Elementary Patterns of Resemblance

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Abstract

We will study patterns which occur when considering how Σ_1-elementary substructures arise within hierarchies of structures. The order in which such patterns evolve will be seen to be independent of the hierarchy of structures provided the hierarchy satisfies some mild conditions. These patterns form the lowest level of what we call *patterns of resemblance*. They were originally used by the author to verify a conjecture of W. Reinhardt concerning epistemic theories (see [5] and [6]), but their relationship to axioms of infinity and usefulness for ordinal analysis were manifest from the beginning. This paper is the first part of a series which provides an introduction to an extensive program including the ordinal analysis of set theories. Future papers will conclude the introduction and establish, among other things, that notations we will derive from the patterns considered here represent the proof-theoretic ordinal of the theory $KP\ell_0$ or, equivalently, $\Pi^1_1 - CA_0$ (as $KP\ell_0$ is a conservative extension of $\Pi^1_1 - CA_0$).

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1 Introduction

Consider the structure $\mathcal{R}_0 = (ORD, 0, +, \leq)$ on the collection of ordinals ORD, with the ordering $\leq$ of ordinals, and the usual operation of ordinal addition $+$. Using $\mathcal{B} \preceq_{\Sigma_1} \mathcal{C}$ to indicate that $\mathcal{B}$ is a Σ_1-elementary substructure of $\mathcal{C}$, inductively define a binary relation $\leq_1$ on ORD so that

$$\alpha \leq_1 \beta \text{ iff } (\alpha, 0, +, \leq, \leq_1) \preceq_{\Sigma_1} (\beta, 0, +, \leq, \leq_1)$$

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for all ordinals α and β (the restriction of the expanded structure to β is defined by induction on β). To clarify this definition, interpret $+$ by the graph of ordinal addition. Let $\mathcal{R}_1$ be the structure $(ORD, 0, +, \leq, \leq_1)$.

Let $\leq_{pw}$ be the *pointwise partial ordering* of finite sets of ordinals where $A \leq_{pw} B$ iff A and B have the same cardinality and if $\alpha_0, \ldots, \alpha_{n-1}$ enumerates the elements of A in increasing order and $\beta_0, \ldots, \beta_{n-1}$ enumerates the elements of B in increasing order then $\alpha_i \leq \beta_i$ for $i < n$. A substructure $\mathbf{A}$ of $\mathcal{R}_1$ is *closed* if $0 \in \mathbf{A}$ and whenever $\alpha_1 + \cdots + \alpha_m$ is in $\mathbf{A}$ where $\alpha_1, \ldots, \alpha_m$ are indecomposable ordinals with $\alpha_1 \geq \cdots \geq \alpha_m$ then $\alpha_1, \ldots, \alpha_m \in \mathbf{A}$ and $\alpha_1 + \cdots + \alpha_i \in A$ for $i = 1, \ldots, m$. Notice that every finite set of ordinals is contained in a finite set of ordinals which is closed. A finite substructure of $\mathcal{R}_1$ which is minimal in the pointwise ordering of the collection of all finite substructures of $\mathcal{R}_1$ which are isomorphic to it will be called *isominimal.* We will refer to the set of ordinals which occur in some isominimal substructure of $\mathcal{R}_1$ as the *core* of $\mathcal{R}_1$.

We will see that for a fixed finite closed substructure $\mathbf{P}$ of $\mathcal{R}_1$, there is a unique isominimal substructure $\mathbf{P}^*$ of $\mathcal{R}_1$ which is isomorphic to $\mathbf{P}$. Moreover, $\mathbf{P}^*$ is closed. This provides a system of notations for the ordinals which occur in the core of $\mathcal{R}_1$: if α appears as the nth element of some closed isominimal substructure $\mathbf{A}$ of $\mathcal{R}_1$ we can use the pair (τ, n) as a notation for α where τ is the isomorphism type of $\mathbf{A}$ (we will actually use a slight variant of this idea). These notations will allow us to show that every proper initial segment of the core of $\mathcal{R}_1$ is isomorphic to a recursive structure and, under certain set theoretic assumptions, the entire core is isomorphic to a recursive structure.

$KP + Infinity$, our base theory for set-theoretic results, is not enough to guarantee that the notation system derived above is recursive. However, one is naturally lead to an extension which is recursive. Assuming $KP\ell_0$ is Π^1_1-sound (by an observation of M. Rathjen, the exact condition under which the notion of the ordinal of a theory is significant), the two systems coincide. This will be established elsewhere. We will prove a weaker result here: the two systems coincide if we assume ZFC.

Which ordinals appear in the core? If κ is the least ordinal such that $\kappa \leq_1 \infty$, i.e. $\kappa \leq_1 \beta$ whenever $\kappa \leq \beta$, then one easily sees that any ordinal in the core must be below κ. Conversely, we will show that every ordinal below κ is in the core so that the core is exactly κ. Elsewhere, we will show that κ is the ordinal of $KP\ell_0$.

The core of $\mathcal{R}_1$ is robust in the sense that, assuming ZFC, any reasonable analogue of $\mathcal{R}_1$ has a core which is isomorphic to the core of $\mathcal{R}_1$ (though the characterization of the core as an initial segment of the ordinals may not hold). An interesting case is when we interpret Σ_1 in the usual set-theoretic sense by allowing arbitrary bounded quantifiers inside the initial existential quantifiers and define $\alpha \preceq \beta$ iff $L_\alpha \preceq_{\Sigma_1} L_\beta$ (where L_ξ is the ξ^{th} level of the constructible hierarchy). The core of $(ORD, 0, +, \leq, \preceq)$ is isomorphic to the core of $\mathcal{R}_1$.

One can avoid using formulas when proving facts about $\leq_1$ by noticing that for two structures $\mathcal{A}$ and $\mathcal{B}$ for a given finite first-order language with $\mathcal{A}$ a substructure of $\mathcal{B}$, $\mathcal{A} \preceq_{\Sigma_1} \mathcal{B}$ iff for all finite sets $X \subseteq |\mathcal{A}|$ and $Y \subseteq |\mathcal{B}| - |\mathcal{A}|$ there exists $\tilde{Y} \subseteq |\mathcal{A}|$ and an isomorphism of $X \cup Y$ with $X \cup \tilde{Y}$ which fixes the elements of X. In particular, $\alpha \leq_1 \beta$ iff for all finite sets $X \subseteq \alpha$ and $Y \subsetneq [\alpha, \beta)$ there exists $\tilde{Y} \subseteq \alpha$ with $X < \tilde{Y}$ such that $X \cup Y$ is isomorphic to $X \cup \tilde{Y}$.

To help get a feel for $\leq_1$ the reader might want to verify some of the following:

- If $\alpha \leq_1 \beta$ and $\alpha \leq \gamma \leq \beta$ then $\alpha \leq_1 \gamma$.
- If $\alpha \leq_1 \alpha + 1$ then α is of the form ω^λ where λ is a limit ordinal.
- More generally, for $0 < \xi < \alpha$, if $\alpha \leq_1 \alpha + \xi$ then α has the form $\omega^{\omega^\xi \cdot \beta}$.
- If $0 < \alpha$ and $\alpha \leq_1 \alpha + \alpha$ then α is an epsilon number.
- If $0 < \alpha$ and $\alpha \leq_1 \alpha + \alpha + \alpha$ then α is a fixed point of the function which enumerates the epsilon numbers.
- More generally, if $0 < \xi < \alpha$ and $\alpha \leq_1 \alpha \cdot (1 + \xi)$ then α has the form $\varphi(\xi, \beta)$.
- If $0 < \alpha$ and $\alpha \leq_1 \alpha^2$ then α is of the form Γ_β.

In fact, the converses of these statements are also true, but somewhat more difficult to prove.

There are generalizations of the constructions and the results obtained here when we start with a structure $(ORD, f_i(i \in I), \leq)$ where the f_i generate all of ORD by a closed class of order indiscernibles in a "continuous" way (corresponding to a flower in the terminology of dilators). There are two cases of particular interest. The first is when there are no function symbols whatsoever. Results in [5] show that the core in this case is ε_0. When providing a proof-theoretic analysis of a theory, having the Veblen function φ explicit in the notation system is desirable for calculations related to cut-elimination. Since the notations mentioned above do not include φ, we would need to define a version of φ within our notations in order to carry out a proof-theoretic anlaysis of $KP\ell_0$ (this is possible, but aesthetically unpleasant). This leads us to the second case which interests us: we modify $\mathcal{R}_0$ by adding φ. Elsewhere, where we provide an ordinal analysis of $KP\ell_0$, we will use the notations arising in this case. There is a trade off in that when φ is included many new details arise in the matters we are concerned with in this paper. For that reason, we will first consider the situation where $\mathcal{R}_0$ is defined as above and later describe how the arguments generalize when φ is added. Elsewhere we will establish that whether φ is included or not, the same ordinal arises i.e. the ordinal κ which determines the core is the same in both cases.

The contents are organized as follows. A characterization of the isomorphism types of closed finite substructures of $\mathcal{R}_0$ is given in section 3. A characterization of finite structures which are isomorphic to a closed substructure of $\mathcal{R}_1$ is given in section 4 along with ways of generating new such structures from a given structure of this sort (that the characterization is truly a characterization can be proved in ZF but not in weak theories such as $KP + Infinity$). These structures are called *patterns of resemblance of order one*, or just *patterns* for short. Section 5 characterizes the core of $\mathcal{R}_1$ as the least κ such that $\kappa \leq_1 \infty$ when such κ exists and ORD otherwise. The key notion of *amalgamation* is presented in section 6 and used to show the core is isomorphic to a recursive structure assuming ZF. Existence of amalgamations in general is shown in section 7 by an elementary proof which allows the construction of an analogue $\mathcal{P}_1/=$ of the core in weak theories. $\mathcal{P}_1/=$ is isomorphic to the core assuming one is working in a reasonably strong theory (ZF is more than enough) and constitutes a system of ordinal notations for $KP\ell_0$. The well-founded part of $\mathcal{P}_1/=$ is studied in section 8 under the assumption of $KP+Infinity$. In section 9, we show that the core of any reasonable analogue of $\mathcal{R}_1$ is isomorphic to an initial segment of the core of $\mathcal{R}_1$ and is isomorphic to the core of $\mathcal{R}_1$ itself if ZF is assumed. Section 10 describes how to modify the results of sections 3-9 to the case when $\mathcal{R}_0$ is redefined as $(ORD, \leq)$ and also gives a different proof from that in [5] that the core is the ordinal ε_0 in this case. Sections 11 and 12 describe the modifications that must be made to sections 3-9 when $\mathcal{R}_1$ is redefined by adding the Veblen operation φ. In section 13, we briefly discuss connections of our work with dilators and the construction behind the generalization of patterns of resemblance of order one to all finite orders.

2 Preliminaries

KP will be used to denote Kripke-Platek set theory (see [1] for background) and $KP + Infinity$ is Kripke-Platek set theory with the axiom of infinity. $KP + Infinity$ is the base theory for the results in the paper of a set-theoretic nature. The theory $KP\ell_0$ has an axiomatization consisting of the usual axiomatization for $KP + Infinity$ with Δ_0-collection removed and an additional axiom saying that every set is an element of an admissible set. ZF denotes Zermelo-Fraenkel set theory.

For proof-theoretic results, we will use $I\Sigma_0(exp)$ as our base theory. $I\Sigma_0(exp)$ is the theory in the language of arithmetic, including exponentiation, whose principal axiom is $\Sigma_0(exp)$-induction (see [8]).

We now mention a few concepts which can be formalized in KP. ORD will denote the class of ordinals. 0 is the empty set, the least ordinal under the usual ordering $\leq$ of the ordinals. $+$ will denote the usual operation of ordinal addition. ε_0 is the least ordinal beyond ω which is closed under ordinal exponentiation. φ will be used to denote the *Veblen operation* on the ordinals:

- $\varphi(0, \alpha) = \omega^\alpha$.
- $\alpha \mapsto \varphi(\xi, \alpha)$ enumerates the ordinals which are fixed points of all maps $\alpha \mapsto \varphi(\eta, \alpha)$ for $\eta < \xi$.

For the basic properties of φ see [10].

0, +, φ, and $\leq$ will have the dual roles of being symbols in first-order languages. 0 will always be a constant symbol, $\leq$ a binary relation symbol, and both + and φ will be binary function symbols. In addition, $\leq_1$ will always be a binary relation symbol. If $\mathcal{L}$ is a first-order language and I is a set, we will use $\mathcal{L}_I$ to denote the expansion of $\mathcal{L}$ by the addition of new constants for each element of I. For notational convenience, we will identify each element of I with the corresponding constant. A *closed term* of $\mathcal{L}$ is a term of $\mathcal{L}$ with no occurrences of any variable. If $t_1, \ldots, t_n$ are terms, $t_1 + \cdots + t_n$ is the term obtained by grouping from the left e.g. $t_1 + t_2 + t_3$ is $((t_1 + t_2) + t_3)$.

Contrary to standard practice, we will allow structures for a first-order language $\mathcal{L}$ to interpret the function symbols as partial operations on the universe and to fail to give an interpretation to some constant symbols. In other words, we use the word "structure" to refer to what are called partial structures elsewhere.

We will allow two kinds of structures for a finite first-order language: those whose universe is a proper class and those whose universe is a set. We make the assumption that any structure whose universe is a set is itself a set e.g. the interpretation of any relation symbol must be a set and not simply a definable relation on the universe.

Assume $\mathbf{A}$ is a structure for a first-order language $\mathcal{L}$. If S is a nonlogical symbol of $\mathcal{L}$, we will use $S^{\mathbf{A}}$ to denote the interpretation of S in $\mathbf{A}$. However, we will usually drop the superscript $\mathbf{A}$ and simply write S for $S^{\mathbf{A}}$ when no confusion is likely. Similarly, we will usually allow $\mathbf{A}$ to also denote the universe of $\mathbf{A}$. The definition of when a closed term is defined in a structure is the natural one, proceeding from bottom up, as is the definition of the value of the term in the structure. See the theory of partial terms in [2] for details. We will take the liberty of allowing a closed term to sometimes play the dual role of denoting not only itself but also its value in $\mathbf{A}$ when the intended meaning is clear. For example, if X is a subset of the universe of $\mathbf{A}$ then writing $t \in X$ means that the value of t in $\mathbf{A}$ is an element of X. Also, if ψ is a sentence in $\mathcal{L}_{\mathbf{A}}$ we will write "ψ in $\mathbf{A}$" to indicate that ψ is true in $\mathbf{A}$. For example, if s and t are closed terms then $s = t$ in $\mathbf{A}$ iff both s and t are defined in $\mathbf{A}$ and their values in $\mathbf{A}$ are equal. We will sometimes even drop reference to $\mathbf{A}$ and write ψ to indicate that ψ is true in $\mathbf{A}$ when $\mathbf{A}$ is clear from the context. We expect the reader will have no problem determining the intended meaning in these situations.

We will need a generalization of the notion of structure. A *prestructure* $\mathbf{A}$ for a language $\mathcal{L}$ consists of a nonempty set $|\mathbf{A}|$, the *universe* of $\mathbf{A}$, an interpretation $S^{\mathbf{A}}$ for each nonlogical symbol S of $\mathcal{L}$, and an interpretation $=^{\mathbf{A}}$ of the equality symbol $=$ such that the following conditions hold.

- $=^{\mathbf{A}}$ is an equivalence relation on $|\mathbf{A}|$.
- The interpretation of a constant symbol is either an equivalence class of $=^{\mathbf{A}}$ or $\emptyset$.

- If f is an n-place function symbol then $f^{\mathbf{A}}$ is an $(n+1)$-ary relation on $|\mathbf{A}|$ such that
  - if $a_1, \ldots, a_{n+1}, b_1, \ldots, b_{n+1} \in |\mathbf{A}|$ and $a_i =^{\mathbf{A}} b_i$ for $i = 1, \ldots, n+1$ then $\langle a_1, \ldots, a_{n+1} \rangle \in f^{\mathbf{A}}$ iff $\langle b_1, \ldots, b_{n+1} \rangle \in f^{\mathbf{A}}$ and
  - if $\langle a_1, \ldots, a_{n+1} \rangle \in f^{\mathbf{A}}$, $\langle b_1, \ldots, b_{n+1} \rangle \in f^{\mathbf{A}}$, and $a_i =^{\mathbf{A}} b_i$ for $i = 1, \ldots, n$ then $a_{n+1} =^{\mathbf{A}} b_{n+1}$.
- If R is an n-place relation symbol then $R^{\mathbf{A}}$ is an n-ary relation on $|\mathbf{A}|$ such that
  - if $a_1, \ldots, a_n, b_1, \ldots, b_n \in |\mathbf{A}|$ and $a_i =^{\mathbf{A}} b_i$ for $i = 1, \ldots, n$ then $\langle a_1, \ldots, a_n \rangle \in R^{\mathbf{A}}$ iff $\langle b_1, \ldots, b_n \rangle \in R^{\mathbf{A}}$.

If $\mathbf{A}$ is a prestructure for $\mathcal{L}$ we define the structure $\mathbf{A}/=$ to be the structure whose universe is the set of equivalence classes of $=^{\mathbf{A}}$ such that the interpretations of the nonlogical symbols of $\mathcal{L}$ in $\mathbf{A}/=$ are made so that the map which sends an element of $\mathbf{A}$ into its equivalence class under $=^{\mathbf{A}}$ is a homomorphism (in the obvious sense) for the language obtained from $\mathcal{L}$ by removing the equality symbol.

Suppose A is a set and $\preceq_1$ is a binary relation on A. $\preceq_1$ is a *forest* on A if $\preceq_1$ is a partial ordering on A and for any $a \in A$ the set of predecessors of a with respect to $\preceq_1$, $\{x \in A \mid x \preceq_1 a\}$, is linearly ordered by $\preceq_1$. If $\preceq$ is a linear ordering of A then we say that $\preceq_1$ *respects* $\preceq$ if $\preceq_1$ is a subset of $\preceq$ and $a \preceq_1 x$ whenever $a \preceq_1 b$ and $a \preceq x \preceq b$.

For the rest of this section, suppose $\mathbf{A}$ is a structure for the first-order language $\mathcal{L}$, $\leq$ is one of the symbols of $\mathcal{L}$, and the interpretation of $\leq$ in $\mathbf{A}$ is a linear ordering of $\mathbf{A}$. We will write $<^{\mathbf{A}}$ for the strict part of $\leq^{\mathbf{A}}$ (and usually drop the superscript $\mathbf{A}$ as mentioned above). If X is a finite nonempty subset of $\mathbf{A}$, $max(X)$ will be the largest element of X and $min(X)$ will be the smallest element of X. A sequence $\langle a_1, \ldots, a_n \rangle$ of elements of $\mathbf{A}$ is said to be *descending* if $a_{i+1} \leq a_i$ in $\mathbf{A}$ whenever $1 \leq i < n$.

We will need two partial orderings on the finite subsets of $\mathbf{A}$. The first will be denoted by $\leq_{pw}^{\mathbf{A}}$ and is defined as $\leq_{pw}$ was in the introduction: $X \leq_{pw}^{\mathbf{A}} Y$ iff $card(X) = card(Y)$ and $x_i \leq y_i$ for $i = 1, \ldots, n$ where $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ enumerate X and Y respectively in increasing order. The second partial ordering will be denoted by $\leq_{lex}^{\mathbf{A}}$ and corresponds to the usual lexicographical ordering derived from $\leq^{\mathbf{A}}$ when we identify each finite subset of $\mathbf{A}$ with the sequence which enumerates it in decreasing order. We will omit the superscript $\mathbf{A}$ on both $\leq_{pw}^{\mathbf{A}}$ and $\leq_{lex}^{\mathbf{A}}$ when $\mathbf{A}$ is clear from the context.

A finite substructure $\mathbf{B}$ of $\mathbf{A}$ is *isominimal in* $\mathbf{A}$ if $\mathbf{B} = \mathbf{C}$ whenever $\mathbf{C}$ is a substructure of $\mathbf{A}$ which is isomorphic to $\mathbf{B}$ such that $\mathbf{C} \leq_{pw} \mathbf{B}$. Notice that if $\mathbf{A}$ is well-ordered by $\leq$ and $\mathbf{B}$ is a finite substructure of $\mathbf{A}$ then there is an isominimal substructure $\mathbf{C}$ of $\mathbf{A}$ such that $\mathbf{C} \leq_{pw} \mathbf{B}$ and $\mathbf{C}$ is isomorphic to $\mathbf{B}$. The *core* of $\mathbf{A}$ is the union of all the isominimal substructures of $\mathbf{A}$.

An element a of $\mathbf{A}$ is *decomposable* in $\mathbf{A}$ if there is a function symbol f of $\mathcal{L}$ and $a_1, \ldots, a_n \in \mathbf{A}$ where n is the arity of f such that $f^{\mathbf{A}}(a_1, \ldots, a_n) = a$ and $a_1, \ldots, a_n < a$. If a is not decomposable in $\mathbf{A}$ then a is *indecomposable* in $\mathbf{A}$.

Finally, a word on the style of our proofs. Most of the time we will tacitly assume the hypothesis of what we are proving. For example, if we are trying to prove

Assume A. If B then C.

we will often assume both A and B without saying so explicitly.

3 Arithmetic Structures: Addition

We begin by giving a definition which provides a characterization of the isomorphism types of finite closed substructures of $\mathcal{R}_0$. Except for comments regarding $\mathcal{R}_0$ after definition 3.5 and lemma 3.7, the results of this section can be formalized in $I\Sigma_0(exp)$.

Definition 3.1 A structure $\mathbf{A}$ for the language $\{0, +, \leq\}$ is *additive* provided the following conditions hold.

(1) $\leq$ is a linear ordering of $\mathbf{A}$.

(2) 0 is defined and is the minimal element of $\mathbf{A}$ with respect to $\leq$.

(3) For any $a \in \mathbf{A}$ there is a descending sequence $\langle a_1, \ldots, a_m \rangle$ of indecomposable elements of $\mathbf{A}$ such that $a = a_1 + \cdots + a_m$.

(4) Whenever $\langle a_1, \ldots, a_m \rangle$ and $\langle b_1, \ldots, b_n \rangle$ are descending sequences of indecomposables of $\mathbf{A}$ such that both $a_1 + \cdots + a_m$ and $b_1 + \cdots + b_n$ are defined then

  (a) $a_1 + \cdots + a_m \leq b_1 + \cdots + b_n \quad \text{iff} \quad \langle a_1, \ldots, a_m \rangle \ \leq_{lex} \ \langle b_1, \ldots, b_n \rangle$

  (b) If $n \neq 0$ then either $(a_1 + \cdots + a_m) + (b_1 + \cdots + b_n) \simeq a_1 + \cdots + a_i + b_1 + \cdots + b_n$ where i is maximal such that $b_1 \leq a_i$ or $a_1 < b_1$ in which case $(a_1 + \cdots + a_m) + (b_1 + \cdots + b_n) = b_1 + \cdots + b_n$.

The symbol $\simeq$ in condition (4)(b) is interpretated so that for expressions E_1 and E_2, $E_1 \simeq E_2$ iff either both expressions are defined and have equal values or neither expression is defined. As mentioned earlier, we have suppressed the superscript $\mathbf{A}$ on $\leq^{\mathbf{A}}$, $0^{\mathbf{A}}$, and $+^{\mathbf{A}}$ in the definition and have written $\mathbf{A}$ in places where $|\mathbf{A}|$ would have been proper.

Conditions (3) and (4) apply to 0 under the convention that $a_1 + \cdots + a_n$ is defined to be 0 when $n = 0$. In particular, $a + 0 = 0 + a = a$ for all $a \in \mathbf{A}$. Also, if $a \in \mathbf{A}$ then conditions (1) and (4)(a) imply that the sequence $a_1, \ldots, a_n$ of condition (3) is unique.

In the remainder of this section, we will catalogue many simple facts regarding additive structures.

Lemma 3.2 *Assume* $\mathbf{A}$ *is an additive structure and* $a, b, c \in \mathbf{A}$.

(1) If $b \leq c$, $b + a$ *is defined, and* $c + a$ *is defined then* $b + a \leq c + a$.

(2) If $b < c$, $a + b$ *is defined, and* $a + c$ *is defined then* $a + b < a + c$.

(3) If $a + b = a + c$ *then* $b = c$.

(4) If $a + b$ *and* $b + c$ *are defined then* $a + (b + c) \simeq (a + b) + c$.

Proof. (1), (2), and (4) are straightforward. (3) follows from (2). □

Part (2) of the lemma has the following consequence which we will use implicitly hereafter: if $a_1 + \cdots + a_m$ is defined, $a_m \neq 0$, and $i < m$ then $a_1 + \cdots + a_i < a_1 + \cdots + a_m$.

Definition 3.3 Assume $\mathbf{A}$ is an additive structure, $a \in \mathbf{A}$, and $a_1, \ldots, a_n$ are as in condition (3) of the definition of additive structure. $a_1 + \cdots + a_n$ is called the *decomposition* of a in $\mathbf{A}$ and $a_1, \ldots, a_n$ are called the *components of* a *in* $\mathbf{A}$. If $a = 0$ then $mc^{\mathbf{A}}(a)$ is defined to be 0; otherwise, $mc^{\mathbf{A}}(a)$ is a_1.

We will omit the superscript on $mc^{\mathbf{A}}(a)$ and write $mc(a)$ when this will cause no confusion.

Lemma 3.4 *Assume* $\mathbf{A}$ *is an additive structure and* $a, b \in \mathbf{A}$.

(1) If $a \leq b$ *then* $mc(a) \leq mc(b)$.

(2) If $b \neq 0$ *then* $a + b = b$ *iff* $mc(a) < mc(b)$.

(3) If $a + b$ *is defined then* $mc(a + b)$ *is the maximum of* $mc(a)$ *and* $mc(b)$.

(4) If a *is not indecomposable then each component of* a *is strictly less than* a.

