Recurrence and Rate of Growth

We will develop techniques for finding the rate of growth of a given function $T(n)$ with nonnegative real values which is given by a recurrence relation i.e. where $T(n)$ is given in terms of the values $T(m)$ where $m < n$. We will concentrate on recurrence relations which have one of the following forms

\[ T(n) = aT(n-b) + f(n) \quad (n \geq b) \]

\[ T(n) = aT\left(\left\lfloor \frac{n}{b} \right\rfloor \right) + f(n) \quad (n \geq b) \]

where $a$ and $b$ are positive integers and $b > 1$ in the second relation.

Consider the special case

\[ T(n) = aT(n-1) + f(n) \]

for $n > 0$. This equation can be represented visually by the tree in Figure 1. Notice that the top node has $a$ descendants.

\[ \begin{align*}
  &f(n) \\
  &\quad \quad \quad \ldots \\
  &\quad \quad T(n-1) \quad T(n-1) \quad T(n-1)
\end{align*} \]

Figure 1: First level of recursion tree for $T(n) = aT(n-1) + f(n)$.

We can expand $T(n-1)$ to get

\[ T(n) = f(n) + af(n-1) + a^2T(n-2) \]

which is illustrated in Figure 2. Continuing we eventually get

\[ T(n) = f(n) + af(n-1) + \cdots + a^k f(n-k) + \cdots + a^{n-1} f(1) + a^n T(0) \]
Figure 2: Second level of recursion tree for $T(n) = aT(n-1) + f(n)$.

This can be seen clearly by considering the recursion tree in Figure 3. Once again, notice that each node other than those on the bottom level has a descendants on the next level below.

**Example 1.** Suppose $T(n)$ satisfies

$$T(n) = T(n-1) + n$$

for $n > 0$. The recursion tree in Figure 4 shows that

$$T(n) = n + (n-1) + \cdots + 1 + T(0)$$

which is in $O(n^2)$.

Let’s return to the general recurrence

$$T(n) = aT(n-b) + f(n) \quad (n \geq b)$$

First, consider the case when $n$ is divisible by $b$. Say, $n = q \cdot b$. The line of reasoning above establishes that

$$T(n) = f(n) + af(n-b) + \cdots + a^{k}f(n-k\cdot b) + \cdots + a^{q-1}f(b) + a^{q}T(0)$$

In fact, we can establish a formula for the general case. If $n = q \cdot b + r$ where $0 \leq r < b$ then

$$T(n) = f(n) + af(n-b) + \cdots + a^{k}f(n-k\cdot b) + \cdots + a^{q-1}f(n-(q-1)\cdot b) + a^{q}T(r)$$

Notice that $r = n - q \cdot b$. 

2
Figure 3: Recursion tree for $T(n) = aT(n - 1) + f(n)$.

**Example 2.** Suppose $T(n)$ satisfies

$$T(n) = 4T(n - 2) + n^3$$

for all $n > 1$. Assume that $n$ is divisible by 2 so that $n = 2 \cdot k$ for some natural number $k$. The recursion tree in Figure 5 shows that

$$
T(2 \cdot k) = 8 \cdot k^3 + 4 \cdot 8 \cdot (k - 1)^3 + 4^2 \cdot 8 \cdot (k - 2)^3 + \cdots + 4^{k-1} \cdot 8 \cdot 1^3 + 4^k \cdot T(0) \\
= 8 \cdot 4^k \left( \frac{k^3}{4^0} + \frac{(k-1)^3}{4^1} + \cdots + \frac{1^3}{4^{k-1}} \right) + 4^k \cdot T(0) \\
= (8 \cdot \frac{k^3}{4^0} + \frac{(k-1)^3}{4^1} + \cdots + \frac{1^3}{4^{k-1}}) + T(0) \cdot 4^k
$$

Since the infinite sum $\sum_{i=1}^{\infty} \frac{i^3}{4^i}$ converges (e.g. by the ratio test), the function given by the final expression, recalling that $k = \frac{n}{2}$, is in $\Theta(2^n)$. This seems to indicate that $T(n)$ is in $\Theta(2^n)$. However, the equation only holds for even integers. We can find a similar expression for $T(n)$ which is in $\Theta(2^n)$ when $n$ has the form $2k + 1$. Since every natural number has one of the two forms ($2k$ or $2k + 1$), we conclude that $T(n)$ is $\Theta(2^n)$ (see exercise 1).

