A NOTE ON ADJOINT LINEAR SYSTEMS

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Abstract. In dimension 2 there is a very useful criterion, due to Miles Reid, giving a necessary and sufficient condition for an adjoint linear system on a nonsingular complex projective surface X to be free at a point $p \in X$, in terms of the coherent cohomology of certain divisors on the blowup of X at p. In this note we extend Miles Reid's result to higher dimensions.

Key words and phrases: linear systems, singularities, mixed Hodge structures.

1. Introduction

Let X be a nonsingular projective surface over $\mathbb{C}$, and let D be an effective divisor on X. Fix a point $p \in \operatorname{Sing} D$, and let $\varphi\colon Y \to X$ be the blowup of X at p. Denote by E the exceptional curve, and consider the divisors D_0 and D_1 on Y given by

$$D_0 = \varphi^* D - E \quad \text{and} \quad D_1 = \varphi^* D - 2E.$$

The following theorem, due to Miles Reid [F, Appendix to §2], gives a necessary and sufficient condition for the linear system $|K_X + D|$ to be free at p.

Theorem 1.1. *With the above assumptions and notation, p is not a base point of $|K_X + D|$ if and only if the restriction map $H^0(\mathcal{O}_{D_0}) \to H^0(\mathcal{O}_{D_1})$ is surjective.*

This criterion is strongly related to the classical approach to freeness of linear systems, due to Bombieri, Kodaira, and Ramanujam. The key ingredients of this approach are vanishing theorems and numerical connectedness. Recall that an effective divisor D on X is numerically k-connected if, for every decomposition $D = A + B$ with $A > 0$ and $B > 0$, we have $A \cdot B \geq k$. This notion is an extension of the usual topological connectedness to nonreduced divisors. It provides information about both the topology and the scheme structure of D. See [BHPV] and [R] for references and for an overview of these ideas.

Note that in the above theorem there is no positivity assumption on D. The statement is very precise, in other words it involves exactly the amount of vanishing that is necessary for $p \notin \operatorname{Bs}|K_X + D|$, and not the vanishing of a whole cohomology group. The criterion is therefore very useful in applications, in conjunction with, or instead of, other vanishing results. For an illustration, see [F].

The aim of this paper is to generalize Theorem 1.1 to arbitrary dimension. This is done in Theorem 2.2, which we describe below.

Let X be a nonsingular complex projective variety of dimension n, and let D be an effective divisor on X. Assume that D has a singular point p of multiplicity at least n. Let $\varphi\colon Y \to X$ be the blowup of X at p, and denote by E the exceptional divisor of φ. Consider the divisors D_0 and D_1 on Y given by

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$$D_0 = \varphi^* D - (n-1)E \quad \text{and} \quad D_1 = \varphi^* D - nE.$$

Under these assumptions, we show in Theorem 2.2 that p is not a base point of $|K_X + D|$ if and only if the restriction map

$$H^{n-2}(\mathcal{O}_{D_0}) \to H^{n-2}(\mathcal{O}_{D_1})$$

is surjective. Equivalently, $|K_X + D|$ is free at p if and only if the map

$$H^{n-1}(\mathcal{O}_{D_0}) \to H^{n-1}(\mathcal{O}_{D_1})$$

is not injective.

The idea of the proof is to show that the constant sheaf cohomology dominates the coherent cohomology in this context, and then use a mixed Hodge structures argument to establish the topological version of the statement.

To this end, we first prove that, if $f\colon H^k(X, \mathcal{O}_X) \to H^k(D, \mathcal{O}_D)$ and $g\colon H^k(X, \mathbb{C}) \to H^k(D, \mathbb{C})$ denote the natural maps, for $k \geq 0$ any integer, then

$$\ker f \cong \ker g \cap H^{0,k}(X).$$

This is the content of Lemma 2.1. It implies that there is no contribution to the kernel of f coming from the nonreduced structure of D, which was observed by Ramanujam [Ra] when $n = 2$ and $k = 1$.

Except for Claim 2.3, the proof of Theorem 2.2 runs as in [F, Appendix to §2]. Namely, one first rephrases the condition $p \notin \mathrm{Bs}|K_X + D|$ as a cohomological condition on the blown up variety Y. The conclusion then follows from the claim, via a commutative diagram coming from the structure sequences of D_0 and D_1.

Claim 2.3 says that the restriction maps $f_i\colon H^{n-1}(\mathcal{O}_Y) \to H^{n-1}(\mathcal{O}_{D_i})$, for $i = 0$ and 1, have the same kernel. By Lemma 2.1, this condition can be rephrases as

$$\mathrm{Gr}_F^0 \ker g_0 = \mathrm{Gr}_F^0 \ker g_1,$$

where $g_i\colon H^{n-1}(Y, \mathbb{C}) \to H^{n-1}(D_i, \mathbb{C})$, $i = 0$ and 1, are the restriction maps. This last equality is the topological statement we have alluded to before. It is proved using a mixed Hodge structures argument that involves the reduced schemes D_0 and D_1 and their resolutions.

