On the divisor class group of 3-fold singularities

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Abstract

In this note we calculate the divisor class number of an isolated canonical singularity $0 \in X : (f = 0) \subset \mathbb{C}^4$, which is assumed to be nondegenerate with respect to its Newton polyhedron, in terms of a suitable set of monomials whose residue classes form a basis for the Milnor algebra of f.

1 Introduction

Let $0 \in X : (f = 0) \subset \mathbb{C}^4$ be an isolated hypersurface singularity. Denote by $\operatorname{Cl} X = \operatorname{Cl} \mathcal{O}^{\mathrm{an}}_{X,0}$ the analytic divisor class group of X. This is a very interesting invariant of the singularity, which is related to the geometry of X and of its partial resolutions. It is a finitely generated free Abelian group; set $\rho(X) := \operatorname{rank} \operatorname{Cl} X$. Canonical singularities were introduced and studied by Reid in [8]. For results about canonical singularities and their minimal models, see also [9] and [10]. In the case of a Gorenstein variety, canonical is equivalent to rational.

The aim of this note is to calculate $\rho(X)$ in the case of an isolated canonical hypersurface singularity, which is assumed to be nondegenerate with respect to its Newton polyhedron. The divisor class group of a canonical quasihomogeneous complete intersection 3-fold singularity was computed by Flenner [5]; beyond this class only a few other cases were known, but not a general answer.

Our main motivation for calculating $\rho(X)$ comes from 3-fold geometry. If $\varphi\colon Y \to X$ is a minimal model of X, then $\rho(X)$ is equal to the difference between the second and the fourth Betti numbers of Y. The problem of determining the cohomology of Y was studied in [4], and will be the subject of a forthcoming paper.

[0] *Mathematics Subject Classification 2000:* 14B05, 14J17

The key towards computing the divisor class number $\rho(X)$ is a result of Flenner [5], giving an identification between $\mathrm{Cl}\, X$ and $H^2(L, \mathbb{Z})$, where L is the link of the singularity $(X, 0)$. We recall Flenner's result in Section 2. This section also contains the basic terminology and notation used throughout the paper.

Section 3 contains some known results about the class group $\mathrm{Cl}\, X$ from [3], [15] and [5]. Section 4 is devoted to obtaining such answers in more general circumstances. The principle is always the same. We start from Flenner's result (Proposition 2.1) and use some short-cuts to obtain information about $\rho(X)$. These short-cuts are either singularity theory results, as in Section 3, or some of their refined versions coming from mixed Hodge structures on the vanishing cohomology, as in Section 4.

The isomorphism $\mathrm{Cl}\, X \cong H^2(L, \mathbb{Z})$ in Proposition 2.1 is a consequence of the assumption that the singularity $0 \in X$ is canonical. This assumption is used once more in Section 4 to relate $\rho(X)$ to other invariants of the singularity. Namely, if F denotes the Milnor fiber and $\Delta(t)$ the characteristic polynomial of the monodromy operator on the homology of F, then $\rho(X)$ coincides with the power of $(t-1)$ dividing $\Delta(t)$. This is the key remark of this section. In particular, since $\Delta(t)$ can be calculated from an embedded resolution $\varphi\colon \widetilde{\mathbb{C}}^4 \to \mathbb{C}^4$ of X by a result of A'Campo, the same is possible for $\rho(X)$. More precisely, if E_i are the φ-exceptional divisors and $\widetilde{X}$ is the proper transform of X, then

$$\rho(X) = 1 - e(E^\circ),$$

where $e(E^\circ)$ is the Euler number of $E^\circ = E \setminus \left(\widetilde{X} \cup \bigcup_{i \neq j} E_i \cap E_j\right)$. This is the content of Proposition 4.9. Although embedded resolutions of 3-fold singularities are difficult to control, in simple cases this information is useful (cf. Example 5.3).

The main result of this paper is Theorem 4.5, which calculates $\rho(X)$ for canonical singularities which are nondegenerate with respect to their Newton diagrams. We use a suitable set of monomials whose residue classes form a basis for the Milnor algebra $M(f)$, which we define to be a regular basis for the Newton filtration on the space Ω^4 of germs at $0 \in \mathbb{C}^4$ of holomorphic 4-forms (Definition 4.3). Making use of results about mixed Hodge structures on the vanishing cohomology from [11], [12], [14] and [18], it follows that, for nondegenerate singularities of Newton degree α, the divisor class number $\rho(X)$ can be calculated as

$$\rho(X) = \#\{m \in B : \alpha(m + \mathbf{1}) = 2\},$$

where B is a regular set of monomials whose residue classes give a basis for $M(f)$.