We now turn our attention to recurrences of the second sort

$$T(n) = aT\left(\lfloor \frac{n}{b} \rfloor \right) + f(n) \quad (n \geq b)$$
We again try to estimate the order of growth using a recursion tree. The reader should sketch the tree for the case \( n = b^k \) for some natural number \( k \) (so \( k = \log_b n \)). It will look much like diagram 3. The recursion tree shows that

\[
T(n) = f(b^k) + a \cdot f(b^{k-1}) + \ldots + a_i \cdot f(b^{k-i}) + \ldots + a^{k-1} \cdot f(b) + a^k \cdot T(1)
\]

or, equivalently,

\[
T(n) = f(n) + a \cdot f \left( \frac{n}{b} \right) + \ldots + a^i \cdot f \left( \frac{n}{b^i} \right) + \ldots + a^{k-1} \cdot f \left( \frac{n}{b^{k-1}} \right) + a^k \cdot T(1)
\]

In fact, for general \( n \) we have the following expression for \( T(n) \)

\[
f(n) + a \cdot f \left( \left\lfloor \frac{n}{b^1} \right\rfloor \right) + \ldots + a^i \cdot f \left( \left\lfloor \frac{n}{b^i} \right\rfloor \right) + \ldots + a^{k-1} \cdot f \left( \left\lfloor \frac{n}{b^{k-1}} \right\rfloor \right) + a^k \cdot T \left( \left\lfloor \frac{n}{b^k} \right\rfloor \right)
\]

where \( k = \lfloor \log_b n \rfloor \). In the following examples, we will estimate the order of growth of the given function using a recursion tree.

**Example 3.** Suppose \( T(n) \) satisfies

\[
T(n) = 2T \left( \left\lfloor \frac{n}{2} \right\rfloor \right) + c \cdot n
\]

where \( c > 0 \). Consider the case \( n = 2^k \) for some natural number \( k \). The recursion tree in diagram 6 shows that

\[
T(n) = c \cdot n + 2 \cdot c \cdot \frac{n}{2} + 2^2 \cdot c \cdot \frac{n}{2^2} + \ldots + 2^{k-1} \cdot c \cdot \frac{n}{2^{k-1}} + 2^k \cdot T(1)
\]

implying that \( T(n) = k \cdot n \cdot c + 2^k \cdot T(1) = n(k \cdot c + T(1)) \). Since \( k = \lg n \)

\[
T(n) = n(c \cdot \lg n + T(1))
\]
for \( n \) of the form \( 2^k \). The final expression determines a function in \( \Theta(n \lg n) \). So we estimate that \( T(n) \) is in \( \Theta(n \lg n) \).

**Example 4.** Suppose \( T(n) \) satisfies

\[
T(n) = 4T\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + c \cdot n^2
\]

where \( c > 0 \). Consider the case \( n = 2^k \) for some natural number \( k \). The recursion tree shows that

\[
T(n) = c \cdot n^2 + 4 \cdot c \cdot \left(\frac{n}{2}\right)^2 + 4^2 \cdot c \cdot \left(\frac{n}{2^2}\right)^2 + \cdots + 4^{k-1} \cdot c \cdot \left(\frac{n}{2^{k-1}}\right)^2 + 4^k \cdot T(1)
\]

implying that

\[
T(n) = k \cdot n^2 \cdot c + 4^k \cdot T(1) = n^2(k \cdot c + T(1))
\]

Since \( k = \lg n \)

\[
T(n) = n^2(c \cdot \lg n + T(1))
\]

for \( n \) of the form \( 2^k \). The final expression determines a function in \( \Theta(n^2 \lg n) \). So we estimate that \( T(n) \) is in \( \Theta(n^2 \lg n) \).
Example 5. Suppose \( T(n) \) satisfies
\[
T(n) = 2T\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + c \cdot n^2
\]
where \( c > 0 \). Consider the case \( n = 2^k \) for some natural number \( k \). The recursion tree shows that
\[
T(n) = c \cdot n^2 + 2 \cdot c \cdot \left(\frac{n}{2}\right)^2 + 2^2 \cdot c \cdot \left(\frac{n}{2^2}\right)^2 + \cdots + 2^{k-1} \cdot c \cdot \left(\frac{n}{2^{k-1}}\right)^2 + 2^k \cdot T(1)
\]
implying that
\[
T(n) = c n^2 \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{k-1}}\right) + n \cdot T(1)
\]
Since \( \sum_{i=0}^{\infty} \frac{1}{2^i} = 2 \)
\[
cn^2 \leq T(n) \leq 2cn^2 + nT(1)
\]
for any \( n \) of the form \( 2^k \). Therefore, we estimate that \( T(n) \) is in \( \Theta(n^2) \).

Example 6. Suppose \( T(n) \) satisfies
\[
T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + c
\]
where \( c > 0 \). Consider the case \( n = 2^k \) for some natural number \( k \). The recursion tree shows that
\[
T(n) = c + c + \cdots + c + T(1)
\]
where there are \( k \) occurrences of \( c \) in the sum on the right hand side. Therefore, \( T(n) = ck + T(1) = c \lg n + T(1) \). We estimate that \( T(n) \) is in \( \Theta(\lg n) \).