Proof. Immediate. □

Definition 3.5 Assume $\mathbf{A}$ is an additive structure. The *addition tree* of $\mathbf{A}$ is the set of descending sequences $\langle a_1, \ldots, a_n \rangle$ of indecomposables such that $a_1 + \cdots + a_n$ is defined.

Notice that $\langle a \rangle$ is in the addition tree of $\mathbf{A}$ whenever a is indecomposable in $\mathbf{A}$.

Lemma 3.6 *If* $\mathbf{I}$ *is a linear order with universe* I *and* T *is a tree of finite descending sequences from* $\mathbf{I}$ *such that* $\langle a \rangle \in T$ *whenever* $a \in I$ *then there is an additive structure whose addition tree is* T *and whose set of indecomposables is* I *ordered according to* $\mathbf{I}$.

Proof. Straightforward. □

If T is taken to be the tree of all finite descending sequences from $\mathbf{I}$ then $+$ is a total function in the corresponding structure. If $\mathbf{I}$ is an ordinal α with the usual ordering, this structure is isomorphic to $\mathcal{R}_0$ restricted to ω^α where an ordinal $\xi < \alpha$ corresponds to ω^ξ.

Definition 3.7 Assume $\mathbf{A}$ is an additive structure and $\mathbf{B}$ is a substructure of $\mathbf{A}$. $\mathbf{B}$ is a *closed substructure* of $\mathbf{A}$ if whenever $a_1 + \cdots + a_n$ is the decomposition in $\mathbf{A}$ of some element b of $\mathbf{B}$ then $a_1, \ldots, a_n \in \mathbf{B}$ and $b = a_1 + \cdots + a_n$ in $\mathbf{B}$.

Lemma 3.8

(1) *Any closed substructure of an additive structure is additive.*

(2) *Any nonempty initial segment of an additive structure is a closed substructure.*

(3) *Any union of closed substructures of an additive structure is closed.*

(4) *Assume* $\mathbf{A}$, $\mathbf{B}$, *and* $\mathbf{C}$ *are additive structures such that* $\mathbf{A}$ *is a substructure of* $\mathbf{B}$ *and* $\mathbf{B}$ *is a substructure of* $\mathbf{C}$.

   (a) *If* $\mathbf{A}$ *is closed in* $\mathbf{B}$ *and* $\mathbf{B}$ *is closed in* $\mathbf{C}$ *then* $\mathbf{A}$ *is closed in* $\mathbf{C}$.

   (b) *If* $\mathbf{A}$ *is closed in* $\mathbf{C}$ *then* $\mathbf{A}$ *is closed in* $\mathbf{B}$.

Proof. Straightforward. □

Notice that by part (1) of the lemma, any closed substructure of $\mathcal{R}_0$ is additive.

Lemma 3.9 *Assume* $\mathbf{A}$ *and* $\mathbf{B}$ *are additive structures and* h *is an order preserving function from the set of indecomposables of* $\mathbf{A}$ *into the set of indecomposables of* $\mathbf{B}$. *If* $\langle h(a_1), \ldots, h(a_n) \rangle$ *is in the addition tree of* $\mathbf{B}$ *whenever* $\langle a_1, \ldots, a_n \rangle$ *is in the addition tree of* $\mathbf{A}$ *then there is a unique embedding* h^+ *of* $\mathbf{A}$ *into* $\mathbf{B}$ *extending* h *which maps* $\mathbf{A}$ *onto a closed substructure of* $\mathbf{B}$.

Proof. Straightforward. □

Whenever $\mathbf{A}$ is a closed substructure of an additive structure $\mathbf{B}$, an element of $\mathbf{A}$ is indecomposable in $\mathbf{A}$ iff it is indecomposable in $\mathbf{B}$. So any isomorphism of an additive structure onto a closed substructure of another additive structure is induced by an embedding of the indecomposables as in the lemma.

We need the following generalization of condition (4)(b) in the definition of an additive structure.

Lemma 3.10 *Assume* $\mathbf{A}$ *is an additive structure,* $\langle a_1, \ldots, a_m\rangle$ *and* $\langle b_1, \ldots, b_n\rangle$ *are descending sequences of indecomposables, and* $i \leq m$ *has the property that* $a_j + b_1 = b_1$ *whenever* $i < j \leq m$*. If* $b_1 + \cdots + b_n$ *is defined then* $(a_1 + \cdots + a_m) + (b_1 + \cdots + b_n) \simeq a_1 + \cdots + a_i + b_1 + \cdots + b_n$*.*

Proof. Part (4) of lemma 3.2 generalizes to give $a+(b_1+\cdots+b_n) \simeq a+b_1+\cdots+b_n$ for any a. In particular, $(a_1 + \cdots + a_m) + (b_1 + \cdots + b_n) \simeq a_1 + \cdots + a_m + b_1 + \cdots + b_n$. Using part (4) of lemma 3.2 again, if $i < j \leq m$ then $a_1+\cdots+a_{j-1}+b_1 \simeq a_1+\cdots+a_j+b_1$. Therefore, $a_1+\cdots+a_i+b_1 \simeq a_1+\cdots+a_m+b_1$. The conclusion of the lemma follows. □

Lemma 3.11 *Assume* $\mathbf{A}$ *is an additive structure and* I *is a set of nonzero elements of* $\mathbf{A}$ *with the property that* $mc(a) < mc(b)$ *whenever* $a, b \in I$ *and* $a < b$*. If* $\mathbf{S}$ *is the substructure of all finite sums of elements of* I *which are defined (including the empty sum,* 0*) then* $\mathbf{S}$ *is the universe of an additive substructure of* $\mathbf{A}$ *whose set of indecomposables is* I*.*

Proof. We need to show $\mathbf{S}$ satisfies the four conditions in the definition of additive structure. Notice that any substructure of $\mathbf{A}$ satisfies condition (1). Since 0 is defined in $\mathbf{S}$, condition (2) holds also.

Before establishing conditions (3) and (4), we show that every element of I is indecomposable in $\mathbf{S}$. Suppose $a \in I$. Let $X = \{x \in \mathbf{S} \mid \text{ if } x < a \text{ then } mc(x) < mc(a)\}$. $0 \in X$ and, by assumption, each element of I is in X. Part (3) of lemma 3.4 implies that $b+c$ is in X whenever $b, c \in X$ and $b+c$ is defined. Therefore, $X = \mathbf{S}$. By part (3) of lemma 3.4 again, a is indecomposable in $\mathbf{S}$.

Let Y be the collection of all $a \in \mathbf{A}$ such that $a = a_1 + \cdots + a_m$ for some descending sequence $\langle a_1, \ldots, a_m\rangle$ of elements of I. The previous lemma shows that if $a, b \in Y$ and $a + b$ is defined then $a + b \in Y$. Therefore, $Y = \mathbf{S}$ and, by the previous paragraph, condition (3) is established. In addition, we see that every indecomposable of $\mathbf{S}$ is in I thus completing the proof that the set of indecomposables of $\mathbf{S}$ is I.

Suppose $a = a_1 + \cdots + a_m$ and $b = b_1 + \cdots + b_n$ where $\langle a_1, \ldots, a_m\rangle$ and $\langle b_1, \ldots, b_n\rangle$ are descending sequences of elements of I. We claim that

$$\langle a_1, \ldots, a_m\rangle <_{lex} \langle b_1, \ldots, b_n\rangle \;\Rightarrow a < b$$

This will establish (4)(a). Assume $\langle a_1, \ldots, a_m\rangle <_{lex} \langle b_1, \ldots, b_n\rangle$. If $\langle a_1, \ldots, a_m\rangle$ is an initial segment of $\langle b_1, \ldots, b_n\rangle$ then $a < b$ is immediate. So suppose otherwise and let i be minimal such that $a_i \neq b_i$. We see that $a_i < b_i$ and $a_j + b_i = b_i$ whenever $i \leq j \leq m$. We have $b = b_1 + \cdots + b_n = a_1+\cdots+a_{i-1}+b_i+\cdots+b_n =$

$a_1 + \cdots + a_m + b_i + \cdots + b_n = a + b_i + \cdots + b_n$ where the next to last equality follows from the previous lemma. Therefore, $a < b$.

Finally, part (b) of condition (4) follows from the previous lemma. □

Lemma 3.12 *Assume* **A** *is an additive structure and* **B** *is a substructure of* **A**.

(1) **B** *is an additive structure iff*

(a) $mc(a) < mc(b)$ *whenever* a *and* b *are indecomposable elements of* **B** *with* $a < b$ *and*

(b) for any $b \in \mathbf{B}$ *there is a descending sequence* $b_1, \ldots, b_n$ *of indecomposables of* **B** *such that* $b = b_1 + \cdots + b_n$ *in* **B**.

(2) If **B** *is an additive structure then* **B** *is closed in* **A** *iff every indecomposable of* **B** *is indecomposable in* **A**.

Proof. (1) The left to right direction is immediate. Assume conditions (a) and (b). Let I be the set of indecomposables of **B** and define **S** to be the substructure of **A** consisting of all finite sums of elements of I which are defined. By the previous lemma, **S** is an additive structure and I is the set of indecomposables of **S**. Clearly, **B** is a closed substrucure of **S** and, therefore, an additive structure.

Part (2) is immediate. □

We will need another version of lemma 3.9.

Lemma 3.13 *Assume* **A** *and* **B** *are additive structures and* h *is an order preserving map of the indecomposables of* **A** *into* $\mathbf{B} - \{0\}$ *such that* $mc(a) < mc(b)$ *whenever* a *and* b *are in the range of* h *and* $a < b$. *If* $h(a_1) + \cdots + h(a_m)$ *is defined whenever* $\langle a_1, \ldots, a_m \rangle$ *is in the addition tree of* **A** *then there is a unique embedding of* **A** *into* **B** *extending* h.

Proof. Let **S** be the substructure of **B** consisting of all finite sums of elements of the range of h which are defined. By lemma 3.11, **S** is an additive structure whose set of indecomposables is the range of h. By lemma 3.9, there is an embedding of **A** onto a closed substructure of **S** which extends h. □

We will next consider two ways of constructing a new additive structure from a given additive structure.

Definition 3.14 Assume **A** is an additive structure and $\langle a_1, \ldots, a_{n+1} \rangle$ is a descending sequence of indecomposables **A** such that $a_1 + \cdots + a_n$ is defined but $\langle a_1, \ldots, a_{n+1} \rangle$ is not defined. An additive structure $\mathbf{A}^+$ is an *extension* of **A** to $a_1 + \cdots + a_{n+1}$ if the universe of $\mathbf{A}^+$ contains exactly one element a not in the universe of **A** and $a = a_1 + \cdots + a_{n+1}$ in $\mathbf{A}^+$.

Notice that $\mathbf{A}^+$ is an extension of $\mathbf{A}$ to $a_1 + \cdots + a_{n+1}$ iff $\langle a_1, \ldots, a_{n+1} \rangle$ is not in the addition tree of $\mathbf{A}$ and the elements of the addition tree of $\mathbf{A}^+$ consist of those of the addition tree of $\mathbf{A}$ along with $\langle a_1, \ldots, a_{n+1} \rangle$.

If $\mathbf{A}$ and $\langle a_1, \ldots, a_{n+1} \rangle$ are as in the definition then the extensions of $\mathbf{A}$ to $a_1 + \cdots + a_{n+1}$ are unique up to isomorphism over $\mathbf{A}$.

Lemma 3.15 *Assume $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{B}$ are additive structures such that $\mathbf{B}$ is an initial segment of both $\mathbf{A}_1$ and $\mathbf{A}_2$ and, setting $X_i = \mathbf{A}_i - \mathbf{B}$ for $i = 1, 2$, X_1 and X_2 are disjoint. If $mc(x) \in X_2$ whenever $x \in X_2$ then there is a unique additive structure $\mathbf{C}$ with universe $\mathbf{B} \cup X_1 \cup X_2$ such that $\mathbf{B} < X_1 < X_2$ and $\mathbf{A}_i$ is a closed substructure of $\mathbf{C}$ for $i = 1, 2$.*

Proof. Let J be the indecomposables in $\mathbf{B}$. For $i = 1, 2$, let I_i be the indecomposables in X_i and let T_i be the addition tree of $\mathbf{A}_i$. The indecomposables of $\mathbf{C}$ must be $J \cup I_1 \cup I_2$ where J is ordered as in $\mathbf{A}_1$ and $\mathbf{A}_2$, I_i is ordered as in $\mathbf{A}_i$ for $i = 1, 2$, and $J < I_1 < I_2$.

Let $\mathbf{C}$ be an additive structure with indecomposables $J \cup I_1 \cup I_2$ ordered as above and with addition tree $T_1 \cup T_2$. Let f_i be the embedding of $\mathbf{A}_i$ into $\mathbf{C}$ which is the identity on $J \cup I_i$. Notice that f_i maps $\mathbf{A}_i$ onto a closed substructure of $\mathbf{C}$ and that $f_1 \cup f_2$ is a $1-1$ function. This allows us to identify $\mathbf{A}_i$ with its image under f_i and assume that $\mathbf{A}_i$ is a closed substructure of $\mathbf{C}$ for $i = 1, 2$.

To see that $X_1 < X_2$, suppose $x_i \in X_i$ for $i = 1, 2$. Since $mc(x_2) \in I_2$ and $mc(x_1) \in J \cup I_1$, we see that $mc(x_1) < mc(x_2)$. Therefore, $x_1 < x_2$.

The uniqueness of $\mathbf{C}$ should now be fairly clear. Details are left to the reader. □

Definition 3.16 Assume $\mathbf{A}$ is an additive structure, a is an indecomposable element of $\mathbf{A}$, and X is a subset of $\mathbf{A}$ such that $a \leq X$ and $[0, a)^{\mathbf{A}} \cup X$ is a closed substructure of $\mathbf{A}$. An additive structure $\mathbf{A}^+$ is *obtained* from $\mathbf{A}$ *by reflecting* X below a provided $\mathbf{A}$ is a closed substructure of $\mathbf{A}^+$ and the universe of $\mathbf{A}^+$ is $\mathbf{A} \cup \tilde{X}$ for some $\tilde{X}$ such that

(1) $[0, a)^{\mathbf{A}} < \tilde{X} < a$ and

(2) $[0, a)^{\mathbf{A}} \cup \tilde{X} \cong [0, a)^{\mathbf{A}} \cup X$.

Lemma 3.17 *If $\mathbf{A}$, a, and X are as in the assumption of the definition then there exists a structure which is obtained from $\mathbf{A}$ by reflecting X below a. Moreover, any two structures which are obtained from $\mathbf{A}$ by reflecting X below a are isomorphic over $\mathbf{A}$.*

Proof. By lemma 3.15. □

Lemma 3.18 *Assume $\mathbf{A}$ is an additive structure and $B, X_1, X_2 \subseteq \mathbf{A}$ where $B < X_1 < X_2$ and both $B \cup X_1$ and $B \cup X_2$ are additive substructures of $\mathbf{A}$. If $mc(x_1) < mc(x_2)$ whenever $x_i \in X_i$ for $i = 1, 2$ then $B \cup X_1 \cup X_2$ is an additive substructure of $\mathbf{A}$ and $B \cup X_i$ is a closed substructure of $B \cup X_1 \cup X_2$ for $i = 1, 2$.*

Proof. The assumption implies that the indecomposables of $B \cup X_1 \cup X_2$ are the union of the indecomposables of $B \cup X_1$ with those of $B \cup X_2$. Part (1) of lemma 3.12 implies that $B \cup X_1 \cup X_2$ is an additive substructure of $\mathbf{A}$. By part (2) of lemma 3.12, $B \cup X_i$ is a closed substructure of $B \cup X_1 \cup X_2$ for $i = 1, 2$. □

4 Additive Patterns of Resemblance of Order One

Except for the comment concerning $\mathcal{R}_1$ just before definition 4.3, this section can be formalized in $I\Sigma_0(exp)$.

Definition 4.1 If $\mathbf{A}$ is a structure in a language extending $\{\leq, 0, +\}$ then the *arithmetic part* of $\mathbf{A}$ is the restriction of $\mathbf{A}$ to $\{0, +, \leq\}$.

Definition 4.2 A finite structure $\mathbf{P}$ for the language $\{0, +, \leq, \leq_1\}$ is an *additive pattern of resemblance of order one* provided

(1) the arithmetic part of $\mathbf{P}$ is an additive structure,

(2) $\leq_1$ is a forest respecting $\leq$, and

(3) if $a, b \in \mathbf{P}$ and $a <_1 b$ then a is indecomposable.

Until section 10 where we begin to discuss alternative choices for $\mathcal{R}_0$,we will refer to additive patterns of resemblance of order one simply as patterns.

We will often carry over concepts defined in terms of additive structures to patterns without making explicit comments to that effect. For example, when we talk about the components of an element a of a pattern $\mathbf{P}$ we mean the components of a in the arithmetic part of $\mathbf{P}$. And when $\mathbf{Q}$ is a structure for the language $\{0, +, \leq, \leq_1\}$, we say that a substructure $\mathbf{P}$ of $\mathbf{Q}$ is a closed substructure of $\mathbf{Q}$ if the arithmetic part of $\mathbf{P}$ is a closed substructure of the arithmetic part of $\mathbf{Q}$.

Every finite closed substructure of a structure satisfying conditions (1)-(3) of the definition above is a pattern. In particular, any finite closed substructure of $\mathcal{R}_1$ is a pattern. More generally, any finite substructure of a structure satisfying conditions (1)-(3) is a pattern provided its restriction to $\{0, +, \leq\}$ is additive. In the future, any substructure of a pattern $\mathbf{P}$ which is a pattern will be referred to as a subpattern of $\mathbf{P}$.

Definition 4.3 Assume $\mathbf{P}$ is a pattern. If $a \in \mathbf{P}$ define $mc^{\mathbf{P}}(a)$ to be $mc^{\mathbf{A}}(a)$ where $\mathbf{A}$ is the arithmetic part of $\mathbf{P}$. If X is a nonempty subset of $\mathbf{P}$, define $mc^{\mathbf{P}}(X)$ to be the maximal element of $\{mc^{\mathbf{P}}(x) \mid x \in X\}$.

We will drop the superscript and write $mc(a)$ for $mc^{\mathbf{P}}(a)$ when there will be no confusion. Notice that if $\mathbf{P}$ is a closed subpattern of $\mathbf{Q}$ then $mc^{\mathbf{P}}(a) = mc^{\mathbf{Q}}(a)$ allowing us to omit the superscripts in such situations.

We will next describe two methods for constructing new patterns from a given pattern which correspond to the methods of extending additive structures in the previous section.

Definition 4.4 Assume $\mathbf{P}$ is a pattern and $\langle a_1, \ldots, a_{n+1} \rangle$ is a descending sequence of indecomposables. A pattern $\mathbf{P}^+$ extending $\mathbf{P}$ is an *extension* of $\mathbf{P}$ to $a_1 + \cdots + a_{n+1}$ provided the arithmetic part of $\mathbf{P}^+$ is an extension of the arithmetic part of $\mathbf{P}$ to $a_1 + \cdots + a_{n+1}$ and for all $a \in \mathbf{P}$, $a \leq_1 a_1 + \cdots + a_{n+1}$ iff there is $b \in \mathbf{P}$ such that $a \leq a_1 + \cdots + a_{n+1} \leq b$ and $a \leq_1 b$.

Notice that since $\mathbf{P}^+$ of the definition is a pattern, there is no a such that $a_1 + \cdots + a_{n+1} <_1 a$.

Lemma 4.5 *If* $\mathbf{P}$ *is a pattern and* $\langle a_1, \ldots, a_{n+1} \rangle$ *is a descending sequence of indecomposables such that* $a_1 + \ldots + a_n$ *is defined but* $a_1 + \ldots + a_{n+1}$ *is not then there is an extension of* $\mathbf{P}$ *to* $a_1 + \cdots + a_{n+1}$*. Moreover, any two extensions of* $\mathbf{P}$ *to* $a_1 + \cdots + a_{n+1}$ *are isomorphic over* $\mathbf{P}$*.*

Proof. Straightforward. □

Definition 4.6 An extension of a pattern $\mathbf{P}$ is a *simple additive extension* of $\mathbf{P}$ provided it is an extension of $\mathbf{P}$ to $a_1 + \cdots + a_{n+1}$ for some $a_1, \ldots, a_{n+1}$.

Definition 4.7 Assume $\mathbf{P}$ is a pattern, $a, b \in \mathbf{P}$, $a <_1 b$, and X is a nonempty subset of $[a, b)$ with the property that $[0, a) \cup X$ is a closed subpattern of $\mathbf{P}$. A pattern $\mathbf{P}^+$ is *obtained* from $\mathbf{P}$ *by reflecting* X from b to a provided $\mathbf{P}$ is a closed subpattern of $\mathbf{P}^+$ and the universe of $\mathbf{P}^+$ is $\mathbf{P} \cup \tilde{X}$ where

(1) $[0, a)^{\mathbf{P}} < \tilde{X} < a$,

(2) $[0, a)^{\mathbf{P}} \cup X \cong [0, a)^{\mathbf{P}} \cup \tilde{X}$, and

(3) if $x \in \tilde{X}$ and $x \leq_1 y$ then $y \in \tilde{X}$.

$\mathbf{P}^+$ is *obtained* from $\mathbf{P}$ by reflection if $\mathbf{P}^+$ is obtained from $\mathbf{P}$ by reflecting X from b to a for some X, b, and a.

Lemma 4.8 *Let* $\mathbf{P}$*,* a*,* b*, and* X *be as in the assumption of the previous definition. There exists a structure which is obtained from* $\mathbf{P}$ *by reflecting* X *from* b *to* a*. Moreover, any two patterns which are obtained from* $\mathbf{P}$ *by reflecting* X *from* b *to* a *are isomorphic over* $\mathbf{P}$*.*

Proof. Let **A** be the arithmetic part of **P** and let $\mathbf{A}^+$ be obtained from **A** by reflecting X below a. Let $\tilde{X}$ be the elements of $\mathbf{A}^+$ which are not in **A** and fix an isomorphism h between the substructures of $\mathbf{A}^+$ with universes $[0,a)^{\mathbf{P}} \cup \tilde{X}$ and $[0,a)^{\mathbf{P}} \cup X$. Let $\mathbf{P}^+$ be the extension of $\mathbf{A}^+$ to $\leq_1$ such that $x \leq_1 y$ iff $x \leq y$ and either

$$x, y \in \mathbf{A} \text{ and } x \leq_1 y$$

or

$$x, y \in [0,a)^{\mathbf{P}} \cup \tilde{X} \text{ and } h(x) \leq_1 h(y).$$

Notice that for $x, y \in [0,a)^{\mathbf{P}}$ the two conditions above are equivalent since $h(x) = x$ and $h(y) = y$. Notice h is an isomorphism between $[0,a)^{\mathbf{P}} \cup \tilde{X}$ and $[0,a)^{\mathbf{P}} \cup X$. Conditions (1) and (3) of the previous definition are clear and **P** is a closed substructure of $\mathbf{P}^+$ by the choice of $\mathbf{A}^+$.

All that remains is to show that $\mathbf{P}^+$ is a pattern. That $\leq_1$ is a forest which respects $\leq$ is straightforward. Suppose $x, y \in \mathbf{P}^+$ and $x <_1 y$. We will show that x is indecomposable. If $x \in \mathbf{P}$ then either $y \in \mathbf{P}$ or $y \in \tilde{X}$ in which case $x = h(x) <_1 h(y)$. Hence, If $x \in \mathbf{P}$ then $x <_1 z$ for some $z \in \mathbf{P}$ which implies x is indecomposable in **P**, hence in $\mathbf{P}^+$. Now suppose $x \in \tilde{X}$. In this case, y must also be an element of $\tilde{X}$ and $h(x) <_1 h(y)$. Therefore, $h(x)$ is indecomposable. Since h is an isomorphism, x is indecomposable.

Suppose **Q** is also obtained from **P** by reflecting X from b to a. By lemma 3.17, we may assume that **Q** has the same arithmetic part as $\mathbf{P}^+$. Moreover, if $x \leq y$ then a routine argument establishes that $x \leq_1 y$ in $\mathbf{P}^+$ iff $x \leq_1 y$ in **Q**. □

Definition 4.9 Assume $\mathbf{P}^+$ is a pattern and **P** is a subpattern of $\mathbf{P}^+$. $\mathbf{P}^+$ is an *immediate extension* of **P** if $\mathbf{P}^+$ is either a simple additive extension of **P** or is obtained from **P** by reflection. $\mathbf{P}^+$ is *exactly generated* from **P** if $\mathbf{P}^+$ can be obtained from **P** by a finite sequence of immediate extensions. $\mathbf{P}^+$ is *generated* from **P** if $\mathbf{P}^+$ is a subpattern of some structure which is exactly generated from **P**.

We will sometimes say that **P** immediately generates, exactly generates, or generates **Q** to indicate that **Q** is an immediate extension of **P**, is exactly generated from **P**, or is generated from **P** respectively.

We remark that not every extension of a pattern **P** which is generated from **P** is exactly generated from **P**. For example, let **P** consist of three indecomposables which form a chain under $\leq_1$ and let $\mathbf{P}^+$ be obtained from **P** by adding three indecomposables below those of **P** which form a tree whose root has two immediate successors. $\mathbf{P}^+$ is generated from **P** but not exactly generated from **P**.

The following is true but not obvious: if **P** generates an extension **Q** and **Q** generates an extension **R** then **P** generates **R**. We will not prove this fact in full generality until section 7.