We illustrate these formulas in the final section.

Acknowledgements

This paper is part of my University of Warwick Ph.D. thesis. It is a great pleasure to thank my research advisor, Miles Reid, for his constant support and guidance. This work is a reflection of his ideas, explanations and criticism during many hours of mathematical discussions.

The final version of this paper was written during a postdoctoral position at University of North Carolina at Chapel Hill. I thank both institutions for hospitality, and Alessio Corti, David Mond and Jonathan Wahl for useful conversations.

2 Preliminaries

This section contains the basic terminology and notation used in this paper. We also recall a result of Flenner, which relates the divisor class number of a 3-fold singularity to singularity theory invariants.

Let $0 \in X : (f = 0) \subset \mathbb{C}^{n+1}$ be an isolated hypersurface singularity. Denote by $\mathcal{O}_{n+1} = \mathbb{C}\{x_1, \ldots, x_{n+1}\}$ the $\mathbb{C}$-algebra of analytic function germs, and by $J(f) = (\partial f/\partial x_i)_{1\leq i\leq n+1}$ the Jacobian ideal of f. The Milnor algebra of f, $M(f) = \mathcal{O}_{n+1}/J(f)$, is finite dimensional. Its dimension, $\mu(f) = \dim_{\mathbb{C}} M(f)$, is the *Milnor number*.

Denote by L the *link* of f. It is a compact $(2n-1)$-dimensional, $(n-1)$-connected submanifold $L \subset S^{2n+1}$. Denote also by F the *Milnor fiber*, and let $m_g \colon F \to F$ be the *geometric monodromy*. At the homology level, m_g induces an automorphism

$$h = (m_g)_* \colon H_n(F, \mathbb{Z}) \to H_n(F, \mathbb{Z}),$$

called the *(homology) monodromy operator*, which is uniquely determined. Let $\Delta(t) = \det(h - t \cdot \mathrm{id})$ be its characteristic polynomial and

$$H_n(F)_1 = \{v \in H_n(F) : \exists m \in \mathbb{N} \text{ s.t. } (h - \mathrm{id})^m v = 0\}$$

the generalised eigenspace of the monodromy belonging to the eigenvalue 1. Suppose $\Delta(t) = \prod_{i=1}^{\mu(f)}(t - \zeta_i)$, where ζ_i are the eigenvalues of h. A useful way of encoding the information about the eigenvalues of h is

$$\operatorname{Div}(f) := \sum_{i=1}^{\mu(f)} (\zeta_i) \in \mathbb{Z}[S^1].$$

The homology of L can be described as follows (see [6]). If $n \geq 2$, then

$$H_n(L, \mathbb{Z}) \cong \ker(h - \mathrm{id}) \subset H_n(F, \mathbb{Z}). \tag{1}$$

Similarly, for $n = 1$, the reduced cohomology of L satisfies

$$\widetilde{H}^0(L) \cong \ker(h - \mathrm{id}). \tag{2}$$

Denote by $b_n(F)_1$ the exponent of $(t-1)$ in $\Delta(t)$. Then, for $n \geq 2$, the interesting Betti numbers of L satisfy

$$b_{n-1}(L) = b_n(L) = \dim \ker(h - \mathrm{id}) \leq \dim H_n(F)_1 = b_n(F)_1, \tag{3}$$

with equality if and only if the Jordan matrix of the monodromy operator h belonging to the eigenvalue 1 is diagonal. As it will turn out, canonical 3-fold singularities have this property (see Section 4).

A mixed Hodge structure on the vanishing cohomology $H^n(F)$ was constructed by Steenbrink [13], using the resolution of the singularity and alternatively by Varchenko [17], using the asymptotic expansion of integrals over vanishing cycles. Denote by $W_\bullet$ the weight filtration and by $F^\bullet$ the Hodge filtration. The geometric monodromy m_g induces an action $T = (m_g^*)^{-1}$ on the cohomology group $H^n(F)$, called the *cohomological monodromy operator.* If λ is an eigenvalue of T, denote also by $h_\lambda^{p,q}$ the corresponding *Hodge numbers.* For any $r \in \mathbb{Q}$, let $\lambda = \exp(-2\pi i r)$ and define the integers n_r in terms of the Hodge numbers $h_\lambda^{p,q}$ corresponding to the eigenvalue λ by

$$n_r = \sum_q h_\lambda^{n+1+[-r],q}.$$

One way of encoding the relation between the semisimple part of the monodromy and the Hodge filtration is the invariant

$$\mathrm{S}(f) = \sum n_r(r),$$

defined as an element of the free Abelian group on $\mathbb{Q}$. This is essentially a more precise version of the invariant $\operatorname{Div}(f)$ defined above; the relation between them is given by taking exponentials $r \mapsto \lambda = \exp(-2\pi i r)$. The invariant $\mathrm{S}(f)$ is the spectrum of f, as defined in [13], shifted by 1 to the right; in [11] and [12], the spectral numbers r are called the *exponents* of the singularity.