In [F, Appendix to §2], the above claim is proved in the case $n = 2$ using an idea that involves Abelian varieties. The reason that Theorem 1.1 can be extended to arbitrary dimension is that this idea admits a mixed Hodge structures interpretation.

2. An extension of Theorem 1.1 to dimension n

Let X be a nonsingular complex projective variety of dimension n, and let D be an effective divisor on X. For any integer $k \geq 0$, consider the following diagram

$$\begin{array}{ccc} H^k(X, \mathcal{O}_X) & \xrightarrow{f} & H^k(D, \mathcal{O}_D) \\ \uparrow & & \uparrow \\ H^k(X, \mathbb{C}_X) & \xrightarrow{g} & H^k(D, \mathbb{C}_D) \end{array} \tag{2.1}$$

The vertical maps are the natural homomorphisms induced by inclusions of sheaves. The horizontal maps are just restrictions. By Hodge Theory, the map $H^k(X, \mathbb{C}_X) \to H^k(X, \mathcal{O}_X)$ can be identified with the projection $H^k(X, \mathbb{C}) \to H^{0,k}(X)$. We use this identification in the sequel, both in Lemma 2.1 below and in the proof of Theorem 2.2.

The following lemma gives a relation between the coherent cohomology and the constant sheaf cohomology of D. It implies that $\ker f$ depends only on the reduced structure of D. When $n = 2$ and $k = 1$, the latter statement is a very useful result of Ramanujam [Ra, Lemma 6].

Lemma 2.1. *With the above assumptions and notation, we have*

$$\ker f = \ker g \cap H^{0,k}(X).$$

Proof. Let D_{red} be the reduced subscheme of D, and denote by f' the composite map

$$\begin{array}{ccc} H^k(X, \mathcal{O}_X) & \xrightarrow{f} & H^k(D, \mathcal{O}_D) \\ & \searrow^{f'} & \downarrow \\ & & H^k(D, \mathcal{O}_{D_{\mathrm{red}}}) \end{array}$$

The first point to note is that $\ker g \cap H^{0,k}(X) \subseteq \ker f \subseteq \ker f'$, which follows just by diagram chasing. In particular

$$\ker f' = \ker g \cap H^{0,k}(X) \Longrightarrow \ker f = \ker g \cap H^{0,k}(X).$$

We may assume, therefore, that D is reduced. Let then $\psi \colon \widetilde{D} \to D$ be a resolution of singularities. Complete diagram 2.1 to

$$\begin{array}{ccccc} H^k(X, \mathcal{O}_X) & \xrightarrow{f} & H^k(D, \mathcal{O}_D) & \xrightarrow{\psi^*} & H^k(\widetilde{D}, \mathcal{O}_{\widetilde{D}}) \\ \uparrow & & \uparrow & & \uparrow \\ H^k(X, \mathbb{C}) & \xrightarrow{g} & H^k(D, \mathbb{C}) & \xrightarrow{h} & H^k(\widetilde{D}, \mathbb{C}) \end{array} \tag{2.2}$$

and set $f'' := \psi^* \circ f$.

The inclusion $\ker f \supseteq \ker g \cap H^{0,k}(X)$ follows from the commutativity of the first square of diagram (2.2). For the opposite inclusion, we need to show that $\ker f \subset \ker g$. To this end, note first that

$$\ker f \subset \ker f'' \subset \ker(h \circ g).$$

The second row of diagram (2.2) is an exact sequence of mixed Hodge structures. Use the weight filtration and the fact that $H^k(X, \mathbb{C})$ is pure to obtain

$$0 = g(W_{k-1} H^k(X, \mathbb{C})) = \operatorname{im} g \cap W_{k-1} H^k(D, \mathbb{C}) = \operatorname{im} g \cap \ker h.$$

Thus $\ker(h \circ g) = \ker g$, which concludes the proof. $\square$

The following result generalizes Theorem 1.1 to arbitrary dimension. The idea of the proof is to first use the previous lemma to reduce the statement to a topological one, and then establish the topological version by a mixed Hodge structures argument.