Lemma 4.10 *Assume* $\mathbf{P}$ *is a subpattern of* $\mathbf{Q}$ *and* $\mathbf{Q}$ *is a subpattern of* $\mathbf{R}$*. If* $\mathbf{P}$ *generates* $\mathbf{R}$ *then* $\mathbf{P}$ *generates* $\mathbf{Q}$*.*

Proof. Immediate. □

Lemma 4.11 *Assume* $\mathbf{P}$ *is a pattern and* $\mathbf{P}^+$ *is generated from* $\mathbf{P}$*.*

(1) $\mathbf{P}$ *is a closed subpattern of* $\mathbf{P}^+$*.*

(2) *If* $a \in \mathbf{P}$ *and* $b \in \mathbf{P}^+$ *then* $a \leq_1 b$ *iff there is* $c \in \mathbf{P}$ *such that* $a \leq b \leq c$ *and* $a \leq_1 c$*.*

(3) *If* $a \in \mathbf{P}$*,* $b \in \mathbf{P}^+$*, and* $b \leq_1 a$ *then* $b \in \mathbf{P}$*.*

(4) $mc(\mathbf{P}) = mc(\mathbf{P}^+)$*.*

(5) *If there is an indecomposable* $c \in \mathbf{P}^+$ *such that* $[0,a)^{\mathbf{P}} < c < a$ *then* $a <_1 b$ *for some* $b \in \mathbf{P}$*.*

Proof. We may assume that $\mathbf{P}^+$ is exactly generated from $\mathbf{P}$. The lemma is now clear since it is true for immediate extensions. □

Definition 4.12 Assume $\mathbf{P}$ and $\mathbf{Q}$ are structures for a language containing $\leq$ and $\leq_1$ such that the interpretation of $\leq$ in $\mathbf{Q}$ is a linear ordering of the universe of $\mathbf{Q}$ and $\mathbf{P}$ is a substructure of $\mathbf{Q}$. $\mathbf{P}$ is *correct* in $\mathbf{Q}$ if whenever $a \leq_1 b$ where $a \in \mathbf{P}$ and $b \in \mathbf{Q}$ there exists $c \in \mathbf{P}$ such that $b \leq c$ and $a \leq_1 c$.

Lemma 4.13

(1) *If* $\mathbf{P}$ *is correct in* $\mathbf{Q}$ *and* $\mathbf{Q}$ *is correct in* $\mathbf{R}$ *then* $\mathbf{P}$ *is correct in* $\mathbf{R}$*.*

(2) *If* $\mathbf{P}$ *is a substructure of* $\mathbf{Q}$*,* $\mathbf{Q}$ *is a substructure of* $\mathbf{R}$*, and* $\mathbf{P}$ *is correct in* $\mathbf{R}$ *then* $\mathbf{P}$ *is correct in* $\mathbf{Q}$*.*

(3) *Any union of correct substructures of a fixed structure* $\mathbf{P}$ *is correct in* $\mathbf{P}$*.*

(4) *If* $\mathbf{P}$ *generates* $\mathbf{P}^+$ *then* $\mathbf{P}$ *is correct in* $\mathbf{P}^+$*.*

(5) *Assume* $\mathbf{P}$ *is a pattern,* a *and* b *are elements of* $\mathbf{P}$ *with* $a \leq b$*, and* $[a,b]^{\mathbf{P}}$ *is correct in* $\mathbf{P}$*. If* $\mathbf{P}$ *generates* $\mathbf{P}^+$ *then* $[a,b]^{\mathbf{P}^+}$ *is correct in* $\mathbf{P}^+$*.*

Proof. (1)-(3) are immediate.

Parts (1) and (2) allow us to assume that $\mathbf{P}$ immediately generates $\mathbf{P}^+$ in part (4). The argument is now straightforward.

For part (5), suppose $x \in [a,b]^{\mathbf{P}^+}$ and $y \in \mathbf{P}^+$ where $x \leq_1 y$. Suppose also that $b \leq y$. Since $x \leq_1 b$, part (3) of the previous lemma implies that $x \in \mathbf{P}$. By part (2) of the previous lemma, $y \leq b$. □

Lemma 4.14 *Assume* $\mathbf{P}$ *is a pattern and let* $T^{\mathbf{P}}$ *be the addition tree of* $\mathbf{P}$. *If* T *is a finite tree of finite descending sequences of indecomposables of* $\mathbf{P}$ *such that* $T^{\mathbf{P}} \subseteq T$ *then* $\mathbf{P}$ *exactly generates some* $\mathbf{P}^+$ *such that the addition tree of* $\mathbf{P}^+$ *is* T. *In particular, if* $a, b \in \mathbf{P}$ *then there is an exact extension of* $\mathbf{P}$ *in which* $a+b$ *is defined.*

Proof. Clearly, $\mathbf{P}^+$ can be obtained by a finite sequence of simple additive extensions. □

5 Characterizing the Core

We will work in $KP + Infinity$ in this section and the next.

Definition 5.1 Assume $\mathbf{P}$ and $\mathbf{Q}$ are structures for the language $\{0, +, \leq, \leq_1\}$. A *covering* of $\mathbf{P}$ into $\mathbf{Q}$ is an embedding h of the arithmetic part of $\mathbf{P}$ into the arithmetic part of $\mathbf{Q}$ with the property that $h(a) \leq_1 h(b)$ whenever $a, b \in \mathbf{P}$ and $a \leq_1 b$. If h also maps $\mathbf{P}$ onto $\mathbf{Q}$ then $\mathbf{Q}$ is called a *cover* of $\mathbf{P}$.

Since each pattern is finite and its arithmetic part includes a linear ordering, if $\mathbf{P}$ is a pattern then a covering of $\mathbf{P}$ into $\mathbf{Q}$ is determined by its range.

Lemma 5.2 *If* f *is a covering of* $\mathbf{P}$ *into* $\mathbf{Q}$ *and* g *is a covering of* $\mathbf{Q}$ *into* $\mathbf{R}$ *then* $g \circ f$ *is a covering of* $\mathbf{P}$ *into* $\mathbf{R}$.

Proof. Immediate. □

Of course, the previous definition and lemma can be formalized in $I\Sigma_0(exp)$.

Lemma 5.3 *Assume* $\mathbf{P}$ *is a pattern and* h *is a covering of* $\mathbf{P}$ *into* $\mathcal{R}_1$. *If* $\mathbf{P}^+$ *is generated from* $\mathbf{P}$ *then there is a covering* h^+ *of* $\mathbf{P}^+$ *into* $\mathcal{R}_1$ *which extends* h.

Proof. Without loss of generality, we may assume that $\mathbf{P}^+$ is exactly generated from $\mathbf{P}$ and, hence, we may further assume that $\mathbf{P}^+$ is an immediate extension of $\mathbf{P}$.

Case 1: $\mathbf{P}^+$ is an extension of $\mathbf{P}$ to $a_1 + \cdots + a_{m+1}$ for some $a_1, \ldots, a_{m+1}$.

By part (2) of lemma 3.2, $mc(h(a)) < mc(h(b))$ whenever a and b are indecomposable in $\mathbf{P}$ and $a < b$. By lemma 3.13, there is an extension h^+ of h which embeds the arithmetic part of $\mathbf{P}^+$ into $\mathcal{R}_0$.

To establish that h^+ is a covering of $\mathbf{P}^+$, Suppose $b \in \mathbf{P}^+$ and $b <_1 a_1 + \cdots + a_{m+1}$. b must be in $\mathbf{P}$ and there exists $c \in \mathbf{P}$ such that $b < a_1 + \cdots + a_{m+1} < c$ and $b \leq_1 c$. Since h is a covering of $\mathbf{P}$, we see that $h^+(b) = h(b) \leq_1 h(c) = h^+(c)$. Therefore, $h^+(b) \leq_1 h^+(a_1 + \cdots + a_{m+1})$.

Case 2: $\mathbf{P}^+$ is obtained from $\mathbf{P}$ by reflecting X from b to a for some X, b, and a.

Let $\tilde{X} = \mathbf{P}^+ - \mathbf{P}$. Define $\mathbf{R}$ to be the substructure of $\mathcal{R}_1$ whose universe is the range of h, and let Y be the image of X under h. Since $h(a) \leq_1 h(b)$, there exists a set of ordinals $\tilde{Y}$ and a function f such that $[0, h(a))^{\mathbf{R}} < \tilde{Y} < h(a)$ and f is an isomorphism of $[0, h(a))^{\mathbf{R}} \cup Y$ with $[0, h(a))^{\mathbf{R}} \cup \tilde{Y}$. Let $\mathbf{R}^+$ be the substructure of $\mathcal{R}_1$ with universe $\mathbf{R} \cup \tilde{Y}$.

Notice that since $h(a) <_1 h(b)$, $h(a)$ is indecomposable in $\mathcal{R}_1$. By lemma 3.18, the arithmetic part of $\mathbf{R}^+$ is an additive structure and $\mathbf{R}$ is a closed substructure of $\mathbf{R}^+$. Therefore, the arithmetic part of $\mathbf{R}^+$ is obtained from the arithmetic part of $\mathbf{R}$ by reflecting Y below $h(a)$. By lemma 3.17 which says that such extensions are unique up to isomorphism over the arithmetic part of $\mathbf{R}$, there is an isomorphism h^+ of the arithmetic part of $\mathbf{P}^+$ and the arithmetic part of $\mathbf{R}^+$ which extends h.

We claim that h^+ is a covering of $\mathbf{P}^+$. To establish this, assume that $x, y \in \mathbf{P}^+$ and $x \leq_1 y$. Since h^+ extends h and h is a covering of $\mathbf{P}$, we may assume that not both of x and y are in $\mathbf{P}$. Moreover, if $x \in \tilde{X}$ then $y \in \tilde{X}$ so we may assume that $x, y < a$ i.e. $x, y \in [0, a)^{\mathbf{P}} \cup \tilde{X}$. Let g be the isomorphism between $[0, a)^{\mathbf{P}} \cup \tilde{X}$ and $[0, a)^{\mathbf{P}} \cup X$. Since $f \circ h \circ g$ is an isomorphism of the arithmetic parts of $[0, a)^{\mathbf{P}} \cup \tilde{X}$ and $[0, h(a))^{\mathbf{R}} \cup \tilde{Y}$, it must be the restriction of h^+ to $[0, a)^{\mathbf{P}} \cup \tilde{X}$. In addition, since h is a covering and both g and f are isomorphisms, $f \circ h \circ g$ is a covering of $[0, a)^{\mathbf{P}} \cup \tilde{X}$. Therefore, $h^+(x) \leq_1 h^+(y)$. □

Definition 5.4 A pattern $\mathbf{P}$ is *covered* if there is a covering of $\mathbf{P}$ into $\mathcal{R}_1$.

Lemma 5.5 *Assume* $\mathbf{P}$ *is a covered pattern and that* $\mathbf{P}$ *generates* $\mathbf{P}^+$. *If* $\mathbf{Q}$ *is a subpattern of* $\mathbf{P}^+$ *which is a cover of* $\mathbf{P}$ *then* $\mathbf{P} \leq_{pw} \mathbf{Q}$.

Proof. Assume to the contrary that $\mathbf{P} \not\leq_{pw} \mathbf{Q}$. Choose i such that the i^{th} element of $\mathbf{Q}$ is less than the i^{th} element of $\mathbf{P}$. Consider the nonempty collection of all ordinals which occur as the i^{th} element of some substructure of $\mathcal{R}_1$ which is a cover of $\mathbf{P}$. By the previous lemma, this is a nonempty class of ordinals without a minimal element – contradiction. □

Definition 5.6 Assume $\mathbf{P}_n$ $(n \in \omega)$ is an increasing sequence of patterns such that $\mathbf{P}_n$ generates $\mathbf{P}_{n+1}$ for each $n \in \omega$. Let $\mathbf{P}_\infty$ be the union of $\mathbf{P}_n$ $(n \in \omega)$. $\mathbf{P}_n$ $(n \in \omega)$ is *fair* if $+$ is a total function in $\mathbf{P}_\infty$ and for each $n \in \omega$, if $a, b \in \mathbf{P}_n$ and $a <_1 b$ then there exists $m > n$ such that $\mathbf{P}_{m+1}$ is obtained from $\mathbf{P}_m$ by reflecting $[a, b)^{\mathbf{P}_m}$ from b to a.

Lemma 5.7 *Assume* $\mathbf{P}_n$ $(n \in \omega)$ *is an increasing sequence of patterns such that* $\mathbf{P}_n$ *exactly generates* $\mathbf{P}_{n+1}$ *for each* n. *Let* $\mathbf{P}_\infty$ *be the union of* $\mathbf{P}_n$ $(n \in \omega)$.

(1) $\mathbf{P}_n$ *is a closed substructure of* $\mathbf{P}_\infty$ *for each* $n \in \omega$ *and* $\mathbf{P}_\infty$ *satisfies conditions (1)-(3) of the definition of pattern (definition 4.2).*

(2) *If h is a covering of $\mathbf{P}_0$ into $\mathcal{R}_1$ then there is a covering of $\mathbf{P}_\infty$ into $\mathcal{R}_1$ which extends h.*

(3) *If $\mathbf{P}_0$ is covered then $\mathbf{P}_\infty$ is order isomorphic to an ordinal.*

(4) *If $\mathbf{P}_n$ $(n \in \omega)$ is fair, $a, b \in \mathbf{P}_\infty$, and $a \leq_1 b$ then $[0,a)^{\mathbf{P}_\infty} \preceq_{\Sigma_1} [0,b)^{\mathbf{P}_\infty}$.*

(5) *If $\mathbf{P}_0$ is covered, $\mathbf{P}_n$ $(n \in \omega)$ is fair, $a, b \in \mathbf{P}_\infty$, and $[0,a)^{\mathbf{P}_\infty} \preceq_{\Sigma_1} [0,b)^{\mathbf{P}_\infty}$ then $a \leq_1 b$.*

Proof. Part (1) is clear.

To prove part (2), use lemma 5.3 to find a nested sequence of functions h_n $(n \in \omega)$ such that $h_0 = h$ and h_n is a covering of $\mathbf{P}_n$ into $\mathcal{R}_1$ for $n \in \omega$. The union of the h_n is the desired covering.

Part (3) follows from part (2).

To establish part (4), assume X and Y are finite subsets of $\mathbf{P}_\infty$ such that $X < a$ and $a \leq Y < b$. Fix n such that $a, b \in \mathbf{P}_n$ and $X, Y \subseteq \mathbf{P}_n$. Fix $m > n$ such that $\mathbf{P}_{m+1}$ is obtained from $\mathbf{P}_m$ by reflecting $[a,b)^{\mathbf{P}_m}$ from b to a. There is a subset $\tilde{Y}$ of $\mathbf{P}_{m+1}$ such that $X < \tilde{Y} < a$ and $X \cup \tilde{Y} \cong X \cup Y$.

For part (5), argue by contradiction and assume that $[0,a)^{\mathbf{P}_\infty} \preceq_{\Sigma_1} [0,b)^{\mathbf{P}_\infty}$ while $a \not\leq_1 b$. Since $[0,a)^{\mathbf{P}_\infty} \preceq_{\Sigma_1} [0,b)^{\mathbf{P}_\infty}$, a is indecomposable in $\mathbf{P}_\infty$. Choose n such that $a, b \in \mathbf{P}_n$. Let c be the largest element of $\mathbf{P}_n$ such that $a \leq_1 c$. Notice that $c < b$. Let $X = [a,c]^{\mathbf{P}_n}$. Since $[0,a)^{\mathbf{P}_\infty} \preceq_{\Sigma_1} [0,b)^{\mathbf{P}_\infty}$, there is a subset $\tilde{X}$ of $\mathbf{P}_\infty$ and a function f such that $[0,a)^{\mathbf{P}_n} < \tilde{X} < a$ and f is an isomorphism of $[0,a)^{\mathbf{P}_n} \cup \tilde{X}$ and $[0,a)^{\mathbf{P}_n} \cup X$. By lemma 3.18, $\mathbf{P}_n \cup \tilde{X}$ is an additive substructure of $\mathbf{P}_\infty$. This and the fact that $+$ is total in $\mathbf{P}_\infty$ imply that the hypotheses of lemma 3.13 are satisfied. Therefore, there is an embedding h of the arithmetic part of $\mathbf{P}_n$ into the arithmetic part of $\mathbf{P}_\infty$ such that $h(x) = x$ if x is an indecomposable of $\mathbf{P}_n$ which is not in X and $h(x) = f(x)$ if x is an indecomposable of $\mathbf{P}_n$ in X. Since $h(x) \leq x$ for all indecomposables, we see that $h(x) \leq x$ for all $x \in \mathbf{P}_n$. Notice that h extends f.

We claim that h is a covering of $\mathbf{P}_n$. Suppose $x, y \in \mathbf{P}_n$ and $x <_1 y$. If $x \in X$ then $y \in X$ (by the choice of c) and $h(x) = f(x) <_1 f(y) = h(y)$. So, we may suppose $x \notin X$. Since x is indecomposable, $h(x) = x$ so that $h(x) = x <_1 y$. Since $h(x) < h(y) \leq y$, this implies $h(x) <_1 h(y)$.

Let $\mathbf{R}$ be the substructure of $\mathbf{P}_\infty$ which is the image of h. Since $h(a) < a$, $\mathbf{P}_n \not\leq_{pw} \mathbf{R}$. Since $\mathbf{R} \subseteq \mathbf{P}_m$ for some $m > n$ and $\mathbf{P}_n$ generates $\mathbf{P}_m$(this is the only point in the proof which uses the assumption that $\mathbf{P}_n$ exactly generates $\mathbf{P}_{n+1}$), this contradicts the previous lemma. □

The proof of part (5) of the lemma requires only that f be a covering. This fact and the following lemma imply that for α and β in the core with $\alpha \not\leq_1 \beta$ the failure of $[0,\alpha) \preceq_{\Sigma_1} [0,\beta)$ is witnessed by a formula in a restricted class of Σ_1 formulas, namely, those formulas which are positive in $\leq_1$. This shows that there is some freedom in the definition of $\leq_1$.

Lemma 5.8 *Assume $\mathbf{P}_n$ $(n \in \omega)$ is a fair sequence of patterns where $\mathbf{P}_{n+1}$ is exactly generated from $\mathbf{P}_n$ for each $n \in \omega$. If $\mathbf{P}_0$ is covered then the union $\mathbf{P}_\infty$ of $\mathbf{P}_n$ $(n \in \omega)$ is isomorphic to an initial segment $\mathbf{P}^*_\infty$ of $\mathcal{R}_1$ which is correct in $\mathcal{R}_1$. Moreover, if $\mathbf{P}^*_n$ is the image of $\mathbf{P}_n$ under the isomorphism of $\mathbf{P}_\infty$ and $\mathbf{P}^*_\infty$ then $\mathbf{P}^*_n \leq_{pw} \mathbf{Q}$ whenever $\mathbf{Q}$ is a substructure of $\mathcal{R}_1$ which is a cover of $\mathbf{P}_n$.*

Proof. By part (2) of the previous lemma, there is a covering of $\mathbf{P}_\infty$ into $\mathcal{R}_1$. By collapsing the range of this covering, we may assume that the universe of $\mathbf{P}_\infty$ is an ordinal λ and the ordering of $\mathbf{P}_\infty$ is the usual ordering of ordinals. A straightforward induction shows that the restriction of $\mathbf{P}_\infty$ to α is the same as the restriction of $\mathcal{R}_1$ to α for $\alpha \leq \lambda$. To verify the step from α to $\alpha + 1$, use parts (4) and (5) of the previous lemma to see that the interpretations of $\leq_1$ agree and show that the decomposition of α must be equal in both structures to see that addition agrees. Thus, $\mathbf{P}_\infty$ is (isomorphic to) an initial segment of $\mathcal{R}_1$.

To see that $\mathbf{P}_\infty$ is correct in $\mathcal{R}_1$, suppose $\alpha \in \mathbf{P}_\infty$ and $\alpha <_1 \beta$. We must show that $\beta \in \mathbf{P}_\infty$. Choose n such that $\alpha \in \mathbf{P}_n$. Since $\alpha \neq 0$, $1 \in \mathbf{P}_\infty$. Since $+$ is total in $\mathbf{P}_\infty$, $max(\mathbf{P}_n) + 1 \in \mathbf{P}_\infty$. Choose m such that $n < m$ and $max(\mathbf{P}_n) + 1 \in \mathbf{P}_m$. Since $\mathbf{P}_n$ generates $\mathbf{P}_m$, $\mathbf{P}_n$ is correct in $\mathbf{P}_m$ and $\alpha \not\leq_1 max(\mathbf{P}_n) + 1$. Therefore, $\beta \leq max(\mathbf{P}_n)$ implying $\beta \in \mathbf{P}_\infty$.

Under our assumptions, $\mathbf{P}^*_n = \mathbf{P}_n$ and $\mathbf{P}_\infty = \mathbf{P}^*_\infty$. Suppose $\mathbf{Q}$ is a substructure of $\mathcal{R}_1$ which is a cover of $\mathbf{P}_n$. By part (2) of the previous lemma, there is a covering h of $\mathbf{P}_\infty$ into $\mathcal{R}_1$ which maps $\mathbf{P}_n$ onto $\mathbf{Q}$. Since h is order preserving, $\mathbf{P}_n \leq_{pw} \mathbf{Q}$. □

Notice that the conclusion of the lemma implies that $\mathbf{P}^*_n$ is an isominimal substructure of $\mathcal{R}_1$.

Theorem 5.9 *If $\mathbf{P}$ is a covered pattern then there is a substructure $\mathbf{P}^*$ of $\mathcal{R}_1$ such that*

(1) $\mathbf{P}^*$ is isomorphic to $\mathbf{P}$,

(2) $\mathbf{P}^*$ is a closed substructure of $\mathcal{R}_1$,

(3) $\mathbf{P}^*$ is correct in $\mathcal{R}_1$,

(4) $\mathbf{P}^* \leq_{pw} \mathbf{Q}$ whenever $\mathbf{Q}$ is a substructure of $\mathcal{R}_1$ which is a cover of $\mathbf{P}$, and

(5) if $\mathbf{Q}$ is a pattern which is a substructure of $\mathcal{R}_1$ such that $\mathbf{P}^*$ is a subpattern of $\mathbf{Q}$ and $mc^{\mathcal{R}_1}(\mathbf{Q}) \leq mc^{\mathcal{R}_1}(\mathbf{P}^*)$ then $\mathbf{P}^*$ generates $\mathbf{Q}$.

Proof. If $\mathbf{P}$ is a trivial pattern with only one element (the interpretation of 0) the theorem is obvious. So assume that $\mathbf{P}$ contains a nonzero element. Let $\mathbf{P}_n$ $(n \in \omega)$ be a fair sequence of patterns with $\mathbf{P} = \mathbf{P}_0$ such that $\mathbf{P}_n$ exactly generates $\mathbf{P}_{n+1}$ for $n \in \omega$. Let $\mathbf{P}_\infty$ be the union of $\mathbf{P}_n$ $(n \in \omega)$. Fix an isomorphism f of $\mathbf{P}_\infty$ with an initial segment $\mathbf{P}^*_\infty$ of $\mathcal{R}_1$ as in the previous lemma. Let $\mathbf{P}^*_n$ be the the image of $\mathbf{P}_n$ under f for $n \in \omega$ and set $\mathbf{P}^* = \mathbf{P}^*_0$. Notice that $\mathbf{P}^*_n$ $(n \in \omega)$ is a fair sequence such that $\mathbf{P}^*_n$ exactly generates $\mathbf{P}^*_{n+1}$ for $n \in \omega$.

Since $\mathbf{P}^*_0$ is a closed substructure of $\mathbf{P}^*_\infty$ and $\mathbf{P}^*_\infty$ is an initial segment of $\mathcal{R}_1$, $\mathbf{P}^*_0$ is a closed substructure of $\mathcal{R}_1$.

For part (3), notice that since $\mathbf{P}^*_\infty$ is correct in $\mathcal{R}_1$, showing that $\mathbf{P}^*$ is correct in $\mathbf{P}^*_\infty$ is sufficient. This follows easily from the fact that $\mathbf{P}^*$ is correct in $\mathbf{P}^*_n$ for all n.

(4) follows from the choice of f.

For (5), let λ be the universe of $\mathbf{P}^*_\infty$. $\lambda \neq 0$ and λ is closed under addition i.e. λ is an indecomposable. Since $mc^{\mathcal{R}_1}(\mathbf{Q}) \leq mc^{\mathcal{R}_1}(\mathbf{P}^*_0) < \lambda$, $\mathbf{Q} \subseteq \lambda$. Therefore, $\mathbf{Q} \subseteq \mathbf{P}^*_n$ for some n implying that $\mathbf{Q}$ is generated from $\mathbf{P}^*_0$. □

Notice that conditions (1) and (4) of the theorem imply that $\mathbf{P}^*$ is isominimal in $\mathcal{R}_1$.

Theorem 5.10 *Assume $\mathbf{P}$ is a covered pattern and $\mathbf{P}^*$ is the isominimal substructure of $\mathcal{R}_1$ which is isomorphic to $\mathbf{P}$. If $\mathbf{P}$ is a substructure of a pattern $\mathbf{Q}$ then $\mathbf{P}$ generates $\mathbf{Q}$ iff*

(1) $mc^{\mathbf{Q}}(\mathbf{Q}) = mc^{\mathbf{P}}(\mathbf{P})$,

(2) $\mathbf{Q}$ *is covered, and*

(3) if $\mathbf{Q}^*$ is the isominimal substructure of $\mathcal{R}_1$ which is isomorphic to $\mathbf{Q}$ then the isomorphism of $\mathbf{Q}$ and $\mathbf{Q}^*$ extends the isomorphism of $\mathbf{P}$ and $\mathbf{P}^*$.

Proof. ($\Rightarrow$) Part (4) of lemma 4.11 implies that condition (1) holds.