For an arbitrary polynomial, $\mathrm{S}(f)$, regarded as a subset of $\mathbb{R}$, is contained in the interval $(0, n+1)$. If f defines a canonical singularity, M. Saito proved in [11] that

$$\mathrm{S}(f) \subset (1, n). \tag{4}$$

Recall standard toric terminology. Let M be the free Abelian group $\mathbb{Z}^{n+1}$, $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ its dual, $M_k = M \otimes_{\mathbb{Z}} k$ and $N_k = N \otimes_{\mathbb{Z}} k$ the vector spaces obtained by extending scalars to k, where $k = \mathbb{Q}$ or $\mathbb{R}$.

Let $e_1, \dots, e_{n+1}$ denote the standard basis of $N_{\mathbb{R}}$, $\sigma = \sum_{i=1}^{n+1} \mathbb{R}_{\geq 0} e_i \subset N_{\mathbb{R}}$ the positive quadrant, and $\check{\sigma} = \sum_{i=1}^{n+1} \mathbb{R}_{\geq 0} e_i^*$ the dual quadrant. We identify $m \in M$ with the monomial $x^m = \prod_{i=1}^{n+1} x_i^{m_i}$. It is convenient to write $m \in f$ if m appears in the polynomial $f = \sum_{m \in M} a_m x^m$ with $a_m \neq 0$; for each $\alpha \in \sigma \cap N$, define then $\alpha(m) = \sum_{i=1}^{n+1} \alpha_i m_i$.

Let $0 \in X : (f = 0) \subset \mathbb{C}^{n+1}$ be an isolated hypersurface singularity. The *Newton polyhedron* of f, $\Gamma_+(f)$, is the convex hull in $M_{\mathbb{R}}$ of the set $\bigcup_{m \in f} (m + \check{\sigma})$; the union of all its compact facets, $\Gamma(f)$, is called the *Newton diagram* of f. For any face $\Gamma \prec \Gamma(f)$, denote $f_\Gamma = \sum_{m \in \Gamma \cap M} a_m x^m$. The polynomial f is *nondegenerate* with respect to its Newton diagram if, for each face Γ of $\Gamma(f)$, the hypersurface defined by $f_\Gamma = 0$ is nonsingular on $(\mathbb{C}^*)^{n+1}$.

Let $0 \in X : (f = 0) \subset \mathbb{C}^4$ be an isolated hypersurface singularity, and denote by $\mathrm{Cl}\, X$ its local analytic divisor class group. $\mathrm{Cl}\, X$ is a finitely generated free Abelian group; set $\rho(X) := \mathrm{rank}\, \mathrm{Cl}\, X$. The key towards computing $\rho(X)$ is the following result of Flenner.

Proposition 2.1 ([5]) *If $0 \in X$ is as above, then*

(1) *the first Chern class map $c_1 \colon \mathrm{Pic}(X \setminus 0) \to H^2(X \setminus 0, \mathbb{Z})$ is an injection;*

(2) *if, moreover, $0 \in X$ is canonical, then c_1 is an isomorphism.*

Thus, via the retraction $(X \setminus 0) \searrow L$ coming from the cone structure of analytic sets, one has, in the case of a canonical singularity, an isomorphism

$$\mathrm{Cl}\, X \cong H^2(L, \mathbb{Z}).$$

Example 2.2 Singularities of the form $xy = g(z,t)$

Suppose $f = xy - g(z,t)$. In this case F_f, the Milnor fiber of f, is a double suspension of F_g and, since the monodromy of xy is trivial, by (1) and (2), we can write

$$H_3(L_f) = \ker(h_f - \mathrm{id}) = \ker(h_g \otimes h_{xy} - \mathrm{id}) = \ker(h_g - \mathrm{id}) = \widetilde{H}^0(L_g).$$

