structures.

It now follows that  $H^3(E) = H^3(Y)$  is pure of weight 3. Thus, for any resolution  $\tilde{E}$  of  $E$ , we have  $H^3(E) = H^3(\tilde{E})$ , which finishes the proof.  $\square$

**Remark 7.3** The conclusion of Theorem 7.2 can be rephrased as

$$b_3(Y) = 2 \sum_{i=1}^c q(\tilde{E}_i),$$

where  $q(\tilde{E}_i)$  is the irregularity of  $\tilde{E}_i$ . Since, by [R1, Corollary 2.14], each  $\tilde{E}_i$  is rational or ruled, this is further equivalent to

$$b_3(Y) = 2 \sum g(C_i),$$

where the summation is taken over the surfaces  $\tilde{E}_i$  which are ruled over  $C_i$ .

For the remainder of this section, suppose that  $0 \in X : (f = 0) \subset \mathbb{C}^4$  is an isolated canonical nondegenerate singularity. Let  $\varphi : (Y, E) \rightarrow (X, 0)$  be a crepant partial resolution of  $X$  from  $Y$  with terminal analytically  $\mathbb{Q}$ -factorial singularities. Theorem 5.2, together with the formulas from [Ca1] and [Ca2] for the number of crepant divisors  $c(X)$  and the divisor class number  $\rho(X)$  of  $X$ , calculate the 2nd and 4th Betti numbers of  $Y$ . As a consequence of Theorem 7.2, we will prove a similar combinatorial formula for  $b_3(Y)$ . Before stating the result, recall some of the terminology and notation used in [Ca1] (see also Section 2).

For a weighting  $\alpha$ , denote by  $\alpha(\mathbf{1})$  the sum of the weights; denote also by  $\alpha(f)$  the degree of the  $\alpha$ -tangent cone of  $X$ . A primitive weighting  $\alpha \in \sigma \cap N$ , not belonging to any proper face of  $\sigma$ , is called *crepant* if it satisfies the equation

$$\alpha(\mathbf{1}) = \alpha(f) + 1.$$

Denote by  $W(f)$  the set of crepant weightings. For any weighting  $\alpha \in W(f)$ , let

$$\Gamma_\alpha := \{m \in \Gamma_+(f) : \alpha(m) = \alpha(f)\}$$

be the face of  $\Gamma(f)$  corresponding to  $\alpha$ . Let  $E(f)$  be the set consisting of the exceptional prime divisors  $E$  on  $\alpha$ -blowups of  $X$  with  $\alpha$  crepant satisfying  $v_E(x_i) = \alpha_i$ , where  $v_E$  is the valuation of  $E$ .

The following result gives a formula for  $b_3(Y)$  in terms of the crepant weightings  $\alpha$  with  $\dim \Gamma_\alpha = 2$  and the number of integral points in the interior  $\Gamma_\alpha^\circ$  of  $\Gamma_\alpha$ .

**Corollary 7.4** *Let  $0 \in X : (f = 0) \subset \mathbb{C}^4$  be an isolated canonical nondegenerate singularity, and let  $\varphi : Y \rightarrow X$  be a crepant partial resolution of  $X$  from  $Y$  with terminal analytically  $\mathbb{Q}$ -factorial singularities. Then*

$$b_3(Y) = 2 \sum_{\substack{\alpha \in W(f) \\ \dim \Gamma_\alpha = 2}} \#(\Gamma_\alpha^\circ \cap N).$$

**Proof** By [Ca1, Theorem 4.3],  $E(f)$  is a set of representatives of the crepant valuations of  $X$ . Let  $E \in E(f)$  be a prime divisor on the  $\alpha$ -blowup of  $X$ , for some  $\alpha \in W(f)$ .

Suppose first that  $\dim \Gamma_\alpha = 3$ . Let  $\tilde{E}$  be a resolution of  $E$ . By [O, Cor. 7.8], we have  $b_1(\tilde{E}) = 0$ . In particular,  $E$  is rational.

If  $\dim \Gamma_\alpha = 1$ , then  $E$  is also rational (see [O, Lemma 4.8]).

Finally, if  $\dim \Gamma_\alpha = 2$ , then  $E$  is birationally ruled over some curve  $C$ . It follows from [O, Theorem 8.5] and from the proof of [O, Cor. 5.4] that the genus of this curve is given by  $g(C) = \#(\Gamma_\alpha^\circ \cap N)$ .

□

## References

[BBD] A. A. Beilinson, J. Bernstein and P. Deligne, in *Analysis and topology on singular spaces, I (Luminy, 1981)*, 5–171, Astérisque, 100, Soc. Math. France, Paris, 1982

- [BS] J. Bingener and U. Storch, Zur Berechnung der Divisorenklassengruppen kompletter lokaler Ringe, *Nova Acta Leopoldina (N.F.)* **52** (1981), no. 240, 7–63
- [Ca1] M. Caibăr, On the number of crepant valuations of canonical singularities, *J. London Math. Soc. (2)*, Vol. 68 (2003), 307–316
- [Ca2] M. Caibăr, On the divisor class group of 3-fold singularities, *Internat. J. of Math.*, Vol. 14, No. 1 (2003), 105–117
- [C] A. Corti, Personal communication
- [D] P. Deligne, Théorie de Hodge II and III, *Inst. Hautes Études Sci. Publ. Math.* No. **40** (1971), 5–57 and **44** (1974), 5–77
- [F] H. Flenner, Divisorenklassengruppen quasihomogener Singularitäten, *J. Reine Angew. Math.* **328** (1981), 128–160
- [G] M. Gross, Personal communication
- [H] R. Hartshorne, *Local cohomology*, A seminar given by A. Grothendieck, Harvard University, Fall 1961, *Lecture Notes in Mathematics*, No. 41, Springer-Verlag, Berlin, 1967
- [Ka] Y. Kawamata, Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces, *Ann. of Math. (2)* **127** (1988), no. 1, 93–163
- [Ki] F. Kirwan, *An introduction to intersection homology theory*, Longman Sci. Tech., Harlow, 1988
- [K1] J. Kollár, Flops, *Nagoya Math. J.* **113** (1989), 15–36
- [K2] J. Kollár, Flips, flops, minimal models, etc, in *Surveys in differential geometry (Cambridge, MA, 1990)*, 113–199
- [O] M. Oka, On the resolution of the hypersurface singularities, in *Complex analytic singularities*, 405–436, *Adv. Stud. Pure Math.*, 8, North-Holland, Amsterdam, 1987
- [R1] M. Reid, Canonical 3-folds, in *Journées de géométrie algébrique d’Angers*, 273–310, Sijthoff and Noordhoff, Alphen aan den Rijn, 1980
- [R2] M. Reid, Minimal models of canonical 3-folds, in *Algebraic varieties and analytic varieties (Tokyo, 1981)*, 131–180, *Adv. Stud. Pure Math.*, 1, North-Holland, Amsterdam, 1983

- [R3] M. Reid, Young person's guide to canonical singularities, in *Algebraic geometry, Bowdoin, 1985*, 345–414, Proc. Sympos. Pure Math., Part 1, 1987
- [S] M. Saito, Modules de Hodge polarisables, Publ. Res. Inst. Math. Sci. **24** (1988), no. 6, 849–995
- [SB] N. Shepherd-Barron, The topology of rational 3-fold singularities, Notes of a lecture, 1983
- [St] J. H. M. Steenbrink, Mixed Hodge structures associated with isolated singularities, in *Singularities, Part 2 (Arcata, Calif., 1981)*, 513–536, Proc. Sympos. Pure Math., 40, 1983

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