Theorem 2.2. *Let X be a nonsingular complex projective variety of dimension n, and let D be an effective divisor on X. Assume that D has a singular point p of multiplicity at least n. Let $\varphi\colon Y \to X$ be the blowup of X at p, and denote by E the exceptional divisor of φ. Consider the divisors D_0 and D_1 on Y given by*

$$D_0 = \varphi^* D - (n-1)E \quad \text{and} \quad D_1 = \varphi^* D - nE.$$

Then p is not a base point of $|K_X + D|$ if and only if the restriction map

$$H^{n-2}(\mathcal{O}_{D_0}) \to H^{n-2}(\mathcal{O}_{D_1})$$

is surjective. Equivalently, $p \notin \mathrm{Bs}|K_X + D|$ if and only if the map

$$H^{n-1}(\mathcal{O}_{D_0}) \to H^{n-1}(\mathcal{O}_{D_1})$$

is not injective.

Proof. Before going to the main part of the proof, note that the conditions involving the cohomology of the divisors D_0 and D_1 in degrees $n-2$ and $n-1$ are indeed equivalent. This follows from the decomposition sequence for $D_0 = D_1 + E$

$$0 \to \mathcal{O}_E(-D_1) \to \mathcal{O}_{D_0} \to \mathcal{O}_{D_1} \to 0,$$

once we notice that $E \cong \mathbb{P}^{n-1}$ and $\mathcal{O}_E(-D_1) \cong \mathcal{O}(-n)$. For then part of the associated long exact sequence in cohomology reads

$$0 \to H^{n-2}(\mathcal{O}_{D_0}) \to H^{n-2}(\mathcal{O}_{D_1}) \to \mathbb{C} \to H^{n-1}(\mathcal{O}_{D_0}) \to H^{n-1}(\mathcal{O}_{D_1}) \to 0,$$

and thus the (injective) map

$$H^{n-2}(\mathcal{O}_{D_0}) \to H^{n-2}(\mathcal{O}_{D_1})$$

is surjective if and only if the (surjective) map

$$H^{n-1}(\mathcal{O}_{D_0}) \to H^{n-1}(\mathcal{O}_{D_1})$$

is not injective.

For the main part of the proof, recall first that the condition $p \notin \mathrm{Bs}|K_X + D|$ can be rephrased as a cohomological condition on the blown up variety Y, as follows. Start from the structure sequence of p on X, tensored by $\mathcal{O}_X(K_X + D)$. By taking the associated cohomology exact sequence, $|K_X + D|$ is free at p if and only if the map

$$H^1(X, \mathcal{O}_X(K_X + D) \otimes \mathcal{I}_p) \to H^1(X, \mathcal{O}_X(K_X + D))$$

is injective. As $K_Y + D_0 = \varphi^*(K_X + D)$ and $D_0 = D_1 + E$, the last condition is equivalent to the injectivity of the map

$$H^1(Y, \mathcal{O}_Y(K_Y + D_1)) \to H^1(Y, \mathcal{O}_Y(K_Y + D_0)).$$

Finally, use Serre duality to obtain that $p \notin \mathrm{Bs}|K_X + D|$ if and only if the map

$$H^{n-1}(Y, \mathcal{O}_Y(-D_1)) \leftarrow H^{n-1}(Y, \mathcal{O}_Y(-D_0)$$

is surjective.

As in the surface case, the main point of the proof is to justify the following.

Claim 2.3. *Let $f_i\colon H^{n-1}(\mathcal{O}_Y) \to H^{n-1}(\mathcal{O}_{D_i})$, for $i = 0$ and 1, be the restriction maps. Then*

$$\ker f_0 = \ker f_1.$$

Assuming the claim, the proof of the theorem can be concluded in the same way as in [F, Appendix to §2]. We reproduce the argument below. The commutative diagram

$$\begin{array}{ccccccccc} H^{n-2}(\mathcal{O}_Y) & \longrightarrow & H^{n-2}(\mathcal{O}_{D_0}) & \longrightarrow & H^{n-1}(\mathcal{O}_Y(-D_0)) & \longrightarrow & \ker f_0 & \longrightarrow & 0 \\ \| & & \downarrow & & \downarrow & & \| & & \\ H^{n-2}(\mathcal{O}_Y) & \longrightarrow & H^{n-2}(\mathcal{O}_{D_1}) & \longrightarrow & H^{n-1}(\mathcal{O}_Y(-D_1)) & \longrightarrow & \ker f_1 & \longrightarrow & 0 \end{array}$$

shows that the map

$$H^{n-1}(Y, \mathcal{O}_Y(-D_0)) \to H^{n-1}(Y, \mathcal{O}_Y(-D_1)$$

is surjective if and only if the map

$$H^{n-2}(\mathcal{O}_{D_0}) \to H^{n-2}(\mathcal{O}_{D_1})$$

is surjective.

In [F, Appendix to §2], the above claim is proved in the case $n = 2$ using an idea that involves Abelian varieties. In the proof below, we replace that argument by an argument involving mixed Hodge structures, which works in any dimension.