If r denotes the number of branches of g, we have $L_g = \bigsqcup_r S^1$ and, since f defines a canonical singularity, Proposition 2.1 implies that

$$\rho(X) = \operatorname{rank} \widetilde{H}^0(L_g) = r - 1.$$

In fact, this class of singularities is very special since, in this case, it is possible to write explicitly generators of $\mathrm{Cl}\, X$ in terms of a factorisation. Namely, if g_i are the irreducible components of g and $D_i : (x = g_i = 0)$ for $1 \leq i \leq r$, then it is known that

$$\mathrm{Cl}\, X = \frac{\bigoplus \mathbb{Z}[D_i]}{(\sum D_i)}.$$

3 Quasihomogeneous singularities

This section contains some formulas for $\rho(X)$ in the case of quasihomogeneous singularities. The class of (semi)quasihomogeneous singularities is the largest class of singularities for which a formula for the divisor class number is known (see [5]). The treatment here emphasizes the relation with singularity theory coming from Proposition 2.1.

Definition 3.1 *A hypersurface $X : (f = 0) \subset \mathbb{C}^{n+1}$ is* quasihomogeneous *if there exists $\alpha \in N_{\mathbb{Q}}$ such that $\alpha(m) = 1$ for every $m \in f$.*

We write $\alpha = (\alpha_1, \dots, \alpha_{n+1}) \in N_{\mathbb{Q}}$ and $\alpha_i = u_i/v_i$ with $\mathrm{hcf}(u_i, v_i) = 1$. Denote also $\alpha_I = \prod_{i \in I} \alpha_i$ and $[v_I] = \mathrm{lcm}(v_i,\, i \in I)$. With the notation

$$\Lambda_m := \sum_{k=1}^{m} (e^{2\pi i k/m}) \in \mathbb{Z}[S^1] \quad \text{for} \quad m \geq 1,$$

the characteristic polynomial of a quasihomogeneous singularity is given, in terms of its weights, by the following result of Milnor and Orlik.

Theorem 3.2 ([7]) *If f defines an α-quasihomogeneous isolated singularity, then $\Delta(t)$ is determined by:*

$$\mathrm{Div}(f) = \prod_{i=1}^{n+1} (u_i^{-1}\Lambda_{v_i} - (1)).$$

There is a larger class of singularities which behave topologically exactly like the quasihomogeneous ones.

Definition 3.3 *A polynomial f is called* α-semiquasihomogeneous*, for some $\alpha \in N_{\mathbb{Q}}$, if its α-tangent cone defines an isolated singularity.*

It is known that the monodromy operator of a semiquasihomogeneous singularity has finite order. In particular, one has equality in (3). Thus, a direct consequence of Proposition 2.1 and Theorem 3.2 is the following formula for the rank of the divisor class group in the three-dimensional canonical case.

Proposition 3.4 *Let $0 \in X \subset \mathbb{C}^4$ be an isolated α-semiquasihomogeneous singularity. Then the Picard number of X satisfies*

$$\rho(X) \leq \sum_{I \subset \{1,2,3,4\}} \frac{(-1)^{|I|}}{\alpha_I \cdot [v_I]}, \tag{5}$$

where, by convention, $\alpha_\phi = [\alpha_\phi] = 1$. Furthermore, equality holds if $0 \in X$ is canonical.

Remark 3.5 In the case of a Brieskorn–Pham singularity, formula (5) was obtained in [15] using different methods (see also [3]).

The divisor class number of a canonical quasihomogeneous complete intersection singularity was calculated by Flenner [5]. We state Flenner's result in the hypersurface case. In the next section we will give a similar formula for nondegenerate singularities. Both formulas can be derived from results on mixed Hodge structures on the vanishing cohomology.

Theorem 3.6 ([5]) *Let $0 \in X : (f = 0) \subset \mathbb{C}^4$ be an isolated canonical semiquasihomogeneous singularity of type α, and B any set of monomials whose residue classes give a basis for $M(f)$. Then*

$$\rho(X) = \#\{m \in B : \alpha(m + \mathbf{1}) = 2\}.$$

4 Canonical singularities

It is possible to calculate explicitly the spectrum for large classes of isolated singularities. In the semiquasihomogeneous case there is the following answer, due to Steenbrink.

Theorem 4.1 ([14]) *Suppose that* $0 \in X : (f = 0) \subset \mathbb{C}^{n+1}$ *is an isolated* α*-semiquasihomogeneous singularity, and let* B *be a set of monomials whose residue classes form a basis of the Milnor algebra of* f*. Then*

$$\mathrm{S}(f) = \sum_{m \in B} \alpha(m + \mathbf{1}). \tag{6}$$

A similar description holds for nondegenerate singularities, for a suitable choice of a basis for the Milnor algebra $M(f)$.