Example 7. Suppose \( T(n) \) satisfies
\[
T(n) = 9T\left(\left\lfloor \frac{n}{3} \right\rfloor \right) + c \cdot n
\]
where \( c > 0 \). Consider the case \( n = 3^k \) for some natural number \( k \). The recursion tree shows that
\[
T(n) = c \cdot n + 9 \cdot c \cdot \frac{n}{3} + 9^2 \cdot c \cdot \frac{n}{3^2} + \cdots + 9^{k-1} \cdot c \cdot \frac{n}{3^{k-1}} + 9^k \cdot T(1)
\]
implying that
\[
T(n) = cn(1 + 3 + 3^2 + \cdots + 3^{k-1}) + n^2 \cdot T(1)
\]
\[
= cn\left(\frac{3^k-1}{3-1}\right) + n^2 \cdot T(1)
\]
\[
= cn\left(\frac{n^k}{2^k}\right) + n^2 \cdot T(1)
\]
Since the final expression defines a function in $\Theta(n^2)$, we estimate that $T(n)$ is in $\Theta(n^2)$.

To prove our estimates were correct in the previous examples would be somewhat tedious. Instead we can use the following theorem.

**Theorem** Assume $a \geq 1$ and $b > 1$ are integers, $f(n)$ is a given function with nonnegative real values and $T(n)$ is defined on the natural numbers to satisfy the recurrence

$$T(n) = aT\left(\left\lfloor \frac{n}{b} \right\rfloor \right) + f(n)$$

for $n \geq b$.

1. If $f(n)$ is in $O(n^{k-ba-\epsilon})$ for some $\epsilon > 0$, then $T(n)$ is in $\Theta(n^{k-ba})$.
2. If $f(n)$ is in $\Theta(n^{k-ba})$, then $T(n)$ is in $\Theta(n^{k-ba} \log n)$.
3. If $f(n)$ is in $\Omega(n^{k-ba+\epsilon})$ for some $\epsilon > 0$ and if $a f\left(\left\lfloor \frac{n}{b} \right\rfloor \right) \leq c f(n)$ for some constant $c < 1$ and sufficiently large $n$, then $T(n)$ is in $\Theta(f(n))$.

The proof of the theorem is beyond the scope of these notes.

**Exercises.**

1. Suppose $q$ is a positive natural number and $f_r(n)$ is in $\Theta(g(n))$ for all natural numbers $0 \leq r < q$. Show that if $T(n) = f_r(n)$ whenever $n \mod q$ is $r$ then $T(n)$ is in $\Theta(g(n))$.

   For exercises 2-5, draw a recursion tree for the recurrence, find a sum which equals $T(n)$ and estimate the rate of growth of $T(n)$.

2. Suppose $T(n)$ satisfies

$$T(n) = T(n-2) + 5n^3 + 3n$$

for $n \geq 2$.

3. Suppose $T(n)$ satisfies

$$T(n) = T(n-3) + 14n + 6$$

for $n \geq 3$.

4. Suppose $T(n)$ satisfies

$$T(n) = T(n-4) + 3\log n$$

for $n \geq 4$. 

7
5. Suppose \( T(n) \) satisfies
\[
T(n) = 2T(n - 1) + 3n
\]
for \( n \geq 1 \).

For the remaining exercises, draw a recursion tree for the recurrence, find a sum which equals \( T(n) \) when \( n \) has the form \( b^k \) (your sum should not involve \( k \) – rewrite it as \( \log_b n \)) and estimate the rate of growth of \( T(n) \).

6. Suppose \( T(n) \) satisfies
\[
T(n) = T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + 5
\]
for \( n \geq 3 \).

7. Suppose \( T(n) \) satisfies
\[
T(n) = T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + 4n
\]
for \( n \geq 3 \).

8. Suppose \( T(n) \) satisfies
\[
T(n) = T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + \log n
\]
for \( n \geq 3 \).

9. Suppose \( T(n) \) satisfies
\[
T(n) = 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 3\log n
\]
for \( n \geq 2 \).

10. Suppose \( T(n) \) satisfies
\[
T(n) = 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 2n + \log n
\]
for \( n \geq 2 \).

11. Suppose \( T(n) \) satisfies
\[
T(n) = 4T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 2n^3 + n^2
\]
for \( n \geq 2 \).

12. Suppose \( T(n) \) satisfies
\[
T(n) = 4T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 2n + 1
\]
for \( n \geq 2 \).
13. Suppose $T(n)$ satisfies

$$T(n) = 9T\left(\left\lfloor \frac{n}{3} \right\rfloor \right) + 4n^2 + 1$$

for $n \geq 3$.

14. Suppose $T(n)$ satisfies

$$T(n) = 4T\left(\left\lfloor \frac{n}{3} \right\rfloor \right) + 4n^2 + 1$$

for $n \geq 3$. 