Denote by g_i the restriction maps $g_i\colon H^{n-1}(Y, \mathbb{C}) \to H^{n-1}(D_i, \mathbb{C})$ for $i = 0, 1$. By Lemma 2.1, the conclusion of the claim is equivalent to

$$\ker g_0 \cap H^{0,k}(Y) = \ker g_1 \cap H^{0,k}(Y) \tag{2.3}$$

This is clearly true if $\mathrm{Supp}\, D_0 = \mathrm{Supp}\, D_1$. Denote by D' the proper transform of D, and by m the multiplicity of D at p. Then $D_0 = D' + (m-n+1)E$, $D_1 = D' + (m-n)E$, and the divisors D_0 and D_1 have the same support unless $m = n$. For the remainder of the proof, suppose $m = n$.

We may assume that D is reduced. Both $\ker g_0$ and $\ker g_1$ are Hodge substructures of $H^{n-1}(Y, \mathbb{C})$, and (2.3) can be rephrased as

$$\mathrm{Gr}_F^0 \ker g_0 = \mathrm{Gr}_F^0 \ker g_1.$$

Let $\psi_0\colon \widetilde{D_0} \to D_0$ be a resolution of singularities which factors through the disjoint union of the irreducible components of D_0. Choose ψ_0 such that E is one of the components of the resolution, and denote $\widetilde{D_1} := \widetilde{D_0} \setminus E$. The restriction map $\psi_1\colon \widetilde{D_1} \to D_1$ is then a resolution of singularities of D_1.

For the remainder of the proof, all cohomology groups considered are with $\mathbb{C}$ coefficients. The following diagram is a diagram of mixed Hodge structures:

$$\begin{array}{ccccccc}
\ker g_0 & \hookrightarrow & H^{n-1}(Y) & \xrightarrow{g_0} & H^{n-1}(D_0) & & \\
\downarrow & & \| & & \downarrow h & \searrow h_0 & \\
\ker g_1 & \hookrightarrow & H^{n-1}(Y) & \xrightarrow{g_1} & H^{n-1}(D_1) & & \\
 & & & & \downarrow h_1 & & \\
 & & & & H^{n-1}(\widetilde{D_1}) & \longleftarrow & H^{n-1}(\widetilde{D_0})
\end{array}$$

Here $\ker g_0 \subset \ker g_1 \subset H^{n-1}(Y)$ and $H^{n-1}(\widetilde{D_i})$, for $i = 0, 1$, are pure Hodge structures of weight $n-1$. The mixed Hodge structures on $H^{n-1}(D_i)$, for $i = 0, 1$, satisfy $W_{n-2}H^{n-1}(D_i) = \ker h_i$. If we apply Gr^W_{n-1} to the above diagram, the maps induced by the h_i become injective. The projection map

$$H^{n-1}(\widetilde{D_0}) = H^{n-1}(\widetilde{D_1}) \oplus H^{n-1}(E) \to H^{n-1}(\widetilde{D_1})$$

is not an isomorphism, unless n is even. It becomes an isomorphism, however, if we apply Gr^0_F, since the cohomology classes of $H^{n-1}(E)$ for n odd are of type $((n-1)/2, (n-1)/2)$.

Combine the above observations, and denote the induced maps to the corresponding graded quotiens by the same symbols, to obtain the following diagram:

$$\begin{array}{ccccccc}
\mathrm{Gr}^0_F \ker g_0 & \hookrightarrow & \mathrm{Gr}^0_F H^{n-1}(Y) & \xrightarrow{g_0} & \mathrm{Gr}^0_F \mathrm{Gr}^W_{n-1} H^{n-1}(D_0) & & \\
\downarrow & & \| & & \downarrow h & \searrow h_0 & \\
\mathrm{Gr}^0_F \ker g_1 & \hookrightarrow & \mathrm{Gr}^0_F H^{n-1}(Y) & \xrightarrow{g_1} & \mathrm{Gr}^0_F \mathrm{Gr}^W_{n-1} H^{n-1}(D_1) & & \\
 & & & & \downarrow h_1 & & \\
 & & & & \mathrm{Gr}^0_F H^{n-1}(\widetilde{D_1}) & \xleftarrow{\cong} & \mathrm{Gr}^0_F H^{n-1}(\widetilde{D_0})
\end{array}$$

The quotient vector space $\mathrm{Gr}^0_F \ker g_1 / \mathrm{Gr}^0_F \ker g_0$ can be identified with a subspace of $\ker h$. The conclusion of the theorem now follows since

$$\frac{\mathrm{Gr}^0_F \ker g_1}{\mathrm{Gr}^0_F \ker g_0} \subset \ker h \subset \ker h_0 = 0.$$

□

Remark 2.4. It would be very useful to have a correct higher dimensional interpretation of the numerical connectedness of surfaces. There are certain notions in the literature, but they do not seem to fit well in the present context.

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