To verify condition (3), notice that lemma 5.3 implies that there is a covering h of $\mathbf{Q}$ into $\mathcal{R}_1$ which extends the isomorphism of $\mathbf{P}$ and $\mathbf{P}^*$. Let $\mathbf{Q}'$ be the image of $\mathbf{Q}$ under h, let $\mathbf{Q}^*$ be the isominimal substructure of $\mathcal{R}_1$ which is isomorphic to $\mathbf{Q}$, and let $\mathbf{P}'$ be the image of $\mathbf{P}$ under the isomorphism of $\mathbf{Q}$ with $\mathbf{Q}^*$. Since part (4) of the previous theorem implies that $\mathbf{Q}^* \leq_{pw} \mathbf{Q}'$, we see that $\mathbf{P}' \leq_{pw} \mathbf{P}^*$. Since $\mathbf{P}^*$ is isominimal, $\mathbf{P}^* = \mathbf{P}'$. Therefore, the isomorphism of $\mathbf{Q}$ with $\mathbf{Q}^*$ extends the isomorphism of $\mathbf{P}$ with $\mathbf{P}^*$.

Notice that in the course of verifying condition (3) we have also verified (2).

($\Leftarrow$) Let $\mathbf{Q}^*$ be the isominimal substructure of $\mathcal{R}_1$ which is isomorphic to $\mathbf{Q}$. By condition (3), it suffices to show that $\mathbf{Q}^*$ is generated from $\mathbf{P}^*$.

$$\begin{aligned} mc^{\mathcal{R}_1}(\mathbf{Q}^*) &\leq mc^{\mathbf{Q}^*}(\mathbf{Q}^*) && \text{(using (3) and (4) of lemma 3.4)} \\ &= mc^{\mathbf{P}^*}(\mathbf{P}^*) && \text{(by condition (1))} \\ &= mc^{\mathcal{R}_1}(\mathbf{P}^*) && \text{(since } \mathbf{P}^* \text{ is a closed substructure of } \mathcal{R}_1\text{)} \end{aligned}$$

By part (5) of the previous theorem, $\mathbf{P}^*$ generates $\mathbf{Q}^*$. □

Corollary 5.11 *Assume* $\mathbf{P}$ *is covered. If* $\mathbf{P}$ *generates* $\mathbf{Q}$ *and* $\mathbf{Q}$ *generates* $\mathbf{R}$ *then* $\mathbf{P}$ *generates* $\mathbf{R}$.

Proof. Immediate from the theorem. □

As mentioned earlier, we will generalize the corollary to all patterns in section 7.

Theorem 5.12 *If there exists a* κ *such that* $\kappa \leq_1 \infty$ *then the least such* κ *is the core of* $\mathcal{R}_1$*. Otherwise, the core of* $\mathcal{R}_1$ *is* ORD.

Proof. First notice that the core of $\mathcal{R}_1$ is an initial segment of ORD. To see this, suppose $\mathbf{P}$ is an isominimal substructure of $\mathcal{R}_1$. Fix a fair sequence $\mathbf{P}_n$ $(n \in \infty)$ such that $\mathbf{P}_0 = \mathbf{P}$ and $\mathbf{P}_{n+1}$ is exactly generated from $\mathbf{P}_n$ for $n \in \omega$. Let $\mathbf{P}_\infty$ be the union of $\mathbf{P}_n$ $(n \in \omega)$. By lemma 5.8, there is an isomorphism f of $\mathbf{P}_\infty$ with an initial segment $\mathbf{P}^*_\infty$ of $\mathcal{R}_1$ which fixes $\mathbf{P}$. Lemma 5.8 also states that the image of each $\mathbf{P}_n$ under f is isominimal in $\mathcal{R}_1$. Therefore, $\mathbf{P}^*_\infty$ is contained in the core. Since both $\mathbf{P}$ and its image under f are isominimal, they must be equal implying that $[0, max(\mathbf{P})]$ is contained in the core.

Suppose there is an ordinal which is not in the core and let κ be the least such ordinal. We will show that $\kappa \leq_1 \infty$.

Suppose X and Y are finite sets of ordinals such that $X < \kappa \leq Y$. We will show that there is $\tilde{Y}$ such that $X < \tilde{Y} < \kappa$ and $X \cup \tilde{Y} \cong X \cup Y$. Without loss of generality, we may assume that $X \cup Y$ is a closed substructure of $\mathcal{R}_1$. Notice that any finite union of isominimal substructures of $\mathcal{R}_1$ is isominimal. Since X is contained in a finite union of isominimal patterns, we may assume that X is isominimal. Now let $\tilde{X} \cup \tilde{Y}$ be the isominimal copy of $X \cup Y$ where $\tilde{X}$ corresponds to X and $\tilde{Y}$ corresponds to Y under the isomorphism. We must have $\tilde{X} \leq_{pw} X$. Since X is isominimal, $\tilde{X} = X$. Since $\tilde{X} \cup \tilde{Y}$ is contained in the core, $\tilde{Y} < \kappa$. □

6 Amalgamations and the Core

A modification of the first part of the argument for theorem 5.12 shows that every proper initial segment of the core of $\mathcal{R}_1$ is isomorphic to a recursive structure. To see this, let $\mathbf{Q}$ be an isominimal substructure of $\mathcal{R}_1$. We want to find a recursive structure which is isomorphic to an initial segment of the core containing $\mathbf{Q}$. Let $\mathbf{P}_n$ $(n \in \omega)$ be a fair sequence such that $\mathbf{P}_0$ is isomorphic to $\mathbf{Q}$ and $\mathbf{P}_{n+1}$ is exactly generated from $\mathbf{P}_n$ for $n \in \omega$. We can choose such a sequence with the additional properties that the universe of each $\mathbf{P}_n$ is a subset of ω, $\mathbf{P}_n < (\mathbf{P}_{n+1} - \mathbf{P}_n)$ where here $<$ is the usual ordering on ω, and the sequence itself is recursive. These conditions imply that the union $\mathbf{P}_\infty$ of $\mathbf{P}_n$ $(n \in \omega)$ is recursive. And as before, we see that $\mathbf{P}_\infty$ is isomorphic to an initial segment of the core which contains $\mathbf{Q}$.

Of course, this is not a very elegant way of seeing that proper initial segments of the core are recursive. Rather, one would like to be able to recursively reconstruct the relations and functions of $\mathcal{R}_1$ simply from knowing the isomorphism types of isominimal substructures of $\mathcal{R}_1$ containing each of the ordinals involved along with the position of each ordinal in its isominimal structure i.e. one would like to see that the relations and functions of $\mathcal{R}_1$ act on the notations described in the introduction in a recursive way. We will see that this is true in this section however we will not show that the notations which arise from a proper initial segment of the core themselves form a recursive set until section 7.

We are left with the obvious question of whether the core of $\mathcal{R}_1$ itself is isomorphic to a recursive structure or, equivalently, is a recursive ordinal. We will see that this is true under the assumption of ZF. In fact, a slightly more refined argument would show that going just beyond $KP\ell_0$ suffices. However, weaker theories like $KP + Infinity$ do not imply that the core is a recursive ordinal. These results will be established elsewhere.

Most of the results in this section are proven only for covered patterns. However, the next lemma implies that if one is willing to accept ZF then this is not a real restriction. Unrestricted versions of many of the results here will be proved under weaker assumptions in the following section.

Lemma 6.1 *Assuming ZF, every pattern is covered.*

Proof. Using the reflection principle, there are cofinally many ordinals κ such that $\kappa \leq_1 \infty$. If $\mathbf{P}$ is a pattern let h be an embedding of the arithmetic part of $\mathbf{P}$ into $\mathcal{R}_0$ which maps the indecomposables of $\mathbf{P}$ to such κ. Such an embedding is a covering of $\mathbf{P}$ into $\mathcal{R}_1$. □

One can strengthen the previous result by showing that every pattern is covered iff $KP\ell_0$ is Π^1_1-sound. This will be established elsewhere.

Definition 6.2 Assume $\mathbf{P}$ is a subpattern of $\mathbf{Q}$. $\mathbf{P}$ is *exact* in $\mathbf{Q}$ if $\mathbf{P}$ is correct in $\mathbf{Q}$ and $\mathbf{P}$ generates $[0, max(\mathbf{P})]^{\mathbf{Q}}$. If f is an embedding of $\mathbf{P}$ into a pattern $\mathbf{R}$ then f is said to be *exact* with respect to $\mathbf{R}$ if the range of f is an exact subpattern of $\mathbf{R}$.

When $\mathbf{R}$ is clear from the context, we will say that f is exact when f is exact with respect to $\mathbf{R}$.

Notice that the proofs of the following two lemmas can be formalized in $I\Sigma_0(exp)$.

Lemma 6.3

(1) If $\mathbf{P}$ generates $\mathbf{P}^+$ then $\mathbf{P}$ is exact in $\mathbf{P}^+$.

(2) *If* $\mathbf{P}$ *is a subpattern of* $\mathbf{Q}$*,* $\mathbf{Q}$ *is a subpattern of* $\mathbf{R}$*, and* $\mathbf{P}$ *is exact in* $\mathbf{R}$ *then* $\mathbf{P}$ *is exact in* $\mathbf{Q}$.

(3) *If* $\mathbf{P}$ *is exact in* $\mathbf{Q}$ *then* $\mathbf{P}$ *is closed in* $\mathbf{Q}$.

(4) *If* $\mathbf{P}$ *is exact in* $\mathbf{Q}$ *then* $[0, max(\mathbf{P})]^{\mathbf{Q}}$ *is exact in* $\mathbf{Q}$.

Proof. (1) is clear since part (4) of lemma 4.13 says that $\mathbf{P}$ is correct in $\mathbf{P}^+$.

(2) is immediate (see lemma 4.10 and part (2) of lemma 4.13).

(3) Since $\mathbf{P}$ is closed in $[0, max(\mathbf{P})]^{\mathbf{Q}}$ and $[0, max(\mathbf{P})]^{\mathbf{Q}}$ is closed in $\mathbf{Q}$, $\mathbf{P}$ is closed in $\mathbf{Q}$.

(4) Suppose $a \in [0, max(\mathbf{P})]^{\mathbf{Q}}$, $b \in \mathbf{Q}$, and $a \leq_1 b$. We want to show that $b \leq max(\mathbf{P})$. Argue by contradiction by assuming that $max(\mathbf{P}) < b$. Since $a \leq_1 max(\mathbf{P})$ in this case, $a \in \mathbf{P}$ by part (3) of lemma 4.11. This contradicts the fact that $\mathbf{P}$ is correct in $\mathbf{Q}$. □

That exactness is transitive will follow when we later prove that the generation relation is transitive.

Lemma 6.4 *Assume* $\mathbf{P}$ *is a pattern and* $b \in \mathbf{P}$ *has the property that* $[0, b]^{\mathbf{P}}$ *is correct in* $\mathbf{P}$*. If* $\mathbf{P}$ *generates* $\mathbf{Q}$ *then* $[0, b]^{\mathbf{P}}$ *generates* $[0, b]^{\mathbf{Q}}$.

Proof. We may assume that $\mathbf{Q}$ is exactly generated from $\mathbf{P}$. Under this assumption, we will show that $[0, b]^{\mathbf{P}}$ exactly generates $[0, b]^{\mathbf{Q}}$. A simple induction allows us to further assume that $\mathbf{Q}$ is an immediate extension of $\mathbf{P}$.

Case 1: $\mathbf{Q}$ is a simple additive extension of $\mathbf{P}$.

Assume $\mathbf{Q}$ is an extension of $\mathbf{P}$ to $a_1 + \cdots + a_{m+1}$. If $a_1 + \cdots + a_{m+1} \leq b$ then $[0, b]^{\mathbf{Q}}$ is an extension of $[0, b]^{\mathbf{P}}$ to $a_1 + \cdots + a_{m+1}$. Otherwise, $[0, b]^{\mathbf{Q}} = [0, b]^{\mathbf{P}}$.

Case 2: $\mathbf{Q}$ is obtained from $\mathbf{P}$ by reflection.

Suppose $\mathbf{Q}$ is obtained from $\mathbf{P}$ by reflecting X from c to a. If $b < a$ then $[0, b]^{\mathbf{Q}} = [0, b]^{\mathbf{P}}$. Suppose $a \leq b$. Since $[0, b]^{\mathbf{P}}$ is correct in $\mathbf{P}$ and $a \leq_1 c$, we have $c \in [0, b]^{\mathbf{P}}$. Therefore, $[0, b]^{\mathbf{Q}}$ is obtained from $[0, b]^{\mathbf{P}}$ by reflecting X from c to a. □

Definition 6.5 Assume $\mathbf{P}_1, \ldots, \mathbf{P}_n$ are patterns. An *amalgamation* of $\mathbf{P}_1, \ldots,$ $\mathbf{P}_n$ is a pattern $\mathbf{Q}$ along with embeddings f_i of $\mathbf{P}_i$ into $\mathbf{Q}$ for $i = 1, \ldots, n$ such that, letting $\mathbf{P}_i^*$ be the image of $\mathbf{P}_i$ under f_i,

(1) $\mathbf{Q} = \mathbf{P}_1^* \cup \cdots \cup \mathbf{P}_n^*$ and

(2) $\mathbf{P}_i^*$ is exact in $\mathbf{Q}$ for $i = 1, \ldots, n$.

We will often say that $\mathbf{Q}, \mathbf{P}_1^*, \ldots, \mathbf{P}_n^*$ is an amalgamation of $\mathbf{P}_1, \ldots, \mathbf{P}_n$ when $\mathbf{P}_i^*$ is a substructure of $\mathbf{Q}$ which is isomorphic to $\mathbf{P}_i$ for $i = 1, \ldots, n$ and $\mathbf{Q}, f_1, \ldots, f_n$ is an amalgamation of $\mathbf{P}_1, \ldots, \mathbf{P}_n$ where f_i is the isomorphism of $\mathbf{P}_i$ with $\mathbf{P}_i^*$ for $i = 1, \ldots, n$. We will also call $\mathbf{Q}$ an amalgamation of

$\mathbf{P}_1, \ldots, \mathbf{P}_n$ if there exist $f_1, \ldots, f_n$ such that $\mathbf{Q}, f_1, \ldots, f_n$ is an amalgamation of $\mathbf{P}_1, \ldots, \mathbf{P}_n$.

There are two simple facts worth mentioning here. The first is that if $\mathbf{Q}, \mathbf{P}_1^*, \ldots, \mathbf{P}_n^*$ is an amalgamation of $\mathbf{P}_1, \ldots, \mathbf{P}_n$ and $\tilde{\mathbf{P}}_i$ is isomorphic to $\mathbf{P}_i$ for $i = 1, \ldots, n$ then $\mathbf{Q}, \mathbf{P}_1^*, \ldots, \mathbf{P}_n^*$ is an amalgamation of $\tilde{\mathbf{P}}_1, \ldots, \tilde{\mathbf{P}}_n$. This will often allow us to make simplifying assumptions e.g. that $\mathbf{P}_i^* = \mathbf{P}_i$ for $i = 1, \ldots, n$. The second fact is that $\mathbf{P}_1 \cup \cdots \cup \mathbf{P}_n, \mathbf{P}_1, \ldots, \mathbf{P}_n$ is an amalgamation of $\mathbf{P}_1, \ldots, \mathbf{P}_n$ whenever each $\mathbf{P}_i$ is exact in some fixed pattern $\mathbf{Q}$.

Lemma 6.6 *If $\mathbf{P}_1, \ldots, \mathbf{P}_n$ are covered patterns and $\mathbf{P}_i^*$ is the isominimal substructure of $\mathcal{R}_1$ which is isomorphic to $\mathbf{P}_i$ for $i = 1, \ldots, n$ then $(\mathbf{P}_1^* \cup \cdots \cup \mathbf{P}_n^*), \mathbf{P}_1^*, \ldots, \mathbf{P}_n^*$ is an amalgamation of $\mathbf{P}_1, \ldots, \mathbf{P}_n$.*

Proof. By part (2) of theorem 5.9, $\mathbf{P}_1^* \cup \cdots \cup \mathbf{P}_n^*$ is closed in $\mathcal{R}_1$ and, consequently, is a pattern. By parts (3) and (5) of lemma 5.9, each $\mathbf{P}_i^*$ is exact in $\mathbf{P}_1^* \cup \cdots \cup \mathbf{P}_n^*$. □

Theorem 6.7 *If $\mathbf{P}_1, \ldots, \mathbf{P}_n$ are covered patterns then there is an amalgamation of $\mathbf{P}_1, \ldots, \mathbf{P}_n$. Moreover, amalgamations of $\mathbf{P}_1, \ldots, \mathbf{P}_n$ are unique up to isomorphism in the sense that if $\mathbf{Q}_1, \mathbf{P}_1^1, \ldots, \mathbf{P}_n^1$ and $\mathbf{Q}_2, \mathbf{P}_1^2, \ldots, \mathbf{P}_n^2$ are both amalgamations of $\mathbf{P}_1, \ldots, \mathbf{P}_n$ then there is an isomorphism f of $\mathbf{Q}_1$ and $\mathbf{Q}_2$ which maps $\mathbf{P}_i^1$ onto $\mathbf{P}_i^2$ for $i = 1, \ldots, n$.*

Proof. The existence of an amalgamation of $\mathbf{P}_1, \ldots, \mathbf{P}_n$ was established in the previous lemma.

To show uniqueness, assume $\mathbf{Q}_1, \mathbf{P}_1^1, \ldots, \mathbf{P}_n^1$ and $\mathbf{Q}_2, \mathbf{P}_1^2, \ldots, \mathbf{P}_n^2$ are amalgamations of $\mathbf{P}_1, \ldots, \mathbf{P}_n$. Notice that $\mathbf{Q}_j$ is generated from some $\mathbf{P}_i^j$ for $j = 1, 2$. By lemma 5.3, each $\mathbf{Q}_j$ is covered. Using the previous lemma again, there is an amalgamation $\mathbf{Q}, \mathbf{Q}_1^*, \mathbf{Q}_2^*$ of $\mathbf{Q}_1$ and $\mathbf{Q}_2$. Without loss of generality, we may assume that $\mathbf{Q}_j = \mathbf{Q}_j^*$ for $j = 1, 2$. Under this assumption, the conclusion of the theorem reduces to showing that $\mathbf{P}_i^1 = \mathbf{P}_i^2$ for $i = 1, \ldots, n$.

Fix i with $1 \leq i \leq n$. Without loss of generality, assume that $\mathbf{P}_i^1 \leq_{lex} \mathbf{P}_i^2$. Since $\mathbf{Q}_2$ generates $[0, max(\mathbf{Q}_2)]^{\mathbf{Q}}$, lemma 6.4 implies that $[0, max(\mathbf{P}_i^2)]^{\mathbf{Q}}$ is generated from $\mathbf{P}_i^2$. Since $[0, max(\mathbf{P}_i^2)]^{\mathbf{Q}}$ contains $\mathbf{P}_i^1$, lemma 5.5 implies that $\mathbf{P}_i^2 \leq_{pw} \mathbf{P}_i^1$ which in turn implies that $\mathbf{P}_i^2 \leq_{lex} \mathbf{P}_i^1$. Therefore, $\mathbf{P}_i^1 = \mathbf{P}_i^2$. □

Definition 6.8 A *pointed pattern* is a pair $(\mathbf{P}, a)$ where $\mathbf{P}$ is a pattern and $a \in \mathbf{P}$. For $(\mathbf{P}, a)$ a pointed pattern where $\mathbf{P}$ is covered, define $\iota(\mathbf{P}, a)$ to be the image of a under the isomorphism of $\mathbf{P}$ with the isominimal substructure of $\mathcal{R}_1$ which is isomorphic to it. $\mathcal{C}_1$ is the prestructure for the language $\{0, +, \leq, \leq_1\}$ whose universe consists of all pointed patterns $(\mathbf{P}, a)$ where $\mathbf{P}$ is covered such that the interpretations of $0, +, \leq$, and $\leq_1$ are made so that the restriction of ι to the universe of $\mathcal{C}_1$ is a homomorphism of $\mathcal{C}_1$ into $\mathcal{R}_1$ e.g. $(\mathbf{P}_1, a_1) + (\mathbf{P}_2, a_2) = (\mathbf{P}_3, a_3)$ in $\mathbf{C}_1$ iff $\iota(\mathbf{P}_1, a_1) + \iota(\mathbf{P}_2, a_2) = \iota(\mathbf{P}_3, a_3)$ in $\mathcal{R}_1$.

$\mathcal{C}_1$ and ι are too large to be sets. Even if we restrict $\mathcal{C}_1$ to the hereditarily finite pointed patterns, $KP + Infinity$ is not strong enough to prove that the restriction exists as a set. However, ι and the universe of $\mathcal{C}_1$ along with the interpretations of the logical symbols and their complements in $\mathcal{C}_1$ are all Σ_1 definable classes.

Lemma 6.9 *$\mathcal{C}_1$ is a prestructure for the language $\{0, +, \leq, \leq_1\}$.*

Proof. Straightforward. □

Theorem 6.10 *If every pattern is covered then the restriction of $\mathcal{C}_1$ to the collection of hereditarily finite pointed patterns is a recursive structure.*

Proof. Let $\mathcal{C}_1{}'$ be the restriction of $\mathcal{C}_1$ described in the hypothesis of the theorem. Clearly, the universe of $\mathcal{C}_1{}'$ is recursive under this assumption, and the interpretation of 0 is the recursive set consisting of all $(\mathbf{P}, a) \in \mathcal{C}_1$ where a is the interpretation of 0 in $\mathbf{P}$.

To see that the interpretation of $\leq$ is recursive notice that by lemma 6.6 and theorem 6.7

$$(\mathbf{P}, a) \leq (\mathbf{Q}, b) \text{ iff } \begin{array}{l}\text{there exists a hereditarily finite amalgamation} \\ \mathbf{R}, f, g \text{ of } \mathbf{P} \text{ and } \mathbf{Q} \text{ such that } f(a) \leq g(b)\end{array}$$

and

$$(\mathbf{P}, a) \not\leq (\mathbf{Q}, b) \text{ iff } \begin{array}{l}\text{there exists a hereditarily finite amalgamation} \\ \mathbf{R}, f, g \text{ of } \mathbf{P} \text{ and } \mathbf{Q} \text{ such that } f(a) \not\leq g(b)\end{array}$$

Similar reasoning shows that the interpretations of $=$, 0, $+$, and $\leq_1$ are recursive. □

Notice that the proof of the theorem establishes that any substructure of $\mathcal{C}_1$ with a recursive domain is recursive.

Corollary 6.11 *If every pattern is covered then the core of $\mathcal{R}_1$ is a recursive ordinal.*

Proof. By the theorem, the restriction $\mathcal{C}_1{}'$ of $\mathcal{C}_1$ to the collection of herditarily finite pointed patterns is a recursive structure. The structure $\mathcal{C}_1{}'/{=}$ exists and is isomorphic to the core. Let X be a recursive subset of the set of hereditarily finite sets such that X contains one element from each equivalence class of $=^{\mathcal{C}_1}$ e.g. let X consist of the first element of each equivalence class under some recursive enumeration of the hereditarily finite sets. The restriction of $\mathcal{C}_1$ to X is clearly isomorphic to $\mathcal{C}_1{}'/{=}$. Therefore, the universe of the core is a recursive ordinal. □

Corollary 6.12 *Assuming ZF, the core is a recursive ordinal.*

Proof. By lemma 6.1, every pattern is covered assuming ZF. □

7 Arbitrary Amalgamations and $\mathcal{P}_1/=$

In this section we will generalize results of earlier sections to show how to amalgamate arbitrary patterns and produce a recursive structure extending $\mathcal{C}_1$ whose universe consists of all pointed patterns. In case every pattern is covered, this structure will be identical to $\mathcal{C}_1$.

This section can be formalized in $I\Sigma_0(exp)$. Thus, we provide elementary proofs of some results in previous sections. On the other hand, without having the use of $\mathcal{R}_1$ the proofs in this section become enmeshed in technical details.

The following lemma will imply that any isominimal substructure of our structure will be a closed substructure.

Lemma 7.1 *Assume* $\mathbf{P}$ *and* $\mathbf{Q}$ *are patterns. If* h *is a covering of* $\mathbf{P}$ *into* $\mathbf{Q}$ *then there exists* $\mathbf{Q}^+$ *which is exactly generated from* $\mathbf{Q}$ *and a covering* g *of* $\mathbf{P}$ *into* $\mathbf{Q}^+$ *such that the range of* g *is a closed subpattern of* $\mathbf{Q}^+$ *and* $g \leq h$. *In fact,* g *can be chosen so that for any indecomposable* a *of* $\mathbf{P}$, $g(a)$ *is the maximal component of* $h(a)$.

Proof. Let I be the set of indecomposables of $\mathbf{P}$ and define $g_0 : I \to \mathbf{Q}$ so that $g_0(a)$ is the maximal component of $h(a)$. Let $\mathbf{Q}^+$ be an exact extension of $\mathbf{Q}$ such that $\langle g_0(a_1), \ldots, g_0(a_m)\rangle$ is in the addition tree of $\mathbf{Q}^+$ whenever $\langle a_1, \ldots, a_m\rangle$ is in the addition tree of $\mathbf{P}$. By lemma 3.13, fix an extension g of g_0 which is an embedding of the additive part of $\mathbf{P}$ onto a closed substructure of of the additive part of $\mathbf{Q}^+$. Since $g_0(a) \leq h(a)$ for $a \in I$, $g \leq h$.

In order to show that g is a covering, suppose $a, b \in \mathbf{P}$ and $a <_1 b$. First notice that $h(a)$ is indecomposable in $\mathbf{Q}$ since $h(a) <_1 h(b)$. Therefore, $g(a) = g_0(a) = h(a) <_1 h(b)$. Since $g(a) < g(b) \leq h(b)$, $g(a) <_1 g(b)$. □

The following lemma allows revisions of a given covering under certain circumstances.