To see this, suppose that f is nondegenerate and *convenient*. The last assumption means that the Newton diagram of f intersects every coordinate axis; this is not restrictive since f is finitely determined and thus, if we change f by adding monomials of the form x_i^k for sufficiently large k, we obtain an equivalent singularity.

Let Ω^p denote the space of germs at $0 \in \mathbb{C}^{n+1}$ of holomorphic p-forms. In [18], a filtration $\mathcal{N}^\bullet$ on Ω^{n+1} is introduced as follows.

Let $\Gamma_i \prec \Gamma(f)$ be the top dimensional faces of the Newton diagram and suppose that each Γ_i is defined by a weighting $\alpha^{(i)} \in N_{\mathbb{Q}}$. The *Newton degree* of f, denoted by α, is defined on monomials $m \in M$ by $\alpha(m) = \min_i \alpha^{(i)}(m)$, and on power series $g \in \mathcal{O}_{n+1}$ by $\alpha(g) = \min_{m \in g} \alpha(m)$. This definition can be extended to forms $\omega = g(\mathbf{x})\mathrm{d}\mathbf{x} \in \Omega^{n+1}$ by $\alpha(\omega) = \alpha(g(\mathbf{x})\mathbf{x})$. The *Newton filtration* $\mathcal{N}^\bullet$ is then the decreasing filtration on Ω^{n+1} associated to α, i.e.,

$$\mathcal{N}^r = \left\{\omega \in \Omega^{n+1} : \alpha(\omega) \geq r\right\}.$$

Let $\overline{\mathcal{N}}^\bullet$ and $\overline{\alpha}$ denote the induced filtration and degree on the quotient space $\Omega_f := \Omega^{n+1}/\mathrm{d}f \wedge \Omega^n$. Denote also by $\mathcal{N}^{>r}$ the set consisting of those forms ω of degree strictly bigger than r; same notation for $\overline{\mathcal{N}}^{>r}$.

Let $\widetilde{\mathrm{S}}(f)$ be the spectrum of the Newton filtration $\overline{\mathcal{N}}^\bullet$ on Ω_f, i.e.,

$$\widetilde{\mathrm{S}}(f) = \sum n_r(r), \text{ where } n_r = \dim \overline{\mathcal{N}}^r / \overline{\mathcal{N}}^{>r}.$$

M. Saito proved in [11] a conjecture of Steenbrink [13] relating the invariants $\mathrm{S}(f)$ and $\widetilde{\mathrm{S}}(f)$. In [18] there is an alternative proof of this. The result is the following:

Theorem 4.2 ([11],[18]) *Under the above assumptions,*

$$\mathrm{S}(f) = \widetilde{\mathrm{S}}(f). \tag{7}$$

[2, 12.7] defines and constructs a *regular basis* for the Milnor algebra of f. We make an analogous definition, except that we work in

$$\Omega_f = \Omega^{n+1}/J_f\Omega^{n+1} = M(f) \otimes_{\mathcal{O}_{n+1}} \Omega^{n+1}.$$

Definition 4.3 *A set of elements $B \subset \mathcal{O}_{n+1}$, whose residue classes form a basis for the Milnor algebra $M(f)$, is* regular *for the filtration $\mathcal{N}^\bullet$ on Ω^{n+1} if, for any r, the elements of the set*

$$\{\omega \in \Omega^{n+1} : \omega = g(\mathbf{x})\mathrm{d}\mathbf{x}, g \in B, \alpha(\omega) = r\}$$

are independent modulo $(\mathrm{d}f \wedge \Omega^n) + \mathcal{N}^{>r}$.

A proof identical to that in [2, 12.7] shows that there exists a monomial basis for $M(f)$, that is regular for the filtration $\mathcal{N}^\bullet$ on Ω^{n+1}.

Lemma 4.4 *Let f and B be as above, $m \in B$ a regular element, and $\omega = \mathbf{x}^m\mathrm{d}\mathbf{x}$. Then $\alpha(\omega) = \bar{\alpha}(\overline{\omega})$.*

Proof The inequality $\overline{\alpha}(\overline{\omega}) \geq \alpha(\omega)$ follows from the definitions of α and $\overline{\alpha}$. Furthermore, if $\overline{\alpha}(\overline{\omega}) > \alpha(\omega)$, then there exists $\eta \in \mathrm{d}f \wedge \Omega^n$ such that $\alpha(\omega + \eta) > \alpha(\omega)$. But this means that ω belongs to the ideal $(\mathrm{d}f \wedge \Omega^n) + \mathcal{N}^{>\alpha(\omega)}$, which is impossible by the regularity assumption.

□

Suppose B is a regular set. By definition, this means that the elements $\overline{\omega}_m$ with $m \in B$, of degree $\overline{\alpha}(\overline{\omega}_m) = r$, form a basis of the quotient vector space $\overline{\mathcal{N}}^r/\overline{\mathcal{N}}^{>r}$. Thus, from the above lemma, one can write

$$\widetilde{\mathrm{S}}(f) = \sum_{m \in B} \alpha(m + \mathbf{1}).$$

The following result is an analog of Theorem 3.6. Next section contains some illustrations of formula (8) below.

Theorem 4.5 *Let $0 \in X : (f = 0) \subset \mathbb{C}^4$ be an isolated canonical nondegenerate singularity of Newton degree α, and B a regular set of monomials whose residue classes give a basis for $M(f)$. Then*

$$\rho(X) = \#\{m \in B : \alpha(m + \mathbf{1}) = 2\}. \tag{8}$$

Proof Let n_r be the multiplicity of the spectral number r. By Theorem 4.2 and Lemma 4.4 , the conclusion is equivalent to $\rho(X) = n_2$.

Since $0 \in X : (f = 0) \subset \mathbb{C}^4$ is a canonical singularity, M. Saito's result (4) implies that $\mathrm{S}(f) \subset (1, 3)$; in particular, the only spectral number associated to the eigenvalue 1 is 2.

For $r \in \mathbb{Z}$, the multiplicities n_r satisfy the relation $n_r = \sum_q h_1^{4-r,q}$. Using the symmetries of the Hodge numbers (see [13]), it follows that $h_1^{p,q} = 0$ unless $(p, q) = (2, 2)$. Therefore $\mathrm{Gr}_l^W H^3(F, \mathbb{C})_1 = 0$ unless $l = 4$, which shows that the Jordan matrix of the monodromy operator belonging to the eigenvalue 1 is diagonal. In particular, (3) implies that

$$b_2(L) = b_3(F)_1 = n_2.$$

The conclusion follows from Proposition 2.1, using again the fact that the singularity is canonical.

□

Remark 4.6 A similar argument shows that Theorem 3.6 follows from Proposition 2.1 and Theorem 4.1.

Remark 4.7 In [16] it is obtained a formula relating the characteristic polynomial of an isolated nondegenerate singularity $0 \in X : (f = 0) \subset \mathbb{C}^{n+1}$ to its Newton diagram $\Gamma(f)$. For any face $\Gamma \prec \Gamma(f)$, denote by $d(\Gamma)$ its dimension and by $V(\Gamma)$ its $(d(\Gamma))$-dimensional volume. In the case of a 3-fold singularity, Varchenko's formula and Proposition 2.1 imply that

$$\rho(X) \leq 1 + \sum_{\Gamma} (-1)^{d(\Gamma)+1} d(\Gamma)! V(\Gamma),$$

where the summation is taken over all faces Γ of $\Gamma(f)$ which are contained in a $(d(\Gamma) + 1)$-dimensional coordinate plane. Equality holds if $0 \in X$ is canonical.

In the remainder of this section we note that it is possible, in principle, to calculate the rank of the divisor class group $\rho(X)$ for *any* isolated canonical singularity $0 \in X : (f = 0) \subset \mathbb{C}^4$. The reason is that $\rho(X)$ is related to the characteristic polynomial $\Delta(t)$ by

$$\rho(X) = b_3(F)_1 \tag{9}$$

as we have seen in the proof of Theorem 4.5, and $\Delta(t)$ can be calculated from the embedded resolution data of X by A'Campo's formula.

Let $0 \in X : (f = 0) \subset \mathbb{C}^{n+1}$ be an isolated singularity, $\varphi\colon \widetilde{\mathbb{C}}^{n+1} \to \mathbb{C}^{n+1}$ an embedded resolution of the singularity of X. Denote by E_i the irreducible components of the exceptional locus $E = \varphi^{-1}(0)$ and by $\widetilde{X}$ the strict transform of X. Let

$$E_m = \{z \in E : f \circ \varphi = z_1^m \text{ in suitable local coordinates near } z\}.$$

The following theorem, due to A'Campo, gives the characteristic polynomial of f in terms of the multiplicities of the exceptional divisors and the topological Euler numbers $e(E_m)$ of the interior loci above.

Theorem 4.8 ([1]) *The characteristic polynomial of the monodromy operator h_f is given by*

$$\Delta(t) = \left(\frac{1}{t-1} \prod_{m \geq 1} (t^m - 1)^{e(E_m)} \right)^{(-1)^n} \tag{10}$$

Denote $E^\circ = E \setminus \left(\widetilde{X} \cup \bigcup_{i \neq j} E_i \cap E_j \right)$. As a consequence of A'Campo's formula, we obtain the following.