Lemma 7.2 (revision) *Assume* h *is a covering of* $\mathbf{Q}$ *into* $\mathbf{R}$. *If* $\mathbf{P}$ *is an initial segment of* $\mathbf{Q}$ *and* f *is a covering of* $\mathbf{P}$ *into* $\mathbf{R}$ *such that for all* $x \in \mathbf{P}$

(a) $f(x) \leq h(x)$ *and*

(b) $f(x) \leq_1 h(x)$ *if* $x <_1 y$ *for some* $y \in \mathbf{Q} - \mathbf{P}$

then there exists a pattern $\mathbf{R}^+$ *which is exactly generated from* $\mathbf{R}$ *and a covering* $\tilde{h}$ *of* $\mathbf{Q}$ *into* $\mathbf{R}^+$ *such that*

(1) $\tilde{h}(x) = h(x)$ *if* x *is indecomposable and* $x \in \mathbf{Q} - \mathbf{P}$,

(2) $\tilde{h} \leq h$, *and*

(3) $\tilde{h}$ *extends* f.

Proof. Define a function $\tilde{h}_0$ on the set of indecomposables of $\mathbf{Q}$ into $\mathbf{R}$ by

$$\tilde{h}_0(x) = \begin{cases} f(x) & \text{if } x \in \mathbf{P} \\ h(x) & \text{if } x \in \mathbf{Q} - \mathbf{P} \end{cases}$$

By lemmas 3.13 and 4.14, there exists $\mathbf{R}^+$ which is exactly generated from $\mathbf{R}$ and an embedding $\tilde{h}$ of the arithmetic part of $\mathbf{P}$ into the arithmetic part of $\mathbf{R}^+$ which extends $\tilde{h}_0$.

Condition (1) holds since $\tilde{h}$ extends $\tilde{h}_0$. Condition (2) follows from the fact that $\tilde{h}(x) \leq h(x)$ for every indecomposable x in $\mathbf{Q}$. Condition (3) holds since $\tilde{h}(x) = f(x)$ for all indecomposables x in $\mathbf{P}$.

In order to see that $\tilde{h}$ is a covering, assume that $x, y \in \mathbf{Q}$ and $x <_1 y$. Notice that x is indecomposable. If $y \in \mathbf{P}$ then $x \in \mathbf{P}$ and $\tilde{h}(x) = f(x) \leq_1 f(y) = \tilde{h}(y)$. So, we may assume that $y \notin \mathbf{P}$.

Case 1: $x \in \mathbf{P}$.

Since $h(x) \leq_1 h(y)$ and assumption (b) implies that $\tilde{h}(x) \leq_1 h(x)$, we see that $\tilde{h}(x) \leq_1 h(y)$. By (2), $\tilde{h}(x) \leq \tilde{h}(y) \leq h(y)$. Therefore, $\tilde{h}(x) \leq_1 \tilde{h}(y)$.

Case 2: $x \notin \mathbf{P}$.

By (1), $\tilde{h}$(x)=h(x). We see that $h(x) = \tilde{h}(x) \leq \tilde{h}(y) \leq h(y)$. Since $h(x) \leq_1 h(y)$, $h(x) \leq_1 \tilde{h}(y)$. □

Lemma 7.3 (extension) *Assume* $\mathbf{P}$ *and* $\mathbf{Q}$ *are patterns and* h *is a covering of* $\mathbf{P}$ *into* $\mathbf{Q}$*. If* $\mathbf{P}^+$ *is exactly generated from* $\mathbf{P}$ *then* $\mathbf{Q}$ *exactly generates some* $\mathbf{Q}^+$ *with the property that there is a covering of* $\mathbf{P}^+$ *into* $\mathbf{Q}^+$ *which extends* h*.*

Proof. The proof is a modification of that of lemma 5.3.

As before, we may assume that $\mathbf{P}^+$ is an immediate extension of $\mathbf{P}$.

Case 1: $\mathbf{P}^+$ is an extension of $\mathbf{P}$ to $a_1 + \cdots + a_{m+1}$ for some $a_1, \ldots, a_{m+1}$.

Fix $\mathbf{Q}^+$ which is exactly generated from $\mathbf{Q}$ such that $h(a_1) + \cdots + h(a_{m+1})$ is defined in $\mathbf{Q}^+$. Now proceed as in lemma 5.3 with $\mathcal{R}_1$ replaced by $\mathbf{Q}^+$.

Case 2: $\mathbf{P}^+$ is obtained from $\mathbf{P}$ by reflecting X from b to a for some X, b, and a.

As before, let $\tilde{X} = \mathbf{P}^+ - \mathbf{P}$. Define $\mathbf{R}$ to be the substructure of $\mathbf{Q}$ whose universe is the range of h, and let Y be the image of X under h. Since $h(a) \leq_1 h(b)$, there is a pattern $\mathbf{Q}^+$ obtained from $\mathbf{Q}$ by reflecting $[h(a), h(b))^{\mathbf{Q}}$ from $h(b)$ to $h(a)$. Let $\tilde{Y}$ be the image of Y under the isomorphism of $[0, h(b))^{\mathbf{Q}}$ with $[0, h(a))^{\mathbf{Q}^+}$. Now proceed as before with $\mathcal{R}_1$ replaced by $\mathbf{Q}^+$. □

Lemma 7.4 *Assume* $\mathbf{P}$ *exactly generates* $\mathbf{Q}$*,* $a \in \mathbf{P}$*, and* b *is the largest element of* $\mathbf{P}$ *such that* $a \leq_1 b$*. If* c *and* d *are elements of* $\mathbf{Q}$ *which satisfy*

(a) $[0, a)^{\mathbf{P}} < c \leq d < a$,

(b) $c \leq_1 d$,

(c) c *is indecomposable, and*

(d) $x \in \mathbf{P}$ *whenever* $x <_1 c$

then there exists $\mathbf{Q}^+$ *which is exactly generated from* $\mathbf{Q}$ *and a covering* h *of* $[0,d]^{\mathbf{Q}}$ *into* $\mathbf{Q}^+$ *such that*

(1) $h(x) = x$ *if* $x < c$,

(2) $a \leq h(c)$, *and*

(3) $h(d) < b$.

Proof. Let $\mathbf{Q}_0, \ldots, \mathbf{Q}_n$ be a sequence of patterns such that $\mathbf{Q}_0 = \mathbf{P}$, $\mathbf{Q}_n = \mathbf{Q}$, and $\mathbf{Q}_{i+1}$ is immediately generated from $\mathbf{Q}_i$ for $i < n$. We will show that the theorem holds with $\mathbf{Q}$ replaced by $\mathbf{Q}_i$ by induction on i.

The theorem trivially holds for $\mathbf{Q}_0$ since there are no c and d satisfying the hypothesis.

Assume $i < n$ and that the theorem holds for $\mathbf{Q}_i$. Let c and d be elements of $\mathbf{Q}_{i+1}$ which satisfy conditions (a)-(d). Without loss of generality, d is the largest element of $\mathbf{Q}_{i+1}$ such that $c \leq_1 d$.

Case 1: $c \notin \mathbf{Q}_i$.

Since c is indecomposable, $\mathbf{Q}_{i+1}$ must be obtained from reflecting X from d' to c' for some X, c', and d'. Without loss of generality, we may assume d' is the largest x in $\mathbf{Q}_i$ such that $c' \leq_1 x$. Let $\tilde{X}$ be $\mathbf{Q}_{i+1} - \mathbf{Q}_i$ and let g be an isomorphism of $[0,c')^{\mathbf{Q}_i} \cup \tilde{X}$ with $[0,c')^{\mathbf{Q}_i} \cup X$. Let g_0 be the restriction of g to $[0,d]^{\mathbf{Q}_{i+1}}$.

Notice that $c' \leq a$ since $c < a$.

Subcase 1 of case 1: $c' = a$.

Let f be the identity function on $[0,c)^{\mathbf{Q}_{i+1}}$. By the the revision lemma (lemma 7.2), there is a covering h of $[0,d]^{\mathbf{Q}_{i+1}}$ into some $\mathbf{Q}^+$ which is exactly generated from $\mathbf{Q}_{i+1}$ such that h extends f, $h \leq g_0$, and $h(x) = g_0(x)$ whenever x is an indecomposable element of $[c,d]^{\mathbf{Q}_{i+1}}$.

Notice that $a \leq g_0(c)$ and $g_0(d) < b$ since $X \subseteq [a,b)^{\mathbf{Q}_i}$. Since c is indecomposable, $h(c) = g_0(c) < b$ implying $a \leq h(c)$. Since $h \leq g_0$, $h(d) \leq g_0(d) < b$.

Subcase 2 of case 1: $c' < a$.

c' and d' satisfy conditions (a)-(d). To see (d) holds, notice that if $x <_1 c'$ then $x <_1 c$. By the induction hypothesis, there is $\mathbf{Q}'$ which is exactly generated from $\mathbf{Q}_i$ and a covering h' of $[0,d']^{\mathbf{Q}_i}$ into $\mathbf{Q}'$ such that h' is the identity on $[0,c')^{\mathbf{Q}_i}$, $a \leq h'(c')$, and $h'(d') < b$. By the extension lemma (lemma 7.3), there exists $\mathbf{Q}''$ which is exactly generated from $\mathbf{Q}_{i+1}$ and a covering h'' of $\mathbf{Q}'$ into $\mathbf{Q}''$ which is the identity on $\mathbf{Q}_i$.

Let h''' be $h'' \circ h' \circ g_0$. h''' is a covering of $[0,d]^{\mathbf{Q}_{i+1}}$ into $\mathbf{Q}''$. Moreover, $a = h''(a) \leq h''(h'(c')) \leq h''(h'(g_0(c))) = h'''(c)$ and $h'''(d) = h''(h'(g_0(d))) < h''(h'(d')) < h''(b) = b$. Letting f be the identity on $[0,c)^{\mathbf{Q}_{i+1}}$, we can proceed as in subcase 1 with g_0 replaced by h''' to find $\mathbf{Q}^+$ which is exactly generated from $\mathbf{Q}''$ and a covering h of $[0,d]^{\mathbf{Q}_{i+1}}$ into $\mathbf{Q}^+$ which satisfy the conclusion of the lemma (with $\mathbf{Q}$ replaced by $\mathbf{Q}_{i+1}$).

Case 2: $c \in \mathbf{Q}_i$.

Since we have assumed d is the maximal x in $\mathbf{Q}_{i+1}$ with $c \leq_1 x$, we see that $d \in \mathbf{Q}_i$. By the induction hypothesis, there is some $\mathbf{Q}'$ which is exactly generated from $\mathbf{Q}_i$ and a covering h' of $[0,d]^{\mathbf{Q}_i}$ into $\mathbf{Q}'$ such that h' is the identity on $[0,c)^{\mathbf{Q}_i}$, $a \leq h'(c)$, and $h'(d) < b$. By the extension lemma, there exists $\mathbf{Q}''$ which is exactly generated from $\mathbf{Q}_{i+1}$ and a covering h'' of $\mathbf{Q}'$ into $\mathbf{Q}''$ which is the identity on $\mathbf{Q}_i$. Let h''' be $h'' \circ h'$. h''' is a covering of $[0,d]^{\mathbf{Q}_i}$ into $\mathbf{Q}''$ which is the identity on $[0,c)^{\mathbf{Q}_i}$. Moreover, $a = h''(a) \leq h''(h'(c)) = h'''(c)$ and $h'''(d) = h''(h'(d)) < h''(b) = b$. If $[0,d]^{\mathbf{Q}_{i+1}} = [0,d]^{\mathbf{Q}_i}$ then $h = h'''$ and $\mathbf{Q}^+ = \mathbf{Q}''$ satisfy the conclusion of the lemma. So, suppose that $[0,d]^{\mathbf{Q}_{i+1}} \neq [0,d]^{\mathbf{Q}_i}$.

Subcase 1 of case 2: $\mathbf{Q}_{i+1}$ is an extension of $\mathbf{Q}_i$ to $a_1 + \cdots + a_{m+1}$ for some $a_1, \ldots, a_{m+1}$.

$[0,d]^{\mathbf{Q}_{i+1}}$ is clearly an extension of $[0,d]^{\mathbf{Q}_i}$ to $a_1 + \cdots + a_{m+1}$. By the extension lemma, there exists $\mathbf{Q}^+$ which is exactly generated from $\mathbf{Q}''$ and a covering h of $[0,d]^{\mathbf{Q}_{i+1}}$ into $\mathbf{Q}^+$ which extends h'''. $\mathbf{Q}^+$ and h are as required.

Subcase 2 of case 2: $\mathbf{Q}_{i+1}$ is obtained from $\mathbf{Q}_i$ by reflecting X from v to u for some X, u, and v.

We first consider the possibility that $u < c$.

Extend h''' to h with domain $[0,d]^{\mathbf{Q}_{i+1}}$ by defining $h(x) = x$ for $x \in \mathbf{Q}_{i+1} - \mathbf{Q}_i$. The conclusion of the lemma holds after setting $\mathbf{Q}^+ = \mathbf{Q}''$. The straightforward proof is left to the reader.

We conclude with the possibility that $c \leq u$.

Using the assumption that d is the largest x in $\mathbf{Q}_{i+1}$ such that $c \leq_1 x$ and the fact that $[0,d]^{\mathbf{Q}_i} \neq [0,d]^{\mathbf{Q}_{i+1}}$, we see that $v \leq d$. This implies that $[0,d]^{\mathbf{Q}_{i+1}}$ is obtained from $[0,d]^{\mathbf{Q}_i}$ by reflecting X from v to u. By the extension lemma, there exists $\mathbf{Q}^+$ which is exactly generated from $\mathbf{Q}''$ and a covering h of $[0,d]^{\mathbf{Q}_{i+1}}$ into $\mathbf{Q}^+$ which extends h'''. $\mathbf{Q}^+$ and h satisfy the conclusion of the lemma. □

The following lemma implies that lemma 5.5 generalizes to all patterns. In particular, $\mathbf{P}$ cannot generate a pattern $\mathbf{Q}$ that contains a copy of $\mathbf{P}$ which is not above $\mathbf{P}$ in the pointwise ordering. For that reason we refer to the lemma as the *nonduplication lemma.*

Lemma 7.5 (nonduplication) *Assume* $\mathbf{P}$ *is a pattern and* a *and* b *are elements of* $\mathbf{P}$ *such that* $a \leq b$ *and* $[a,b]^{\mathbf{P}}$ *is correct in* $\mathbf{P}$. *If* h *is a covering of* $[0,b]^{\mathbf{P}}$ *into some* $\mathbf{Q}$ *which is generated from* $\mathbf{P}$ *such that* $x \leq h(x)$ *whenever* $x < a$ *then* $x \leq h(x)$ *whenever* $x \leq b$.

Proof. With $\mathbf{P}$ fixed, we argue by induction on the cardinality of $[a,b]^{\mathbf{P}}$.

Assume $n \in \omega$ and the lemma holds under the additional condition that $card([a,b]^{\mathbf{P}}) < n$.

Fix $\mathbf{P}$, a, and b as in the assumption of the lemma where $card([a,b]^{\mathbf{P}}) = n$. Let h be a covering of $[0,b]^{\mathbf{P}}$ into $\mathbf{Q}$ where $\mathbf{Q}$ is generated from $\mathbf{P}$ and assume that $x \leq h(x)$ whenever $x < a$. We may assume that $\mathbf{Q}$ is exactly generated from $\mathbf{P}$.

Case 1: $a \leq h(a)$.

Noting that $(a, b]^{\mathbf{P}}$ is correct in $\mathbf{P}$, we may apply the induction hypothesis, if necessary, to see that $x \leq h(x)$ for all $x \leq b$.

Case 2: $h(a) < a$.

Notice that a must be indecomposable since otherwise there would be $a_1, a_2 \in [0, a)^{\mathbf{P}}$ such that $a = a_1 + a_2$ implying $a = a_1 + a_2 \leq h(a_1) + h(a_2) = h(a_1 + a_2) = h(a)$.

Let b' be the largest x in $\mathbf{P}$ such that $a \leq_1 x$. Since $[a, b]^{\mathbf{P}}$ is correct in $\mathbf{P}$, we must have $b' \leq b$.

Subcase 1 of case 2: $b' < b$.

Since $[a, b']^{\mathbf{P}}$ is correct in $\mathbf{P}$, the induction hypothesis implies that $x \leq h(x)$ whenever $x \leq b'$. This contradicts the fact that $h(a) < a$.

Subcase 2 of case 2: $b' = b$.

For $x \in [0, a)^{\mathbf{P}}$, $x + h(a) \leq h(x) + h(a) = h(x + a) = h(a)$ (the last equality follows from the fact that a is indecomposable). By parts (1) and (2) of lemma 3.4, $[0, a)^{\mathbf{P}} < mc(h(a)) < a$. This implies, by part (5) of lemma 4.11, that $a < b'$ i.e. $a < b$. Therefore, $h(a) <_1 h(b)$ implying $h(a)$ is indecomposable. By part (3) of lemma 4.11, $h(a) \not\leq_1 a$ implying $h(b) < a$. Let c be the least x in $\mathbf{Q}$ such that $[0, a)^{\mathbf{P}} < x$ and $x \leq_1 h(a)$. Since $h(a)$ is indecomposable, so is c. Using part (3) of lemma 4.11 again, we see that if $y <_1 c$ then $y \in \mathbf{P}$. The previous lemma applies to provide a covering g of $[0, h(b)]^{\mathbf{Q}}$ into some $\mathbf{Q}^+$ which is exactly generated from $\mathbf{Q}$ such that $g(x) = x$ whenever $x < c$, $a \leq g(c)$, and $g(h(b)) < b$.

Notice that $x \leq g(x)$ for $x \leq h(b)$. Let $\tilde{h}$ be $g \circ h$. For $x < a$, we have $x \leq h(x) \leq g(h(x)) = \tilde{h}(x)$. Moreover, $a \leq g(c) \leq g(h(a)) = \tilde{h}(a)$. As in case 1, the induction hypothesis implies that $x \leq \tilde{h}(x)$ whenever $x \leq b$. This contradicts the fact that $\tilde{h}(b) = g(h(b)) < b$. □

Lemma 7.6 (minimality) *Assume* $\mathbf{P}$ *is an exact subpattern of* $\mathbf{Q}$. *If* $\mathbf{P}'$ *is a subpattern of* $\mathbf{Q}$ *which is a cover of* $\mathbf{P}$ *then* $\mathbf{P} \leq_{pw} \mathbf{P}'$.

Proof. Let h be the covering of $\mathbf{P}$ into $\mathbf{Q}$ with range $\mathbf{P}'$. Since $\mathbf{P}$ generates $[0, max(\mathbf{P})]^{\mathbf{Q}}$, the extension lemma implies there is a covering h' of $[0, max(\mathbf{P})]^{\mathbf{Q}}$ into some $\mathbf{Q}^+$ which is exactly generated from $\mathbf{Q}$ such that h' extends h. Since $[0, max(\mathbf{P})]^{\mathbf{Q}}$ is correct in $\mathbf{Q}$ by part (4) of lemma 6.3, the nonduplication lemma implies that $x \leq h'(x)$ for all x in $[0, max(\mathbf{P})]^{\mathbf{Q}}$. Since h' extends h, $x \leq h(x)$ for all x in $\mathbf{P}$ i.e. $\mathbf{P} \leq_{pw} \mathbf{P}'$. □

Suppose $\mathbf{P}$ and $\mathbf{P}_1, \ldots, \mathbf{P}_n$ are patterns. The previous lemma implies that if there exist $\mathbf{P}_1^*, \ldots, \mathbf{P}_n^*$ such that $\mathbf{P}, \mathbf{P}_1^*, \ldots, \mathbf{P}_n^*$ is an amalgamation of $\mathbf{P}_1, \ldots, \mathbf{P}_n$ then there are unique such $\mathbf{P}_1^*, \ldots, \mathbf{P}_n^*$.

Lemma 7.7 *If* h *is a covering of* $\mathbf{P}$ *into* $\mathbf{Q}$ *then there exists* $\mathbf{Q}^+$ *which is exactly generated from* $\mathbf{Q}$ *and an embedding* $\tilde{h}$ *of* $\mathbf{P}$ *into* $\mathbf{Q}^+$ *such that* $\tilde{h} \leq h$ *and the range of* $\tilde{h}$ *is a correct and closed subpattern of* $\mathbf{Q}^+$.

Proof. By the lemma 7.1, we can assume that the range of h is a closed subpattern of $\mathbf{Q}$.

The following definitions will be in effect only for the current proof. For each a in $\mathbf{P}$, let $m(a)$ be the largest x in $\mathbf{P}$ such that $a \leq_1 x$. Suppose g is a covering of $\mathbf{P}$ into some $\mathbf{R}$. If $a \in \mathbf{P}$, g is *correct at* a if $g(m(a))$ is the largest x in $\mathbf{R}$ such that $g(a) \leq_1 x$. For b in $\mathbf{P}$, we say that g is *correct up to* b if g is correct at a whenever $m(a) \leq b$.

We will show by induction over the ordering of $\mathbf{P}$ that for each b in $\mathbf{P}$ there exist $\mathbf{Q}^+$ which is exactly generated from $\mathbf{Q}$ and a covering $\tilde{h}$ of $\mathbf{P}$ into $\mathbf{Q}^+$ which is correct up to b such that $\tilde{h} \leq h$ and the range of $\tilde{h}$ is a closed subpattern of $\mathbf{Q}^+$. Notice that if b is the largest element of $\mathbf{P}$ then $\mathbf{Q}^+$ and $\tilde{h}$ satisfy the conclusion of the lemma.

Let J be the set of all b in $\mathbf{P}$ such that there exist $\mathbf{Q}^+$ and $\tilde{h}$ as in the previous paragraph.

Assume $b \in \mathbf{P}$ and $b' \in J$ whenever $b' < b$. There exist $\mathbf{Q}'$ which is exactly generated from $\mathbf{Q}$ and a covering h' of $\mathbf{P}$ into $\mathbf{Q}'$ such that h' is correct at a whenever $m(a) < b$, $h' \leq h$, and the range of h' is a closed subpattern of $\mathbf{Q}'$ (if $b = 0$, let $h' = h$).

If h' is correct up to b then $\mathbf{Q}'$ and h' witness that $b \in J$. So, assume h' is not correct up to b. This means there is some a in $\mathbf{P}$ such that $m(a) = b$ and $h'(a) <_1 c$ for some c with $h'(m(a)) < c$. This implies that the least element x of $\mathbf{P}$ with $m(x) = b$ has this property. So, we may assume that a is the least x in $\mathbf{P}$ such that $m(x) = b$. Let X be the image of $[a, b]^{\mathbf{P}}$ under h' and let $\mathbf{Q}''$ be obtained from $\mathbf{Q}'$ by reflecting X from c to $h'(a)$. Setting $\tilde{X} = \mathbf{Q}'' - \mathbf{Q}'$, there is an isomorphism g of $[0, h'(a))^{\mathbf{Q}'} \cup X$ with $[0, h'(a))^{\mathbf{Q}'} \cup \tilde{X}$. Define an covering f of $[0, b]^{\mathbf{P}}$ into $\mathbf{Q}''$ by $f(x) = g(h'(x))$. Notice that the range of f is a closed subpattern of $\mathbf{Q}''$. By the revision lemma, there is some $\mathbf{Q}^+$ which is exactly generated from $\mathbf{Q}''$ and a covering $\tilde{h}$ of $\mathbf{P}$ into $\mathbf{Q}^+$ such that $\tilde{h}$ extends f, $\tilde{h} \leq h'$, and $\tilde{h}(x) = h'(x)$ if x is an indecomposable of $\mathbf{P}$ with $b < x$.

Since the ranges of f and h' are closed subpatterns of $\mathbf{Q}^+$, the range of $\tilde{h}$ is also. In addition, $\tilde{h} \leq h' \leq h$.

In order to show that $\tilde{h}$ is correct up to b, assume $x \in \mathbf{P}$ and $m(x) \leq b$.

Case 1: $m(x) < a$.

Since h' is correct at x, $\tilde{h}(x) = h'(x)$, and $\tilde{h}(m(x)) = h'(m(x))$, we see that $\tilde{h}$ is correct at x.

Case 2: $a \leq m(x) < b$.

Since $a \leq_1 b$, we must have $a < x$. Let y be the least element z of $\mathbf{P}$ with $m(x) < z$. Since $\tilde{h}(y) = f(y)$, $\tilde{h}(y)$ is easily seen to be the least element z of $\mathbf{Q}^+$ with $\tilde{h}(m(x)) < z$. Using the fact that $\tilde{h}$ extends the embedding f again, we see $\tilde{h}(x) \not\leq_1 z$. Therefore, $\tilde{h}$ is correct at x.

Case 3: $m(x) = b$.

By choice of a, we have $a \leq_1 x$. Since $\tilde{h}(b)$ is the largest element of $\tilde{X}$, $h'(a)$ is the least element z of $\mathbf{Q}^+$ with $\tilde{h}(b) < z$. Since $\tilde{h}(x) \in \tilde{X}$, $\tilde{h}(x) \not\leq_1 h'(a)$ implying that $\tilde{h}$ is correct at x. □

Lemma 7.8 (transitivity of generation) *If* $\mathbf{P}$ *generates* $\mathbf{Q}$ *and* $\mathbf{Q}$ *generates* $\mathbf{R}$ *then* $\mathbf{P}$ *generates* $\mathbf{R}$.

Proof. There exists $\mathbf{P}'$ which is exactly generated from $\mathbf{P}$ such that $\mathbf{Q}$ is a subpattern of $\mathbf{P}'$. By the extension lemma, there is a covering h of $\mathbf{R}$ into some $\mathbf{P}''$ which is exactly generated from $\mathbf{P}'$ such that $h(x) = x$ whenever $x \in \mathbf{Q}$. By the previous lemma, there is an embedding f of $\mathbf{R}$ into some $\mathbf{P}'''$ which is exactly generated from $\mathbf{P}''$ such that $f \leq h$. Notice that $f(x) \leq x$ whenever $x \in \mathbf{P}$. By the nonduplication lemma, $x \leq f(x)$ whenever $x \in \mathbf{P}$. Therefore, $x = f(x)$ whenever $x \in \mathbf{P}$ and we see that the range of f is isomorphic to $\mathbf{R}$ over $\mathbf{P}$. This implies that $\mathbf{R}$ is generated from $\mathbf{P}$. □

Lemma 7.9 (transitivity of exactness) *If* $\mathbf{P}$ *is an exact subpattern of* $\mathbf{Q}$ *and* $\mathbf{Q}$ *is an exact subpattern of* $\mathbf{R}$ *then* $\mathbf{P}$ *is an exact subpattern of* $\mathbf{R}$.