Proposition 4.9 *Let $0 \in X : (f = 0) \subset \mathbb{C}^4$ be an isolated singularity. Then, with the above notation,*

$$\rho(X) \leq 1 - e(E^\circ),$$

with equality if the singularity is canonical.

Proof As in the proof of Theorem 4.5, we have $\rho(X) \leq b_3(F)_1$, with equality if $0 \in X$ is canonical. The conclusion follows from Theorem 4.8 and Proposition 2.1. □

Remark 4.10 Although embedded resolutions of 3-fold singularities are difficult to control, in simple cases relation (9) is useful even when X is degenerate. For instance, suppose that $f(x,y,z,t) = g(x,y) + h(z,t)$. The invariants of curve singularities can be computed from their resolution graphs. In particular, via the Thom–Sebastiani type formulas in [7] and [17], one can calculate $\rho(X)$ as well. See Example 5.3 for an illustration.

5 Examples

The following examples are illustrations of the formulas in the previous section.

Example 5.1 Let $g(x,y,z) = x^p + y^q + z^r + xyz$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ be a singularity of type $T_{p,q,r}$, and let

$$f(x,y,z,t) = g(x,y,z) + t^n \quad \text{with} \quad n \geq 2.$$

The polynomial f defines an isolated canonical nondegenerate singularity $0 \in X$. The set

$$B = \Big\{1, xyz, x^i, y^j, z^k;\ 1 \leq i \leq p-1,\ 1 \leq j \leq q-1,\ 1 \leq k \leq r-1\Big\}$$

is regular. In particular,

$$\mathrm{S}(g) = (1) + (2) + \sum_{i=1}^{p-1}\left(1 + \frac{i}{p}\right) + \sum_{j=1}^{q-1}\left(1 + \frac{j}{q}\right) + \sum_{k=1}^{r-1}\left(1 + \frac{k}{r}\right).$$

Since $\mathrm{S}(t^n) = \sum_{l=1}^{n-1}(l/n)$, the spectrum of f can be computed using the Thom–Sebastiani type formula in [17]. It follows that the Picard number of X is given by

$$\rho(X) = (p,n) + (q,n) + (r,n) - 3,$$

where, for any integers a and b, $(a,b) := \mathrm{hcf}(a,b)$.

Example 5.2 A blowup of a factorial singularity need not be locally factorial, as the following example shows. Let $f(x,y,z,t) = g(x,y,z) + t^n$, where $g(x,y,z) = x^3 + x^2z + y^2z + z^4$ and $n \geq 2$. The polynomial f defines an isolated canonical nondegenerate singularity $0 \in X$. The set of monomials

$$B = \Big\{1, x, y, x^2, xy, z^k; 1 \leq k \leq 4\Big\}$$

induces a regular basis on Ω_g. This implies that

$$\mathrm{S}(g) = (1) + \left(\frac{5}{4}\right) + 2\left(\frac{4}{3}\right) + \left(\frac{3}{2}\right) + 2\left(\frac{5}{3}\right) + \left(\frac{7}{4}\right) + (2).$$

From the Thom–Sebastiani type formula in [17] and Theorem 4.5 it follows that

$$\rho(X) = \begin{cases} 0 & \text{if } n \equiv \pm 1 \bmod 6 \\ 1 & \text{if } n \equiv \pm 2 \bmod 12 \\ 3 & \text{if } n \equiv \pm 4 \bmod 12 \\ 4 & \text{if } n \equiv \pm 3 \bmod 12 \\ 5 & \text{if } n \equiv 6 \bmod 12 \\ 7 & \text{if } n \equiv 0 \bmod 12. \end{cases}$$

In particular, for $n = 5$, the singularity $0 \in X$ is (analytically) factorial.

Suppose $n = 5$, and let $X_1 = \mathrm{Bl}_0 X$ be the blowup of X at its singular point. The blown up variety X_1 has only one singular point. Its equation near this point is $x^3 + x^2 z + y^2 z + t z^4 + t^2 = 0$ or, by a change of variable,

$$X_1 : x^3 + x^2 z + y^2 z + z^8 + t^2 = 0.$$

The singularity $0 \in X_1$ is a cD_4 point. Denote $h = x^3 + x^2 z + y^2 z + z^8$. A regular set for Ω_h is

$$B' = \left\{1, x, y, x^2, xy, z^k; 1 \leq k \leq 8\right\}$$

and the spectral numbers of h are given by

$$\mathrm{S}(h) = (1) + 2\left(\frac{4}{3}\right) + 2\left(\frac{5}{3}\right) + \sum_{i=1}^{7}\left(\frac{i+8}{8}\right) + (2).$$

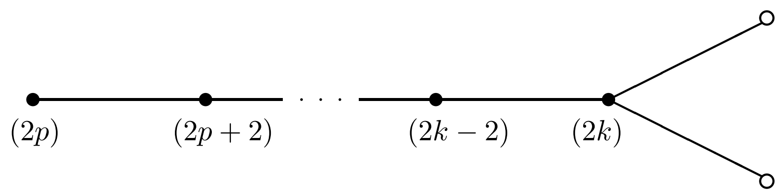
It follows that $\rho(X_1) = 1$.

Example 5.3 This is an example about the calculation of $\rho(X)$ for degenerate singularities. Let $f(x, y, z, t) = x^2 + y^3 + g(z, t)$, where

$$g(z, t) = (z^p + t^p)^2 + t^n \quad \text{with} \quad p \geq 2 \quad \text{and} \quad n \geq 2p + 1.$$

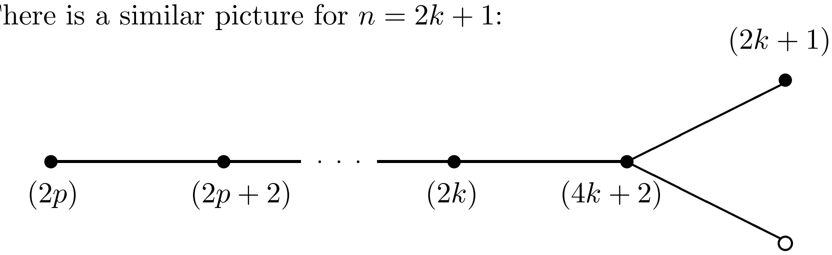
The polynomials g and f define isolated singularities, degenerate with respect to their Newton polyhedrons. This is because the singular locus defined by the hypersurface $g_{\Gamma(g)} = 0$ consists of p lines passing through the origin, each with multiplicity 2.

First we want to compute $\Delta_g(t)$ from the resolution graph of $0 \in C : (g = 0) \subset \mathbb{C}^2$. Blowing up $0 \in \mathbb{C}^2$ we obtain an exceptional divisor $E_1 \cong \mathbb{P}^1$ of multiplicity $2p$. The proper transform of C has p singular points of type A_{n-2p-1}, lying on E_1. The following is one of the p identical branches, having their left vertex in common, of the resolution graph of C in the case $n = 2k$:



Vertices of the graph $\bullet$ correspond to the exceptional curves; the irreducible components of the proper transform of C are denoted by $\circ$. An edge is associated to a pair of vertices with nonempty intersection. The numbers in brackets are the multiplicities of the corresponding exceptional divisors.

There is a similar picture for $n = 2k + 1$:



From A'Campo's formula (10) it follows that

$$\operatorname{Div}(g) = \begin{cases} (1) + (p-2)\Lambda_{2p} + p\Lambda_{2k} & \text{if } n = 2k \\ (1) + (p-2)\Lambda_{2p} + p\Lambda_{4k+2} - p\Lambda_{2k+1} & \text{if } n = 2k+1. \end{cases}$$

For any integers a and b, denote $(a, b) = \operatorname{hcf}(a, b)$ and $[a, b] = \operatorname{lcm}(a, b)$. Using the Thom–Sebastiani formula in [7] we obtain:

$$\operatorname{Div}(f) = \begin{cases} p(3,k)\Lambda_{2[3,k]} + (p-2)(3,p)\Lambda_{2[3,p]} - p\Lambda_{2k} & \\ -(p-2)\Lambda_{2p} + \Lambda_6 - \Lambda_3 - \Lambda_2 + (1) & \text{if } n = 2k \\ p(3,2k+1)\Lambda_{[3,2k+1]} + (p-2)(3,p)\Lambda_{2[3,p]} & \\ -p\Lambda_{2k+1} - (p-2)\Lambda_{2p} + \Lambda_6 - \Lambda_3 - \Lambda_2 + (1) & \text{if } n = 2k+1. \end{cases}$$

In particular, $b_3(F_f)_1 = p((3,n) - 1) + (p-2)((3,p) - 1)$ in both cases. The singularity $0 \in X$ is canonical if and only if $p \leq 5$. Therefore

$$\rho(X) \leq p((3,n) - 1) + (p-2)((3,p) - 1),$$

with equality for $p \leq 5$.

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