Proof. Since the correctness relation is transitive, $\mathbf{P}$ is correct in $\mathbf{R}$.

By part (4) of lemma 6.3, $[0, max(\mathbf{P})]^{\mathbf{Q}}$ is correct in $\mathbf{Q}$. Since $\mathbf{Q}$ generates $[0, max(\mathbf{Q})]^{\mathbf{R}}$, lemma 6.4 implies that $[0, max(\mathbf{P})]^{\mathbf{Q}}$ generates $[0, max(\mathbf{P})]^{\mathbf{R}}$. Since $\mathbf{P}$ generates $[0, max(\mathbf{P})]^{\mathbf{Q}}$, the previous lemma implies that $\mathbf{P}$ generates $[0, max(\mathbf{P})]^{\mathbf{R}}$. □

The following lemma says that amalgamation can be reduced to a binary operation which is associative.

Recall that $\mathbf{P}$ is said to be an amalgamation of $\mathbf{P}_1, \ldots, \mathbf{P}_n$ iff there are $\mathbf{P}_1^*, \ldots, \mathbf{P}_n^*$ such that $\mathbf{P}_1, \mathbf{P}_1^*, \ldots, \mathbf{P}_n^*$ is an amalgamation of $\mathbf{P}_1, \ldots, \mathbf{P}_n$.

Lemma 7.10 (associativity) *If* $\mathbf{Q}_i$ *is an amalgamation of* $\mathbf{P}_1^i, \ldots, \mathbf{P}_{n_i}^i$ *for* $i = 1, 2$ *and* $\mathbf{Q}$ *is an amalgamation of* $\mathbf{Q}_1, \mathbf{Q}_2$ *then* $\mathbf{Q}$ *is an amalgamation of* $\mathbf{P}_1^1, \ldots, \mathbf{P}_{n_1}^1, \mathbf{P}_1^2, \ldots, \mathbf{P}_{n_2}^2$.

Proof. Without loss of generality, we may assume that $\mathbf{Q}, \mathbf{Q}_1, \mathbf{Q}_2$ is an amalgamation of $\mathbf{Q}_1, \mathbf{Q}_2$ and $\mathbf{Q}_i, \mathbf{P}_1^i, \ldots, \mathbf{P}_{n_i}^i$ is an amalgamation of $\mathbf{P}_1^i, \ldots, \mathbf{P}_{n_i}^i$ for $i = 1, 2$.

By the previous lemma, $\mathbf{P}_j^i$ is exact in $\mathbf{Q}$ whenever $1 \leq i \leq 2$ and $1 \leq j \leq n_i$. Also, $\mathbf{Q} = \mathbf{Q}_1 \cup \mathbf{Q}_2 = (\mathbf{P}_1^1 \cup \cdots \cup \mathbf{P}_{n_1}^1) \cup (\mathbf{P}_1^2 \cup \cdots \cup \mathbf{P}_{n_2}^2)$. □

Lemma 7.11 *Assume* $\mathbf{P}$ *is a pattern and* $b \in \mathbf{P}$. *If* $[0, b]^{\mathbf{P}}$ *generates a pattern* $\mathbf{Q}$ *such that* $max(\mathbf{Q}) = b$ *then* $\mathbf{P}$ *generates a pattern* $\mathbf{P}^+$ *such that* $[0, b]^{\mathbf{P}^+}$ *is isomorphic to* $\mathbf{Q}$ *over* $[0, b]^{\mathbf{P}}$ *and* $\mathbf{P}^+ = \mathbf{P} \cup [0, b]^{\mathbf{P}^+}$.

Proof. A simple induction using part (4) of lemma 4.11 shows that whenever a pattern $\mathbf{R}$ exactly generates some $\mathbf{R}^+$ and b is the largest element of $\mathbf{R}$ then $\mathbf{R}$ exactly generates $[0, b]^{\mathbf{R}^+}$. Therefore, we can assume that $[0, b]^{\mathbf{P}}$ exactly generates $\mathbf{Q}$. The lemma is straightforward in case $\mathbf{Q}$ is an immediate extension of $[0, b]^{\mathbf{P}}$. The general case follows by a simple induction. □

Notice that the pattern $\mathbf{P}^+$ of the lemma is uniquely determined up to isomorphism over $\mathbf{P}$ since if $x \leq b < y$ and $x \leq_1 y$ then $x, y \in \mathbf{P}$.

Lemma 7.12 *If* $\mathbf{P}_1, \ldots, \mathbf{P}_n$ *are patterns then there exists an amalgamation of* $\mathbf{P}_1, \ldots, \mathbf{P}_n$.

Proof. By the associativity lemma, proving the case $n = 2$ is sufficient. We will prove a generalization of the lemma under the assumption $n = 2$. This will require several temporary definitions.

If $\mathbf{Q}$ is an initial segment of $\mathbf{P}$, we say that $\mathbf{P}$ is *well-situated over* $\mathbf{Q}$ if for any $x \in \mathbf{Q}$ and $y_1, y_2 \in \mathbf{P} - \mathbf{Q}$, $x \leq_1 y_1$ iff $x \leq_1 y_2$. If $\mathbf{Q}$ is an initial segment of both $\mathbf{P}_1$ and $\mathbf{P}_2$ we say that $\mathbf{P}_1$ and $\mathbf{P}_2$ are *similarly well-situated over* $\mathbf{Q}$ if both $\mathbf{P}_1$ and $\mathbf{P}_2$ are well-situated over $\mathbf{Q}$ and $x \leq_1 y_1$ iff $x \leq_1 y_2$ whenever $x \in \mathbf{Q}$ and $y_i \in \mathbf{P}_i - \mathbf{Q}$ for $i = 1, 2$. (We have included the definition of well-situated to make the definition of similarly well-situated more natural.) If $\mathbf{P}_1$ and $\mathbf{P}_2$ are similarly well-situated over $\mathbf{Q}$, an *amalgamation* of $\mathbf{P}_1, \mathbf{P}_2$ over $\mathbf{Q}$ consists of patterns $\mathbf{P}, \mathbf{P}_1^*, \mathbf{P}_2^*$ such that

(i) $\mathbf{Q}$ is an initial segment of $\mathbf{P}$,

(ii) $\mathbf{Q} \subseteq \mathbf{P}_i^*$ for $i = 1, 2$,

(iii) $\mathbf{P}_i^*$ is isomorphic to $\mathbf{P}_i$ for $i = 1, 2$,

(iv) $\mathbf{P} = \mathbf{P}_1^* \cup \mathbf{P}_2^*$,

(v) $\mathbf{P}_i^* - \mathbf{Q}$ is correct in $\mathbf{P}$ for $i = 1, 2$, and

(vi) $\mathbf{P}_i^*$ generates $[0, max(\mathbf{P}_i^*)]^{\mathbf{P}}$ for $i = 1, 2$.

Notice that if $\mathbf{P}, \mathbf{P}_1^*, \mathbf{P}_2^*$ is an amalgamation of $\mathbf{P}_1, \mathbf{P}_2$ over $\mathbf{Q}$ and $\tilde{\mathbf{P}}_i$ is isomorphic to $\mathbf{P}_i$ over $\mathbf{Q}$ for $i = 1, 2$ then $\mathbf{P}, \mathbf{P}_1^*, \mathbf{P}_2^*$ is an amalgamation of $\tilde{\mathbf{P}}_1, \tilde{\mathbf{P}}_2$ over $\mathbf{Q}$.

Claim: If $\mathbf{P}_1$ and $\mathbf{P}_2$ are similarly well-situated over $\mathbf{Q}$ then there exists an amalgamation of $\mathbf{P}_1, \mathbf{P}_2$ over $\mathbf{Q}$.

Before proving the claim, notice that it implies the lemma for $n = 2$. We may assume that $0^{\mathbf{P}_1} = 0^{\mathbf{P}_2}$ in which case $\mathbf{P}_1$ and $\mathbf{P}_2$ are similarly well-situated over $\{0^{\mathbf{P}_1}\}$. An amalgamation of $\mathbf{P}_1, \mathbf{P}_2$ over $\{0^{\mathbf{P}_1}\}$ is an amalgamation of $\mathbf{P}_1, \mathbf{P}_2$.

We will prove the claim by induction on $card(\mathbf{P}_1 - \mathbf{Q}) + card(\mathbf{P}_2 - \mathbf{Q})$.

Assume $m \in \omega$ and that the claim holds for any $\mathbf{P}_1, \mathbf{P}_2$, and $\mathbf{Q}$ such that $card(\mathbf{P}_1 - \mathbf{Q}) + card(\mathbf{P}_2 - \mathbf{Q}) < m$.

Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be similarly well-situated over $\mathbf{Q}$ where $card(\mathbf{P}_1 - \mathbf{Q}) + card(\mathbf{P}_2 - \mathbf{Q}) = m$. If $\mathbf{P}_1 = \mathbf{Q}$ or $\mathbf{P}_2 = \mathbf{Q}$ then the existence of an amalgamation of $\mathbf{P}_1, \mathbf{P}_2$ over $\mathbf{Q}$ is trivial. So, assume that $\mathbf{P}_i - \mathbf{Q} \neq \emptyset$ for $i = 1, 2$. Let $a_i = min(\mathbf{P}_i - \mathbf{Q})$ and let b_i be the largest x in $\mathbf{P}_i$ such that $a_i \leq_1 x$ for $i = 1, 2$.

Case 1: $b_i \neq max(\mathbf{P}_i)$ for either $i = 1$ or $i = 2$.

Let $\overline{\mathbf{P}}_i$ be $[0, b_i]^{\mathbf{P}_i}$ for $i = 1, 2$. By the induction hypothesis, there is an amalgamation $\overline{\mathbf{P}}, \overline{\mathbf{P}}_1^*, \overline{\mathbf{P}}_2^*$ of $\overline{\mathbf{P}}_1, \overline{\mathbf{P}}_2$ over $\mathbf{Q}$. By replacing $\mathbf{P}_i$ by a structure isomorphic to it over $\mathbf{Q}$ if necessary, we may assume that $\overline{\mathbf{P}}_i^* = \overline{\mathbf{P}}_i$ for $i = 1, 2$. We will freely make such simplifying assumptions henceforth.

Subcase 1 of case 1: $b_1 = b_2$.

Since a_i is the least element x of $\overline{\mathbf{P}}$ such that $\mathbf{Q} < x$ and $x \leq_1 b_i$, we see that $a_1 = a_2$. We will drop subscripts and write a for a_1 and a_2 and b for b_1 and b_2.

Suppose $1 \leq i \leq 2$. By lemma 7.11, $\mathbf{P}_i$ generates some $\tilde{\mathbf{P}}_i$ such that $[0, b]^{\tilde{\mathbf{P}}_i}$ is isomorphic to $\overline{\mathbf{P}}$ over $\overline{\mathbf{P}}_i$ and $\tilde{\mathbf{P}}_i = \mathbf{P}_i \cup [0, b]^{\tilde{\mathbf{P}}_i}$. We may assume that $[0, b]^{\tilde{\mathbf{P}}_i} = \overline{\mathbf{P}}$.

Notice that $\tilde{\mathbf{P}}_1$ and $\tilde{\mathbf{P}}_2$ are similarly well-situated over $\overline{\mathbf{P}}$ since $[a, b]^{\tilde{\mathbf{P}}_i}$ is correct in $\tilde{\mathbf{P}}_i$ for $i = 1, 2$ by part (5) of lemma 4.13. By the induction hypothesis, there is an amalgamation $\mathbf{P}, \tilde{\mathbf{P}}_1^*, \tilde{\mathbf{P}}_2^*$ of $\tilde{\mathbf{P}}_1, \tilde{\mathbf{P}}_2$ over $\overline{\mathbf{P}}$. We may assume that $\tilde{\mathbf{P}}_i^* = \tilde{\mathbf{P}}_i$ for $i = 1, 2$.

We claim that $\mathbf{P}, \mathbf{P}_1, \mathbf{P}_2$ is an amalgamation of $\mathbf{P}_1, \mathbf{P}_2$ over $\mathbf{Q}$. Conditions (i)-(iii) are clear, and (iv) follows from the fact that $\overline{\mathbf{P}} = \overline{\mathbf{P}}_1 \cup \overline{\mathbf{P}}_2 \subseteq \mathbf{P}_1 \cup \mathbf{P}_2$. (vi) is implied by the transitivity of generation since $\mathbf{P}_i$ generates $\tilde{\mathbf{P}}_i$, $\tilde{\mathbf{P}}_i$ generates $[0, max(\tilde{\mathbf{P}}_i)]^{\mathbf{P}}$, and $max(\mathbf{P}_i) = max(\tilde{\mathbf{P}}_i)$. Condition (v) is less immediate.

To establish (v), assume $1 \leq i \leq 2$. To see that $\mathbf{P}_i - \mathbf{Q}$ is correct in $\mathbf{P}$, it is enough to show that both $\mathbf{P}_i - \overline{\mathbf{P}}_i$ and $\overline{\mathbf{P}}_i - \mathbf{Q}$ are correct in $\mathbf{P}$. That $\mathbf{P}_i - \overline{\mathbf{P}}_i$ is correct in $\mathbf{P}$ follows from the facts that $\mathbf{P}_i - \overline{\mathbf{P}}_i = \tilde{\mathbf{P}}_i - \overline{\mathbf{P}}$ and $\tilde{\mathbf{P}}_i - \overline{\mathbf{P}}$ is correct in $\mathbf{P}$. Notice that $\overline{\mathbf{P}}_i - \mathbf{Q} = [a, b]^{\mathbf{P}_i}$. To show that $[a, b]^{\mathbf{P}_i}$ is correct in $\mathbf{P}$, assume that $c \in [a, b]^{\mathbf{P}_i}$ and let d be the largest x in $\mathbf{P}$ such that $c \leq_1 x$. We must show that $d \in [a, b]^{\mathbf{P}_i}$. Fix j such that $max(\tilde{\mathbf{P}}_j) = max(\mathbf{P})$. Since $\tilde{\mathbf{P}}_j$ generates $\mathbf{P}$ and $c \in \overline{\mathbf{P}} \subseteq \tilde{\mathbf{P}}_j$, we see that $d \in \tilde{\mathbf{P}}_j$. Since $[a, b]^{\mathbf{P}_j}$ is correct in $\mathbf{P}_j$, $[a, b]^{\tilde{\mathbf{P}}_j}$ is correct in $\tilde{\mathbf{P}}_j$. Therefore, $d \leq b$ and $d \in \overline{\mathbf{P}}$. Since $\overline{\mathbf{P}}_i$ generates $\overline{\mathbf{P}}$, $d \in \overline{\mathbf{P}}_i$ implying $d \in [a, b]^{\mathbf{P}_i}$.

Subcase 2 of case 1: $b_1 \neq b_2$.

We consider the case $b_1 < b_2$.

We first establish that $b_1 < a_2$. Argue by contradiction and assume that $a_2 \leq b_1$. If $a_1 < a_2$ then $a_1 \leq_1 a_2$ contradicting part (3) of lemma 4.11 since $a_1 \notin \overline{\mathbf{P}}_2$. Therefore, $a_2 \leq a_1$. The assumption $a_2 < a_1$ is similarly contradictory. So, $a_1 = a_2$. But then $a_1 \leq_1 b_2$ contradicting the fact that $[a_1, b_1]^{\mathbf{P}_1}$ is correct in $\overline{\mathbf{P}}$. Thus, we have established that $b_1 < a_2$.

By lemma 7.11, $\mathbf{P}_2$ generates some $\tilde{\mathbf{P}}_2$ such that $[0, b_2]^{\tilde{\mathbf{P}}_2}$ is isomorphic to $\overline{\mathbf{P}}$ over $\overline{\mathbf{P}}_2$ and $\tilde{\mathbf{P}}_2 = \mathbf{P}_2 \cup [0, b_2]^{\tilde{\mathbf{P}}_2}$. We may assume that $[0, b_2]^{\tilde{\mathbf{P}}_2} = \overline{\mathbf{P}}$. Notice that $\overline{\mathbf{P}}_1$ is an initial segment of $\tilde{\mathbf{P}}_2$ and $\mathbf{P}_1$ and $\tilde{\mathbf{P}}_2$ are similarly well-situated over $\overline{\mathbf{P}}_1$. By the induction hypothesis, there is an amalgamation $\mathbf{P}, \mathbf{P}_1^*, \tilde{\mathbf{P}}_2^*$ of $\mathbf{P}_1, \tilde{\mathbf{P}}_2$ over $\overline{\mathbf{P}}_1$. We may assume that $\mathbf{P}_1^* = \mathbf{P}_1$ and $\tilde{\mathbf{P}}_2^* = \tilde{\mathbf{P}}_2$.

An argument similar to that used in subcase 1 shows that $\mathbf{P}, \mathbf{P}_1, \mathbf{P}_2$ is an amalgamation of $\mathbf{P}_1, \mathbf{P}_2$ over $\mathbf{Q}$.

Case 2: $b_i = max(\mathbf{P}_i)$ for $i = 1, 2$.

Subcase 1 of case 2: Either a_1 or a_2 is not indecomposable.

We consider the case when a_1 is not indecomposable. There are $b, c \in \mathbf{Q}$ such that c is indecomposable and $a_1 = b + c$. If $b + c$ is defined in $\mathbf{P}_2$ then set $\mathbf{P} = \mathbf{P}_2$. Otherwise, let $\mathbf{P}$ be a simple additive extension of $\mathbf{P}_2$ in which $b + c$ is defined. Let a be the element of $\mathbf{P}_2$ such that $a = b + c$. A straightforward argument shows that $\mathbf{P}, \mathbf{Q} \cup \{a\}, \mathbf{P}_2$ is an amalgamation of $\mathbf{P}_1, \mathbf{P}_2$ over $\mathbf{Q}$.

Subcase 2 of case 2: a_1 and a_2 are indecomposable.

We may assume that $a_1 = a_2$ in which case the structures $[0, a_1]^{\mathbf{P}_1}$ and $[0, a_2]^{\mathbf{P}_2}$ are identical. We will write a for a_1 and a_2.

By the induction hypothesis, there is an amalgamation $\mathbf{P}, \mathbf{P}_1^*, \mathbf{P}_2^*$ of $\mathbf{P}_1, \mathbf{P}_2$ over $[0, a]^{\mathbf{P}_1}$. We may assume that $\mathbf{P}_i^* = \mathbf{P}_i$ for $i = 1, 2$.

If $b_1 = b_2$, a straightforward argument shows that $\mathbf{P}, \mathbf{P}_1, \mathbf{P}_2$ is an amalgamation of $\mathbf{P}_1, \mathbf{P}_2$ over $\mathbf{Q}$.

Suppose $b_1 \neq b_2$. Consider the case $b_1 < b_2$. Let $\mathbf{P}^+$ be obtained from $\mathbf{P}$ by reflecting $[a, b_1]^{\mathbf{P}_1}$ from b_2 to a. Notice that $[0, a)^{\mathbf{P}^+}$ is isomorphic to $\mathbf{P}_1$. Let $\mathbf{P}'$ be $[0, a)^{\mathbf{P}^+} \cup [a, b]^{\mathbf{P}_2}$. A straightforward argument shows that $\mathbf{P}', [0, a)^{\mathbf{P}^+}, \mathbf{P}_2$ is an amalgamation of $\mathbf{P}_1, \mathbf{P}_2$ over $\mathbf{Q}$. □

Lemma 7.13 *If* $\mathbf{Q}_1$ *and* $\mathbf{Q}_2$ *are amalgamations of* $\mathbf{P}_1, \ldots, \mathbf{P}_n$ *then* $\mathbf{Q}_1$ *and* $\mathbf{Q}_2$ *are isomorphic.*

Proof. Let $\mathbf{Q}_i, \mathbf{P}_1^i, \ldots, \mathbf{P}_n^i$ be an amalgamation of $\mathbf{P}_1, \ldots, \mathbf{P}_n$ for $i = 1, 2$. By the previous lemma, there is an amalgamation $\mathbf{Q}, \mathbf{Q}_1^*, \mathbf{Q}_2^*$ of $\mathbf{Q}_1, \mathbf{Q}_2$. We may assume that $\mathbf{Q}_i^* = \mathbf{Q}_i$ for $i = 1, 2$. Under this assumption, the lemma reduces to $\mathbf{P}_i^1 = \mathbf{P}_i^2$ for $i = 1, \ldots, n$.

Fix j with $1 \leq j \leq n$. By the associativity lemma, $\mathbf{Q}, \mathbf{P}_1^1, \ldots, \mathbf{P}_n^1, \mathbf{P}_1^2, \ldots, \mathbf{P}_n^2$ is an amalgamation of $\mathbf{P}_1^1, \ldots, \mathbf{P}_n^1, \mathbf{P}_1^2, \ldots, \mathbf{P}_n^2$. By the minimality lemma, both $\mathbf{P}_j^1 \leq_{pw} \mathbf{P}_j^2$ and $\mathbf{P}_j^2 \leq_{pw} \mathbf{P}_j^1$. Therefore, $\mathbf{P}_j^1 = \mathbf{P}_j^2$. □

By the minimality lemma, if there exists an exact embedding of a pattern $\mathbf{P}$ into a pattern $\mathbf{Q}$ then that embedding is unique. By the transitivity of exactness, we see that the composition of exact embeddings is exact. So, the system of exact embeddings is a commutative system of embeddings. Moreover, the existence of amalgamations says that this system is directed. The following definition is based on a typical direct limit construction.

Definition 7.14 $\mathcal{P}_1$ is a prestructure for the language $\{0, +, \leq, \leq_1\}$ whose universe is the collection of pointed patterns. The interpretation of $0, +, \leq$, and $\leq_1$ are in accordance with the system of exact embeddings. For example, the interpretation of $\leq$ is given by

$(\mathbf{P}_1, a_1) \leq (\mathbf{P}_2, a_2)$ in $\mathcal{P}_1$ iff there exist $\mathbf{P}, f_1, f_2$ such that f_i is an exact embedding of $\mathbf{P}_i$ into $\mathbf{P}$ for $i = 1, 2$ and $f_1(a_1) \leq f_2(a_2)$.

$\mathcal{P}_1$ is too large to be a set. In fact, the equivalence classes of $=^{\mathcal{P}_1}$ are too large to be sets. So, we cannot form the structure $\mathcal{P}_1/=$ in the usual way. However, by restricting $\mathcal{P}_1$ to an appropriate set, we can produce the desired direct limit of the system of exact embeddings.

Lemma 7.15 *$\mathcal{P}_1$ is a prestructure for the language $\{0, +, \leq, \leq_1\}$.*

Proof. Routine. □

Lemma 7.16 *The restriction of $\mathcal{P}_1$ to the hereditarily finite pointed patterns is a recursive structure.*

Proof. The proof is similar to the proof of theorem 6.10. We will consider the case of $\leq$ since similar arguments show that the interpretations of 0, $+$, and $\leq_1$ are recursive.

First, notice that if f_i is an exact embedding of $\mathbf{P}_i$ into $\mathbf{P}$ for $i = 1, 2$ and $\mathbf{Q}$ is the subpattern of $\mathbf{P}$ whose universe is the union of the ranges of f_1 and f_2 then $\mathbf{Q}, f_1, f_2$ is an amalgamation of $\mathbf{P}_1, \mathbf{P}_2$. Therefore, if $a_i \in \mathbf{P}_i$ for $i = 1, 2$ then $(\mathbf{P}_1, a_1) \leq (\mathbf{P}_2, a_2)$ in $\mathcal{P}_1$ iff there exists an amalgamation $\mathbf{Q}, f_1, f_2$ of $\mathbf{P}_1, \mathbf{P}_2$ such that $f_1(a_1) \leq f_2(a_2)$. The existence and uniqueness of amalgamations proved in lemmas 7.12 and 7.13 allows us to conclude as in the argument of theorem 6.10. □

As mentioned above, we cannot form the structure $\mathcal{P}_1/=$ in the usual way. However, the previous lemma allows us to give an alternative construction. Fix an appropriate recursive set of pointed patterns P_1 such that each equivalence class of $=^{\mathcal{P}_1}$ contains exactly one element of P_1. For example, P_1 could consist of the the least element of each equivalence class with respect to some recursive enumeration of the hereditarily finite sets. Let $\mathcal{P}_1{}'$ be the restriction of $\mathcal{P}_1$ to P_1. Since $\mathcal{P}_1{}'$ is isomorphic to the reduction of $\mathcal{P}_1$ by $=^{\mathcal{P}_1}$ in the obvious sense, we will write $\mathcal{P}_1/=$ for $\mathcal{P}_1{}'$. Also, if $(\mathbf{P}, a)$ is a pointed pattern, we will write $(\mathbf{P}, a)/=$ for the unique element $(\mathbf{Q}, b)$ of P_1 such that $(\mathbf{Q}, b) =^{\mathcal{P}_1} (\mathbf{P}, a)$.

Definition 7.17 For $\mathbf{P}$ a pattern, define the function $\iota_{\mathbf{P}}$ from $\mathbf{P}$ into $\mathcal{P}_1/=$ by $\iota_{\mathbf{P}}(a) = (P, a)/=$ for $a \in \mathbf{P}$.

Lemma 7.18 *$\mathcal{P}_1/=$ along with the maps $\iota_{\mathbf{P}}$ for $\mathbf{P}$ a pattern form a direct limit for the system of exact embeddings.*

Proof. Routine. □

Lemma 7.19 *Assume $\mathbf{P}$ and $\mathbf{Q}$ are patterns and let $\mathbf{P}^*$ and $\mathbf{Q}^*$ be the ranges of the maps $\iota_{\mathbf{P}}$ and $\iota_{\mathbf{Q}}$ respectively.*

(1) If there is an exact embedding of $\mathbf{P}$ into $\mathbf{Q}$ then $\mathbf{P}^*$ is an exact subpattern of $\mathbf{Q}^*$.

(2) *If* $\mathbf{P}^*$ *is an exact subpattern of* $\mathbf{Q}$ *then* $\mathbf{Q}^*$ *is isomorphic to* $\mathbf{Q}$ *over* $\mathbf{P}^*$.

Proof. (1) Let f be the exact embedding of $\mathbf{P}$ into $\mathbf{Q}$. Since $f[\mathbf{P}]$ is an exact subpattern of $\mathbf{Q}$, $\iota_{\mathbf{Q}}[f[\mathbf{P}]]$ is an exact subpattern of $\mathbf{Q}^*$. Since $\iota_{\mathbf{P}} = \iota_{\mathbf{Q}} \circ f$, we see that $\iota_{\mathbf{Q}}[f[\mathbf{P}]]$ is $\mathbf{P}^*$.

(2) Since $\mathbf{P}^*$ is an exact subpattern of $\mathbf{Q}$, we see that $\iota_{\mathbf{P}}$ is an exact embedding of $\mathbf{P}$ into $\mathbf{Q}$. Therefore, $\iota_{\mathbf{P}} = \iota_{\mathbf{Q}} \circ \mathbf{P}$ i.e. $\iota_{\mathbf{Q}}$ is the identity on $\mathbf{P}^*$. □

Theorem 7.20

(1) $\mathcal{P}_1/=$ *is (isomorphic to) a recursive structure.*

(2) *The arithmetic part of* $\mathcal{P}_1/=$ *is an additive structure.*

(3) 0 *is defined in* $\mathcal{P}_1/=$ *and the interpretation of* $+$ *in* $\mathcal{P}_1/=$ *is a total operation on the universe of* $\mathcal{P}_1/=$.

(4) *If* $p, q \in \mathcal{P}_1/=$ *and* $p <_1 q$ *then* p *is indecomposable.*

(5) $\leq_1$ *is a forest respecting* $\leq$ *in* $\mathcal{P}_1/=$.

Proof. (1) follows from lemma 7.16 and our identification of $\mathcal{P}_1/=$ with the substructure of $\mathcal{P}_1$ with universe P_1.

(2) follows from the fact that the range of an exact embedding of $\mathbf{P}$ into $\mathbf{Q}$ is a closed subpattern of $\mathbf{Q}$. One first uses this fact to show that if $\mathbf{P}$ is a pattern and a is an indecomposable element of $\mathbf{P}$ then $\iota_{\mathbf{P}}(a)$ is indecomposable in $\mathcal{P}_1/=$. This implies that the decomposition of an element b of a pattern $\mathbf{P}$ will translate into an appropriate decomposition of $\iota_{\mathbf{P}}(b)$ in $\mathcal{P}_1/=$.

(3) 0 is defined in $\mathcal{P}_1/=$ since 0 is defined in every pattern. To see that $+$ is total in $\mathcal{P}_1/=$, assume $p, q \in \mathcal{P}_1/=$. Choose a pattern $\mathbf{P}$ such that p and q are in $\mathbf{P}^*$ where $\mathbf{P}^*$ is the range of $\iota_{\mathbf{P}}$. Let $\mathbf{Q}$ be a pattern which is generated by $\mathbf{P}^*$ in which $p+q$ is defined. By part (2) of the previous lemma, we may assume that $\mathbf{Q}$ is a substructure of $\mathcal{P}_1/=$. Therefore, $a+b$ is defined in $\mathcal{P}_1/=$.

(4) Suppose $x, y \in \mathcal{P}_1/=$ and $x+y=p$. We want to show that either $x=p$ or $y=p$. There is a pattern $\mathbf{P}$ which is a substructure of $\mathcal{P}_1/=$ such that $p, q, x, y \in \mathbf{P}$. Since $p <_1 q$ in $\mathbf{P}$, p is indecomposable in $\mathbf{P}$. Therefore, $x=p$ or $y=p$.

(5) follows immediately from the fact that $\mathcal{P}_1/=$ is a direct limit of the system of exact embeddings. □

Lemma 7.21 *Assume* $\mathbf{P}$ *is a pattern and* $\mathbf{P}^*$ *is the range of* $\iota_{\mathbf{P}}$. *If* $\mathbf{Q}$ *is a cover of* $\mathbf{P}^*$ *which is a substructure of* $\mathcal{P}_1/=$ *then* $\mathbf{P}^* \leq_{pw} \mathbf{Q}$.

Proof. Choose a pattern $\mathbf{R}$ such that $\mathbf{P}$ is an exact subpattern of $\mathbf{R}$ and $\mathbf{Q} \subseteq \mathbf{R}^*$ where $\mathbf{R}^*$ is the range of $\iota_{\mathbf{R}}$. By part (1) of lemma 7.19, $\mathbf{P}^*$ is an exact subpattern of $\mathbf{R}^*$. By the minimality lemma, $\mathbf{P}^* \leq_{pw} \mathbf{Q}$. □

In the previous lemma, $\mathbf{P}^*$ is the unique isominimal substructure of $\mathcal{P}_1/=$ which is isomorphic to $\mathbf{P}$. Therefore, the isominimal substructures of $\mathcal{P}_1/=$ which are patterns are exactly the ranges of the $\iota_{\mathbf{P}}$. Also, if $\mathbf{P}$ is a pattern which is an isominimal substructure of $\mathcal{P}_1/=$ then $\iota_{\mathbf{P}}$ is the identity on $\mathbf{P}$ i.e. $p = \iota_{\mathbf{P}}(p) = (\mathbf{P}, p)/=$.

The following theorem is an amplification of the fact that $\mathcal{P}_1/=$ is a direct limit of the system of exact embeddings.

Theorem 7.22

(1) If $\mathbf{P}$ *is a pattern which is isominimal in* $\mathcal{P}_1/=$ *and* $\mathbf{Q}$ *is a cover of* $\mathbf{P}$ *which is a substructure of* $\mathcal{P}_1/=$ *then* $\mathbf{P} \leq_{pw} \mathbf{Q}$.

(2) Any pattern is isomorphic to an isominimal substructure of $\mathcal{P}_1/=$.

(3) If X *is a finite subset of* $\mathcal{P}_1/=$ *then* X *is a subset of some pattern which is isominimal in* $\mathcal{P}_1/=$.

(4) If $\mathbf{P}$ *and* $\mathbf{Q}$ *are patterns which are isominimal in* $\mathcal{P}_1/=$ *and* $\mathbf{P} \subseteq \mathbf{Q}$ *then* $\mathbf{P}$ *is exact in* $\mathbf{Q}$.

Proof. (1)-(3) follow immediately from the previous lemma.

(4) Let $\mathbf{R}$ be an amalgamation of $\mathbf{P}, \mathbf{Q}$ and let $\mathbf{R}^*$ be the range of $\iota_{\mathbf{R}}$. By part (1) of lemma 7.19, $\mathbf{P}$ and $\mathbf{Q}$ are exact subpatterns of $\mathbf{R}^*$. Therefore, $\mathbf{P}$ is exact in $\mathbf{Q}$. □

Corollary 7.23 *Assume* $\mathbf{P}$ *and* $\mathbf{Q}$ *are patterns and substructures of* $\mathcal{P}_1/=$.

(1) If $\mathbf{Q}$ *is isominimal in* $\mathcal{P}_1/=$ *and* $\mathbf{P}$ *is an exact subpattern of* $\mathbf{Q}$ *then* $\mathbf{P}$ *is isominimal in* $\mathcal{P}_1/=$.

(2) If $\mathbf{P}$ *is isominimal in* $\mathcal{P}_1/=$ *and* $\mathbf{P}$ *is a subpattern of* $\mathbf{Q}$ *then* $\mathbf{P}$ *is exact in* $\mathbf{Q}$.

(3) If $\mathbf{P}$ *is isominimal in* $\mathcal{P}_1/=$ *then*

- *(a)* $\mathbf{P}$ *is closed in* $\mathcal{P}_1/=$,
- *(b)* $\mathbf{P}$ *is correct in* $\mathcal{P}_1/=$, *and*
- *(c) if* $p \in \mathbf{P}$ *and* $q \leq_1 p$ *then* $q \in \mathbf{P}$.

(4) If $\mathbf{P}$ *and* $\mathbf{Q}$ *are isominimal in* $\mathcal{P}_1/=$ *then* $\mathbf{P} \cup \mathbf{Q}$ *is a pattern which is isominimal in* $\mathcal{P}_1/=$.

(5) If $\mathbf{P}$ *and* $\mathbf{Q}$ *are isominimal in* $\mathcal{P}_1/=$ *then* $\mathbf{P} \cup \mathbf{Q}$ *is an amalgamation of* $\mathbf{P}, \mathbf{Q}$.

Proof. (1) Let $\mathbf{P}'$ be an isominimal substructure of $\mathcal{P}_1/=$ which is isomorphic to $\mathbf{P}$. By part (3) of the theorem, there is a pattern $\mathbf{R}$ which is isominimal in $\mathcal{P}_1/=$ and which contains $\mathbf{P}'$ and $\mathbf{Q}$. By part (4) of the theorem, $\mathbf{P}'$ and $\mathbf{Q}$ are exact in $\mathbf{R}$. By transitivity of exactness, $\mathbf{P}$ is exact in $\mathbf{R}$. The minimality lemma implies that $\mathbf{P}' = \mathbf{P}$.

(2) Choose a pattern $\mathbf{R}$ which is isominimal in $\mathcal{P}_1/=$ such that $\mathbf{Q} \subseteq \mathbf{R}$. Since $\mathbf{P}$ is exact in $\mathbf{R}$, $\mathbf{P}$ is exact in $\mathbf{Q}$.

(3) To verify (a), it is enough to show that every indecomposable of $\mathbf{P}$ is indecomposable in $\mathcal{P}_1/=$. Assume p is indecomposable in $\mathbf{P}$ and $p_1, p_2 \in \mathcal{P}_1/=$ have the property that $p = p_1 + p_2$. Let $\mathbf{R}$ be an isominimal substructure of $\mathcal{P}_1/=$ such that $p_1, p_2 \in \mathbf{R}$ and $\mathbf{P} \subseteq \mathbf{R}$. Since $\mathbf{P}$ is exact in $\mathbf{R}$, either $p_1 = p$ or $p_2 = p$.

Similar arguments prove (b) and (c):

For (b), suppose that $p \in \mathbf{P}$, $q \in \mathcal{P}_1/=$, and $p \leq_1 q$. Let $\mathbf{R}$ be a pattern which is isominimal in $\mathcal{P}_1/=$ such that $\mathbf{P} \subseteq \mathbf{R}$ and $q \in \mathbf{R}$. Since $\mathbf{P}$ is correct in $\mathbf{R}$ by part (4) of the theorem, there exists r in $\mathbf{P}$ such that $q \leq r$ and $p \leq_1 r$.

For (c), suppose that $p \in \mathbf{P}$ and $q \leq_1 p$. Choose a pattern $\mathbf{R}$ which is isominimal in $\mathcal{P}_1/=$ such that $\mathbf{P} \subseteq \mathbf{R}$ and $q \in \mathbf{R}$. Since $\mathbf{P}$ is exact in $\mathbf{R}$, $\mathbf{P}$ generates $[0, max(\mathbf{P})]^{\mathbf{Q}}$. Since $q \leq p$, we see that $q \in [0, max(\mathbf{P})]^{\mathbf{Q}}$. By part (3) of lemma 4.11, $q \in \mathbf{P}$.

(4) Clearly, $\mathbf{P} \cup \mathbf{Q}$ is isominimal in $\mathcal{P}_1/=$. Let $\mathbf{R}$ be a pattern which is isominimal in $\mathcal{P}_1/=$ such that $\mathbf{P}, \mathbf{Q} \subseteq \mathbf{R}$. Since both $\mathbf{P}$ and $\mathbf{Q}$ are closed substructures of $\mathbf{R}$, we see that $\mathbf{P} \cup \mathbf{Q}$ is a closed substructure of $\mathbf{R}$ and, therefore, a pattern.

(5) follows from (4) since $\mathbf{P}$ and $\mathbf{Q}$ are exact in $\mathbf{P} \cup \mathbf{Q}$ by part (2). □

Theorem 7.24 *If $p, q \in \mathcal{P}_1$ then $p \leq_1 q$ iff $[0, p) \preceq_{\Sigma_1} [0, q)$.*

Proof. ($\Rightarrow$) Suppose $p \leq_1 q$. In order to show $[0, p) \preceq_{\Sigma_1} [0, q)$ assume $X \subseteq [0, p)$, $Y \subseteq [p, q)$, and both X and Y are finite. Choose a pattern $\mathbf{P}$ which is isominimal in $\mathcal{P}_1/=$ such that $X \cup Y \cup \{p, q\}$ is a subset of $\mathbf{P}$. Let $\mathbf{P}^+$ be obtained from $\mathbf{P}$ by reflecting $[p, q)^{\mathbf{P}}$ from q to p, and let f be an isomorphism of $[0, q)^{\mathbf{P}}$ with $[0, p)^{\mathbf{P}^+}$. By part (2) of lemma 7.19, we may assume that $\mathbf{P}^+$ is an isominimal substructure of $\mathcal{P}_1/=$. A straightforward argument shows that if $\tilde{Y}$ is the image of Y under f then $X < \tilde{Y} < p$ and $X \cup Y$ is isomorphic to $X \cup \tilde{Y}$.

($\Leftarrow$) Suppose $[0, p) \preceq_{\Sigma_1} [0, q)$. Argue by contradiction and assume $p \not\leq_1 q$. Choose a pattern $\mathbf{P}$ which is isominimal in $\mathcal{P}_1/=$ such that $p, q \in \mathbf{P}$. Let r be the largest x in $\mathbf{P}$ such that $p \leq_1 x$.

Notice that since $p \not\leq_1 q$, $r < q$ and $[p, r]^{\mathbf{P}}$ is correct in $\mathbf{P}$. Since $[0, p) \preceq_{\Sigma_1} [0, q)$, there exist a subset X of $\mathcal{P}_1/=$ and a function h such that $[0, p)^{\mathbf{P}} < X < p$ and h is an isomorphism of $[0, r]^{\mathbf{P}}$ with $[0, p)^{\mathbf{P}} \cup X$. By lemma 3.18, $\mathbf{P} \cup X$ is a pattern. Part (2) of the previous corollary implies that $\mathbf{P}$ generates $\mathbf{P} \cup X$. This fact and the existence of h contradicts the nonduplication lemma. □

8 The Well-Founded Part of $\mathcal{P}_1/=$

In this section we will work in $KP + Infinity$.

By Σ-recursion, there is a Σ-definable operation F such that F maps an initial segment of ORD into $\mathcal{P}_1/=$ such that

$$F(\alpha) \simeq \text{the least } p \text{ in } \mathcal{P}_1/= \text{ such that } p \neq F(\xi) \text{ for all } \xi < \alpha$$

Clearly, F is an order isomorphism of an initial segment of ORD with an initial segment of $\mathcal{P}_1/=$.

Definition 8.1 For F as above, define Δ to be the inverse of F and define $WF(\mathcal{P}_1/=)$ to be the substructure of $\mathcal{P}_1/=$ whose universe is the range of F.

Notice that Δ and $WF(\mathcal{P}_1/=)$ are Σ-definable classes.

Lemma 8.2 *If f is an order isomorphism of $[0, p)$ with an ordinal α then Δ extends f and $\Delta(p) = \alpha$.*

Proof. By a simple induction. □

If $WF(\mathcal{P}_1/=)$ is not all of $\mathcal{P}_1/=$ then the lemma implies that $(\mathcal{P}_1/=) - WF(\mathcal{P}_1/=)$ does not have a least element.

Lemma 8.3 *Δ is an isomorphism of $WF(\mathcal{P}_1/=)$ with an initial segment of $\mathcal{R}_1$.*

Proof. Clearly, Δ preserves the interpretation of 0. The preservation of $+$ and $\leq_1$ are established as in the proof of lemma 5.8. To see the interpretation of $+$ is preserved, establish by induction on α that the decomposition of α in $\mathcal{R}_1$ corresponds under F to the decomposition of $F(\alpha)$ in $\mathcal{P}_1/=$. Similarly, the preservation of the interpretation of $\leq_1$ follows by induction using theorem 7.24. □

Lemma 8.4 *$\mathcal{C}_1$ is a substructure of $\mathcal{P}_1$.*

Proof. Recall that lemma 6.6 says that if $\mathbf{P}_1, \ldots, \mathbf{P}_n$ are covered patterns and $\mathbf{P}_i^*$ is the isominimal substructure of $\mathcal{R}_1$ which is isomorphic to $\mathbf{P}_i$ for $i = 1, \ldots, n$ then $(\mathbf{P}_1^* \cup \cdots \cup \mathbf{P}_n^*), \mathbf{P}_1^*, \ldots \mathbf{P}_n^*$ is an amalgamation of $\mathbf{P}_1, \ldots, \mathbf{P}_n$.

To see that the interpretation of equality in $\mathcal{C}_1$ is the restriction of the interpretation of equality in $\mathcal{P}_1$, assume $(\mathbf{P}, a), (\mathbf{Q}, b) \in \mathcal{C}_1$.

$$\begin{aligned}(\mathbf{P}, a) = (\mathbf{Q}, b) \text{ in } \mathcal{C}_1 \text{ iff } & \iota(\mathbf{P}, a) = \iota(\mathbf{Q}, b) \\ \text{iff } & f(a) = g(b) \text{ for some amalgamation } \mathbf{R}, f, g \text{ of } \mathbf{P}, \mathbf{Q} \\ \text{iff } & \text{there exist a pattern } \mathbf{R} \text{ and exact embeddings } f \text{ of} \\ & \mathbf{P} \text{ into } \mathbf{R} \text{ and } g \text{ of } \mathbf{Q} \text{ into } \mathbf{R} \text{ such that } f(a) = g(b) \\ \text{iff } & (\mathbf{P}, a) = (\mathbf{Q}, b) \text{ in } \mathcal{P}_1\end{aligned}$$

Similar arguments show that the interpretations of 0, $+$, $\leq$, and $\leq_1$ in $\mathcal{C}_1$ are the restrictions of the interpretations of 0, $+$, $\leq$, and $\leq_1$ respectively in $\mathcal{P}_1$. $\Box$

As was remarked earlier, $KP + Infinity$ is not strong enough to guarantee that the equivalence classes of the restriction of $=^{\mathcal{P}_1}$ to the universe of $\mathcal{C}_1$ are sets. Because of this, we will use $\mathcal{C}_1/{=}$ to designate the substructure of $\mathcal{P}_1/{=}$ whose universe consists of all $(\mathbf{P}, a)/{=}$ where $(\mathbf{P}, a) \in \mathcal{C}_1$. $\mathcal{C}_1/{=}$ is a Σ-definable class.

Theorem 8.5

(1) $\mathcal{C}_1/{=}$ *is correct in* $\mathcal{P}_1/{=}$ *.*

(2) $\mathcal{C}_1/{=}$ *is an initial segment of* $WF(\mathcal{P}_1/{=})$ *.*

(3) If $(P, a) \in \mathcal{C}_1$ *then* $\Delta((\mathbf{P}, a)/{=}) = \iota(\mathbf{P}, a)$*. Therefore,* Δ *maps* $\mathcal{C}_1/{=}$ *isomorphically onto the core of* $\mathcal{R}_1$*.*

(4) If $\mathcal{C}_1/{=}$ *is a proper initial segment of* $WF(\mathcal{P}_1/{=})$ *then there is a least element* p *of* $WF(\mathcal{P}_1/{=})$ *which is not in* $\mathcal{C}_1/{=}$ *and* $p \leq_1 q$ *for some* q *which is not in* $WF(\mathcal{P}_1/{=})$ *.*

Proof. By the definitions of $\mathcal{C}_1/{=}$ and $\iota_{\mathbf{P}}$ for $\mathbf{P}$ a pattern, $p \in \mathcal{C}_1/{=}$ iff p is in the range of $\iota_{\mathbf{P}}$ for some covered pattern $\mathbf{P}$. By lemma 7.21 we have

Claim: $\mathcal{C}_1/{=}$ is the union of all covered patterns which are isominimal in $\mathcal{P}_1/{=}$.

(1) Every pattern which is isominimal in $\mathcal{P}_1/{=}$ is correct in $\mathcal{P}_1/{=}$ by part (3) of corollary 7.23. Therefore, $\mathcal{C}_1/{=}$, being a union of correct substructures of $\mathcal{P}_1/{=}$, is correct in $\mathcal{P}_1/{=}$.

Since $\mathcal{C}_1$ is a substructure of $\mathcal{P}_1$, there is a Σ-definable operation Δ_0 from $\mathcal{C}_1/{=}$ into ORD such that $\Delta_0((P, a)/{=}) = \iota(P, a)$ whenever $(P, a) \in \mathcal{C}_1$. The definition of ι and $\mathcal{C}_1$ make clear that Δ_0 is an isomorphism of $\mathcal{C}_1/{=}$ and the core of $\mathcal{R}_1$.

To establish (2), we first show that $\mathcal{C}_1/{=}$ is an initial segment of $\mathcal{P}_1/{=}$. Suppose $p \in \mathcal{C}_1/{=}$, $q \in \mathcal{P}_1/{=}$, and $q \leq p$. There is a covered pattern $\mathbf{P}$ which is isominimal in $\mathcal{P}_1/{=}$ such that $p \in \mathbf{P}$. Choose an pattern $\mathbf{Q}$ which is isominimal in $\mathcal{P}_1/{=}$ such that $\mathbf{P} \subseteq \mathbf{Q}$ and $q \in \mathbf{Q}$. Since $\mathbf{P}$ is exact in $\mathbf{Q}$ by part (4) of theorem 7.22, part (4) of lemma 6.3 implies that $[0, max(P)]^{\mathbf{Q}}$ is exact in $\mathbf{Q}$. Therefore, $[0, max(\mathbf{P})]^{\mathbf{Q}}$ is isominimal in $\mathcal{P}_1/{=}$. Also, since $\mathbf{P}$ generates $[0, max(\mathbf{P})]^{\mathbf{Q}}$, we see that $[0, max(\mathbf{P})]^{\mathbf{Q}}$ is covered. Therefore, $q \in [0, max(\mathbf{P})]^{\mathbf{Q}} \subseteq \mathcal{C}_1/{=}$.

If $WF(\mathcal{P}_1/{=})$ is a proper initial segment of $\mathcal{C}_1/{=}$ then there would be a least element of $(\mathcal{C}_1/{=}) - WF(\mathcal{P}_1/{=})$ (since Δ_0 is a Σ-definable order isomorphism of $\mathcal{C}_1/{=}$ with an initial segment of ORD) – a contradiction. Therefore, $\mathcal{C}_1/{=}$ is an initial segment of $WF(\mathcal{P}_1/{=})$. This concludes the proof of (2).

Using (2), a simple application of the axiom schema of foundation shows that Δ extends Δ_0. This establishes (3).

Notice that a pattern $\mathbf{P}$ which is isominimal in $\mathcal{P}_1/=$ is covered iff $\mathbf{P} \subseteq WF(\mathcal{P}_1/=)$. For, if $\mathbf{P} \subseteq WF(\mathcal{P}_1/=)$ then the restriction of Δ is a covering of $\mathbf{P}$ into $\mathcal{R}_1$.

For (4), assume $p \in WF(\mathcal{P}_1/=) - (\mathcal{C}_1/=)$. Let $\mathbf{P}$ be pattern which is isominimal in $\mathcal{P}_1/=$ such that $p \in \mathbf{P}$. Let q be the least x in $\mathbf{P}$ such that $x \leq_1 p$ and let r be the largest x in $\mathbf{P}$ such that $q \leq_1 r$. By part (3)(c) of corollary 7.23, there is no x such that $x <_1 q$. $[0,r]^{\mathbf{P}}$ is easily seen to be correct in $\mathbf{P}$. Part (1) of corollary 7.23 implies that $[0,r]^{\mathbf{P}}$ is isominimal in $\mathcal{P}_1/=$. If $r \in WF(\mathcal{P}_1/=)$ then $[0,r]^{\mathbf{P}}$ would be covered implying that $p \in \mathcal{C}_1/=$. Therefore, $r \notin WF(\mathcal{P}_1/=)$.

We will be done if we show that q is the least element of $WF(\mathcal{P}_1/=) - (\mathcal{C}_1/=)$. Since $q \leq p$ and $p \in WF(\mathcal{P}_1/=)$, we see that $q \in WF(\mathcal{P}_1/=)$. Since $q \leq_1 r$ and $r \notin WF(\mathcal{P}_1/=)$, the fact that $\mathcal{C}_1/=$ is correct in $\mathcal{P}_1/=$ implies that $q \notin \mathcal{C}_1/=$. We have shown that $q \in WF(\mathcal{P}_1/=) - (\mathcal{C}_1/=)$. To see that $[0,q)^{\mathcal{P}_1/=} \subseteq \mathcal{C}_1/=$, suppose $x < q$. Let $\mathbf{Q}$ be pattern which is isominimal in $\mathcal{P}_1/=$ such that $x \in \mathbf{Q}$. Since there is no y such that $y <_1 q$, we see that $\mathbf{Q} \cap [0,q)^{\mathcal{P}_1/=}$ is correct in $\mathbf{Q}$. Therefore, $\mathbf{Q} \cap [0,q)^{\mathcal{P}_1/=}$ is isominimal in $\mathcal{P}_1/=$. Moreover, $\mathbf{Q} \cap [0,q)^{\mathcal{P}_1/=} \subseteq WF(\mathcal{P}_1/=)$. Therefore, $\mathbf{Q} \cap [0,q)^{\mathcal{P}_1/=}$ is covered. This implies that $x \in \mathcal{C}_1/=$. □

The theorem says that $\mathcal{C}_1/=$ can be characterized as the largest initial segment of $WF(\mathcal{P}_1/=)$ which is correct in $\mathcal{P}_1/=$.

Notice that if $WF(\mathcal{P}_1/=)$ equals $\mathcal{P}_1/=$ then $\mathcal{C}_1/=$ equals $\mathcal{P}_1/=$ or, equivalently, $WF(\mathcal{P}_1/=)$ equals $\mathcal{P}_1/=$ iff every pattern is covered.

We remark that $KP + Infinity$ does not prove that $WF(\mathcal{P}_1/=)$ equals $\mathcal{C}_1/=$. This will be established elsewhere.

Regarding the results on limits of fair sequences in section 6, the obvious generalization that the limit of an arbitrary fair sequence is isomorphic to an initial segment of $\mathcal{P}_1/=$ holds just in case $\mathcal{P}_1/=$ is well-founded (we mean the "set version" of well-foundedness here: every nonempty subset has a minimal element). However, given a pattern $\mathbf{P}$, one can produce a fair sequence starting with $\mathbf{P}$ whose limit is isomorphic to an initial segment of $\mathcal{P}_1/=$. One uses the following fact whose tedious proof follows the lines used to prove the existence of amalgamations: if $\mathbf{P}, \mathbf{P}_1, \mathbf{P}_2$ is an amalgamation of $\mathbf{P}_1, \mathbf{P}_2$ and $max(\mathbf{P}) = max(\mathbf{P}_2)$ then $\mathbf{P}_2$ exactly generates some $\mathbf{Q}$ such that $\mathbf{P}_1$ is an exact subpattern of $\mathbf{Q}$.

9 Categoricity of the Core

In this section we work in $KP + Infinity$.

We begin by describing some notation to be used in this section. If $\mathcal{B}$ is a structure whose universe is an initial segment of the ordinals and α is an ordinal which is a subset of the universe of $\mathcal{B}$ then we will use the notation $\mathcal{B}|\alpha$ for the substructure of $\mathcal{B}$ with universe α. Recall that by our convention regarding the use of the word "structure" we intend that if the universe of $\mathcal{B}$ is an ordinal then $\mathcal{B}$ is a set. If the language of $\mathcal{B}$ includes $\{0, +, \leq\}$ then we say that $\mathcal{B}$ gives the *standard interpretation* to 0, $+$, and $\leq$ provided 0 is interpreted as 0, $+$ is interpreted as the restriction of ordinal addition to the universe of $\mathcal{B}$, and $\leq$ is interpreted as the restriction of the usual ordering of the ordinals to the universe of $\mathcal{B}$. Finally, a formula in the language $\{0, +, \leq, \leq_1\}$ is said to be Σ_1^+ if it is a Σ_1 formula in which every occurrence of $\leq_1$ is positive.

Theorem 9.1 *Assume $\mathcal{B}$ is a structure for the language $\{0, +, \leq, \leq_1\}$ such that the universe is a nonzero ordinal λ and 0, $+$, and $\leq$ are given the standard interpretations. If*

(a) $\mathcal{B}|\alpha \preceq_{\Sigma_1} \mathcal{B}|\beta$ whenever $\alpha \leq_1 \beta$ and

(b) $\leq_1$ is a forest which respects $\leq$

then the core of $\mathcal{B}$ is isomorphic to an initial segment of the core of $\mathcal{R}_1$.

Proof. We will show that the core of $\mathcal{B}$ is isomorphic to an initial segment of $\mathcal{C}_1/{=}$. This is sufficient since $\mathcal{C}_1/{=}$ is isomorphic to the core of $\mathcal{R}_1$.

We will say that a pattern $\mathbf{P}$ is *$\mathcal{B}$-covered* if there is a covering of $\mathbf{P}$ into $\mathcal{B}$. Let $C_{\mathcal{B}}$ be the core of $\mathcal{B}$ i.e. $C_{\mathcal{B}}$ is the union of all the substructures of $\mathcal{B}$ which are isominimal in $\mathcal{B}$.

The argument used to prove lemma 5.3 carries over to the present situation to show that if $\mathbf{P}$ generates $\mathbf{Q}$ and $max(\mathbf{P}) = max(\mathbf{Q})$ then any covering of $\mathbf{P}$ into $\mathcal{B}$ extends to a covering of $\mathbf{Q}$ into $\mathcal{B}$ (if λ were indecomposable we could omit the requirement that $max(\mathbf{P}) = max(\mathbf{Q})$). Therefore, if $\mathbf{P}$ is $\mathcal{B}$-covered, $\mathbf{P}$ generates $\mathbf{Q}$, and $max(\mathbf{P}) = max(\mathbf{Q})$ then $\mathbf{Q}$ is $\mathcal{B}$-covered.

Since any finite union of isominimal substructures of $\mathcal{B}$ is isominimal, every finite subset of $C_{\mathcal{B}}$ is contained in a substructure of $\mathcal{B}$ which is isominimal in $\mathcal{B}$.

Condition (a) implies that α is indecomposable whenever $\alpha <_1 \beta$ in $\mathcal{B}$ for some β (consider the statement $\exists x(\xi + \eta = x)$ for $\xi, \eta < \alpha$). Therefore, any finite closed substructure of $\mathcal{B}$ is a pattern.

Claim 1: If $\mathbf{M}$ is isominimal substructure of $\mathcal{B}$ then $\mathbf{M}$ is a subset of some pattern which is isominimal in $\mathcal{B}$.

Let $\mathbf{Q}$ be a finite closed substructure of $\mathcal{B}$ which contains $\mathbf{M}$. Note that $\mathbf{Q}$ is a pattern. Let $\mathbf{P}$ be an isominimal substructure of $\mathcal{B}$ such that $\mathbf{P}$ is isomorphic to $\mathbf{Q}$ and $\mathbf{P} \leq_{pw} \mathbf{Q}$. Since $\mathbf{P}$ is isomorphic to $\mathbf{Q}$, $\mathbf{P}$ is a pattern. Let $\mathbf{M}'$ be the image of $\mathbf{M}$ under the isomorphism of $\mathbf{Q}$ with $\mathbf{P}$. Since $\mathbf{M}' \leq_{pw} \mathbf{M}$, the fact that $\mathbf{M}$ is isominimal in $\mathcal{B}$ implies that $\mathbf{M}' = \mathbf{M}$. Therefore, $\mathbf{M} \subseteq \mathbf{P}$, proving the claim.

Claim 2: Assume $\mathbf{P}$ is a $\mathcal{B}$-covered pattern. If $\mathbf{P}^*$ is a cover of $\mathbf{P}$ which is a minimal element with respect to $\leq_{pw}$ of the set of substructures of $\mathcal{B}$ which are covers of $\mathbf{P}$ then

(i) if $\mathbf{Q}$ is a cover of $\mathbf{P}$ which is a substructure of $\mathcal{B}$ then $\mathbf{P}^* \leq_{pw} \mathbf{Q}$,

(ii) $\mathbf{P}^* \cong \mathbf{P}$, and

(iii) $\mathbf{P}^*$ is closed in $\mathcal{B}$.

For (i), suppose $\mathbf{Q}$ is a substructure of $\mathcal{B}$ which is a cover of $\mathbf{P}$. Let $\mathbf{R}$ be a finite closed substructure of $\mathcal{B}$ which contains $\mathbf{P}^* \cup \mathbf{Q}$. There is an amalgamation of $\mathbf{R}, \mathbf{P}$ of the form $\mathbf{U}, \mathbf{R}, \mathbf{P}'$. By the minimality lemma, we see that $\mathbf{P}' \leq_{pw} \mathbf{P}^*$ and $\mathbf{P}' \leq_{pw} \mathbf{Q}$. Hence, $max(\mathbf{U}) = max(\mathbf{R})$ implying $\mathbf{R}$ generates $\mathbf{U}$. Let h be a covering of $\mathbf{U}$ into $\mathcal{B}$ which extends the identity on $\mathbf{R}$. Since $h[\mathbf{P}'] \leq_{pw} h[\mathbf{P}^*] = \mathbf{P}^*$, the minimality of $\mathbf{P}^*$ implies that $h[\mathbf{P}'] = \mathbf{P}^*$. Applying h^{-1} yields $\mathbf{P}' = \mathbf{P}^*$. Therefore, $\mathbf{P}^* \leq_{pw} \mathbf{Q}$.

Letting $\mathbf{Q} = \mathbf{P}^*$ in the above argument, we see that $\mathbf{P}' = \mathbf{P}^*$ and $\mathbf{R}, \mathbf{R}, \mathbf{P}^*$ is an amalgamation of $\mathbf{R}, \mathbf{P}$. Therefore, $\mathbf{P} \cong \mathbf{P}^*$ and, since $\mathbf{P}^*$ is closed in $\mathbf{R}$, $\mathbf{P}^*$ is closed in $\mathcal{B}$.

This completes the proof of the claim.

Claim 2 says that if $\mathbf{P}$ is pattern which is isominimal in $\mathcal{B}$ and $\mathbf{Q}$ is a substructure of $\mathcal{B}$ which is a cover of $\mathbf{P}$ then $\mathbf{P} \leq_{pw} \mathbf{Q}$.

Claim 3: If $\mathbf{P}$ and $\mathbf{Q}$ are patterns which are isominimal in $\mathcal{B}$ then $\mathbf{P} \cup \mathbf{Q}, \mathbf{P}, \mathbf{Q}$ is an amalgamation of $\mathbf{P}, \mathbf{Q}$.

By claim 2, $\mathbf{P}$ and $\mathbf{Q}$ are closed in $\mathcal{B}$. Therefore, $\mathbf{P} \cup \mathbf{Q}$ is closed in $\mathcal{B}$ and, consequently, a pattern.

There is an amalgamation of $\mathbf{P} \cup \mathbf{Q}, \mathbf{P}, \mathbf{Q}$ of the form $\mathbf{R}, \mathbf{P} \cup \mathbf{Q}, \mathbf{P}^*, \mathbf{Q}^*$. Since $\mathbf{P}^* \leq_{pw} \mathbf{P}$ and $\mathbf{Q}^* \leq_{pw} \mathbf{Q}$, we see that $max(\mathbf{R}) = max(\mathbf{P} \cup \mathbf{Q})$ and, therefore, that $\mathbf{P} \cup \mathbf{Q}$ generates $\mathbf{R}$. Let h be a covering of $\mathbf{R}$ into $\mathcal{B}$ which extends the identity function on $\mathbf{P} \cup \mathbf{Q}$. Applying h to the inequalities $\mathbf{P}^* \leq_{pw} \mathbf{P}$ and $\mathbf{Q}^* \leq_{pw} \mathbf{Q}$, we see $h[\mathbf{P}^*] \leq_{pw} h[\mathbf{P}] = \mathbf{P}$ and $h[\mathbf{Q}^*] \leq_{pw} h[\mathbf{Q}] = \mathbf{Q}$. By claim 2, $h[\mathbf{P}^*] = \mathbf{P}$ and $h[\mathbf{Q}^*] = \mathbf{Q}$. Applying h^{-1} we see that $\mathbf{P}^* = \mathbf{P}$ and $\mathbf{Q}^* = \mathbf{Q}$. We conclude that $\mathbf{P}$ and $\mathbf{Q}$ are exact in $\mathbf{R} = \mathbf{P} \cup \mathbf{Q}$ i.e. $\mathbf{P} \cup \mathbf{Q}, \mathbf{P}, \mathbf{Q}$ is an amalgamation of $\mathbf{P}, \mathbf{Q}$.

This completes the proof of claim 3.

Define a function f from $C_{\mathcal{B}}$ into $\mathcal{P}_1/=$ so that

$$f(\alpha) = \iota_{\mathbf{P}}(\alpha)$$

whenever $\mathbf{P}$ is a pattern which is isominimal in $\mathcal{B}$ and $\alpha \in \mathbf{P}$. Claim 1 implies that the domain of f is $C_{\mathcal{B}}$. To see that f is well-defined, suppose that $\mathbf{P}_1$ and $\mathbf{P}_2$ are patterns which are isominimal in $\mathcal{B}$ and $\alpha \in \mathbf{P}_i$ for $i = 1, 2$. Since claim 3 implies that $\mathbf{P}_1 \cup \mathbf{P}_2, \mathbf{P}_1, \mathbf{P}_2$ is an amalgamation of $\mathbf{P}_1, \mathbf{P}_2$, we have $\iota_{\mathbf{P}_1}(\alpha) = \iota_{\mathbf{P}_1 \cup \mathbf{P}_2}(\alpha) = \iota_{\mathbf{P}_2}(\alpha)$.

To see that f is 1-1, suppose that $\alpha, \beta \in C_{\mathcal{B}}$ and $\alpha \neq \beta$. Choose a pattern $\mathbf{P}$ which is isominimal in $\mathcal{B}$ such that $\alpha, \beta \in \mathbf{P}$. Since f extends $\iota_{\mathbf{P}}$, $f(\alpha) \neq f(\beta)$. A similar argument shows that f is an embedding of $C_{\mathcal{B}}$ into $\mathcal{P}_1/=$.

Let I be the range of f. To see that I is an initial segment of $\mathcal{P}_1/{=}$, assume $\alpha \in C_{\mathcal{B}}$ and $p \leq f(\alpha)$. Let $\mathbf{P}$ be a pattern which is isominimal in $\mathcal{B}$ such that $\alpha \in \mathbf{P}$ and let $\mathbf{P}' = f[\mathbf{P}]$. Since f extends $\iota_{\mathbf{P}}$, lemma 7.21 implies that $\mathbf{P}'$ is an isominimal substructure of $\mathcal{P}_1/{=}$. Let $\mathbf{Q}$ be a pattern which is isominimal in $\mathcal{P}_1/{=}$ such that $p \in \mathbf{Q}$ and $\mathbf{P}' \subseteq \mathbf{Q}$. By part (4) of theorem 7.22, $\mathbf{P}'$ is exact in $\mathbf{Q}$. Therefore, $[0, max(\mathbf{P}')]^{\mathbf{Q}}$ is exact in $\mathbf{Q}$ and is generated from $\mathbf{P}'$. Also, part (1) of corollary 7.23 implies that $[0, max(\mathbf{P}')]^{\mathbf{Q}}$ is an isominimal substructure of $\mathcal{P}_1/{=}$. Let $\mathbf{R}$ be the isominimal substructure of $\mathcal{B}$ which is isomorphic to $[0, max(\mathbf{P}')]^{\mathbf{Q}}$. By lemma 7.21, $\iota_{\mathbf{R}}$ maps $\mathbf{R}$ onto $[0, max(\mathbf{P}')]^{\mathbf{Q}}$. Since f extends $\iota_{\mathbf{R}}$, we see that $p \in I$.

By collapsing the range of f^{-1} i.e. $C_{\mathcal{B}}$, we see by lemma 8.3 that $I \subseteq WF(\mathcal{P}_1/{=})$. Since the range of $\iota_{\mathbf{P}}$ is isominimal in $\mathcal{P}_1/{=}$ for any pattern $\mathbf{P}$, we see that I is a union of correct subsets of $\mathcal{P}_1/{=}$. Therefore, I is correct in $\mathcal{P}_1/{=}$. Since $\mathcal{C}_1/{=}$ is the longest initial segment of $WF(\mathcal{P}_1/{=})$ which is correct in $\mathcal{P}_1/{=}$ by theorem 8.6, I is an initial segment of $\mathcal{C}_1/{=}$. □

Corollary 9.2 *Assume $\mathcal{B}$ is a structure for a language containing $\{0, +, \leq\}$ such that the universe is an ordinal λ and 0, $+$, and $\leq$ are given the standard interpretation. Let $\preceq$ be the binary relation on λ defined by*

$$\alpha \preceq \beta \text{ iff } \mathcal{B}|\alpha \leq_{\Sigma_1} \mathcal{B}|\beta.$$

If the restriction of $\preceq$ to α is uniformly Δ_1-definable over $\mathcal{B}|\alpha$ for all $\alpha < \lambda$ then the core of $(\lambda, 0, +, \leq, \preceq)$ is isomorphic to an initial segment of the core of $\mathcal{R}_1$.

Proof. The premises of the previous theorem hold for $(\lambda, 0, +, \leq, \preceq)$. □

The theorem and corollary can be generalized to the case when the universe of $\mathcal{B}$ is ORD provided $\mathcal{B}$ is an amenable class in an appropriate sense. The isomorphism of the core of $\mathcal{B}$ with an initial segment of the core of $\mathcal{R}_1$ is then Σ-definable (using a predicate for $\mathcal{B}$). We leave a precise formulation to the reader.

Recall that assuming ZFC, $(\mathcal{C}_1/{=}) = (\mathcal{P}_1/{=})$. If we assume ZFC in the corollary and if λ is large enough, depending on the size of the language of $\mathcal{B}$, then we can conclude that every pattern is $\mathcal{B}$-covered and, consequently, the core of $\mathcal{B}$ is isomorphic to $\mathcal{P}_1/{=}$.

10 Pure Patterns

In this section we will discuss the development undertaken in sections 3-9 starting with $\mathcal{R}_0 = (ORD, \leq)$ rather than $\mathcal{R}_0 = (ORD, 0, +, \leq)$. While the arguments become simpler in many cases, the underlying ideas remain the same. For the remainder of this section $\mathcal{R}_0$ will denote $(ORD, \leq)$.

The notion of an additive structure is replaced by that of a linear ordered structure. There are no analogues for the notions of closed substructure and indecomposables (one can consider every element to be indecomposable). Section 3 can be ignored.

Define the arithmetic part of a structure $\mathbf{P}$ for the language $\{\leq, \leq_1\}$ to be the restriction of $\mathbf{P}$ to the language $\{\leq\}$. The definition of the notion of pattern becomes

Definition 10.1 A finite structure $\mathbf{P}$ for the language $\{\leq, \leq_1\}$ is a *pure pattern of resemblance of order one* provided

(1) the arithmetic part of $\mathbf{P}$ is a linear ordering and

(2) $\leq_1$ is a forest respecting $\leq$.

For the remainder of this section, pure patterns of resemblance of order one will be referred to simply as patterns.

The definition of when $\mathbf{P}^+$ is obtained from $\mathbf{P}$ by reflection should be changed in the obvious way:

Definition 10.2 Assume $\mathbf{P}$ is a pattern, $a, b \in \mathbf{P}$, $a <_1 b$, and X is a nonempty subset of $[a, b)$. A pattern $\mathbf{P}^+$ is *obtained* from $\mathbf{P}$ by reflecting X from b to a provided $\mathbf{P}$ is a subpattern of $\mathbf{P}^+$ and the universe of $\mathbf{P}^+$ is $\mathbf{P} \cup \tilde{X}$ where

(1) $[0, a)^{\mathbf{P}} < \tilde{X} < a$,

(2) $[0, a)^{\mathbf{P}} \cup X \cong [0, a)^{\mathbf{P}} \cup \tilde{X}$, and

(3) if $x \in \tilde{X}$ and $x \leq_1 y$ then $y \in \tilde{X}$.

$\mathbf{P}^+$ is *obtained* from $\mathbf{P}$ by reflection if $\mathbf{P}^+$ is obtained from $\mathbf{P}$ by reflecting X from b to a for some X, b, and a.

Notice that condition (3) is equivalent to saying that $\tilde{X}$ is correct in $\mathbf{P}^+$.

Since there is no notion of simple additive extension here, an *immediate extension* of a pattern $\mathbf{P}$ is a pattern which is obtained from $\mathbf{P}$ by reflection. The remaining definitions and results are as before except that there is nothing corresponding to parts (1) and (4) of lemma 4.11 and lemma 4.14 (they are mainly concerned with $+$).

Skipping ahead to section 7, we see that the results there carry over to our present situation except for lemma 7.1 and parts (2) and (3) of lemma 7.20. The results in [5] imply that, assuming $KP + Infinity$, every pattern is covered and the core of $\mathcal{R}_1$ is ε_0. In fact, using notations for ordinals below ε_0, these results can be formalized in $I\Sigma_0(exp)$. We will present an alternate proof below by providing a recursive order isomorphism between $\mathcal{P}_1/\!=$ and ε_0.

The discussion in the previous paragraph indicates that the results in sections 6 and 8, except for lemmas 6.3 and 6.4 which are needed in section

7, become trivial under the assumption of $KP + Infinity$ since $(\mathcal{C}_1/{=}) = WF(\mathcal{P}_1/{=}) = (\mathcal{P}_1/{=})$. Of course, one could begin with an alternative definition of $WF(\mathcal{P}_1/{=})$ in a weaker theory, but we will not pursue that here.

The following modifications should be made in section 5. In the definition of the notion of fair sequence, the condition that the interpretation of $+$ be total should be dropped. Conclusion (1) of lemma 5.7 should be changed to: $\mathbf{P}_\infty$ satisfies conditions (1) and (2) in the (new) definition of pattern. In statement (5) of theorem 5.9, the requirement that $mc^{\mathcal{R}_1}(\mathbf{Q}) = mc^{\mathcal{R}_1}(\mathbf{P}^*)$ should be replaced by $max(\mathbf{Q}) = max(\mathbf{P}^*)$. Similarly, in theorem 5.10 the condition that $mc^{\mathbf{Q}}(\mathbf{Q}) = mc^{\mathbf{P}}(\mathbf{P})$ should be changed to $max(\mathbf{Q}) = max(\mathbf{P})$. Moreover, since all patterns are covered under the assumption of $KP + Infinity$, conditions in lemmas and theorems stating that various patterns are covered are unnecessary.

Section 9 carries over with obvious modifications e.g. in theorem 9.1 and corollary 9.2, $\mathcal{B}$ is a structure for the language $\{\leq, \leq_1\}$ rather than $\{0, +, \leq, \leq_1\}$.

ε_0 is the least ordinal greater than ω which is closed under ordinal exponentiation. However, for the remainder of this section, we will use ε_0 to denote the usual recursive system of notations for ordinals below the ordinal ε_0 based on Cantor normal forms. So, ε_0 will be a structure for the language $\{0, +, \leq, exp\}$ such that the universe is the set CNF of closed terms which are in "full" Cantor normal form, the interpretations of 0, $+$, and $\leq$ are standard under the identification of ordinals below ε_0 with their Cantor normal forms, and the interpretation of exp corresponds to the function $\alpha \mapsto \omega^\alpha$ on ordinals. As usual, we will write ω^α for $exp(\alpha)$ when α is a notation, 1 will be the term ω^0, and ω will be ω^1. Notice that $(CNF, 0, +, \leq)$ is an additive structure and exp enumerates the indecomposables. For more details see [10].

Theorem 10.3 *There is a recursive order isomorphism of $\mathcal{P}_1/{=}$ with ε_0.*

Proof. Since ε_0 is recursively order isomorphic to the interval $[1, \infty)$ in ε_0, finding a recursive isomorphism of $\mathcal{P}_1/{=}$ with $[1, \infty)$ will suffice. For this proof, the term *ordinal* will refer to an element of ε_0. The proof will involve a series of inductions many of whose proofs we will omit.

A pattern $\mathbf{T}$ is said to be *connected* if $min(\mathbf{T}) \leq_1 max(\mathbf{T})$. Notice that $\mathbf{T}$ is connected iff the interpretation of $\leq_1$ is a tree. If $\mathbf{P}$ is a pattern and $a \in \mathbf{P}$ we will write $\mathbf{P} \uparrow a$ for the subpattern of $\mathbf{P}$ whose universe is $\{x \in \mathbf{P} \ : \ a \leq_1 x\}$.

We begin by defining an auxiliary function ρ on the set of connected patterns into ε_0 such that

$$\rho(\mathbf{T}) = \omega^{\rho(\mathbf{T}\uparrow a_1) + \cdots + \rho(\mathbf{T}\uparrow a_n)}$$

whenever $\mathbf{T}$ is a connected pattern, $a_1, \ldots, a_n$ are the immediate successors of $min(\mathbf{T})$ with respect to $\leq_1$, and $a_1 < \cdots < a_n$.

Claim 1: If α is an indecomposable ordinal then $\alpha = \rho(\mathbf{T})$ for some connected pattern $\mathbf{T}$.

The proof is by induction on terms α in Cantor normal form.

Claim 2: If $\mathbf{T}$ is a connected pattern and a is an element of $\mathbf{T}$ other than $min(\mathbf{T})$ then $\rho(\mathbf{T} \uparrow a) < \rho(\mathbf{T})$.

One can show by reverse induction on the ordering of $\mathbf{T}$ that for any $b \in \mathbf{T}$, if $x \in \mathbf{T}$ and $b{<_1}x$ then $\rho(\mathbf{T} \uparrow x) < \rho(\mathbf{T} \uparrow b)$.

Claim 3: If $\mathbf{T}$ is a connected pattern and $\mathbf{T}'$ is a connected subpattern of $\mathbf{T}$ then $\rho(\mathbf{T}') \leq \rho(\mathbf{T})$. Moreover, if $max(\mathbf{T}) \notin \mathbf{T}'$ then $\rho(\mathbf{T}') < \rho(\mathbf{T})$.

One can show that

- $\rho(\mathbf{T}' \uparrow a) \leq \rho(\mathbf{T} \uparrow a)$ and
- if $max(\mathbf{T} \uparrow a) \notin \mathbf{T}'$ then $\rho(\mathbf{T}' \uparrow a) < \rho(\mathbf{T} \uparrow a)$

for all $a \in \mathbf{T}'$ by reverse induction on the ordering of $\mathbf{T}'$. For the proof, notice that the previous claim implies that if $a \in \mathbf{T}$ and $a{<_1}a_i$ for $i = 1, \ldots, n$ then $\rho(\mathbf{T} \uparrow a_1) + \cdots + \rho(\mathbf{T} \uparrow a_n) < \rho(\mathbf{T} \uparrow a)$.

Claim 4: Assume $\mathbf{T}$ is a connected pattern, $a \in \mathbf{T}$, and $\mathbf{P}$ is a pattern such that $\rho(\mathbf{P} \uparrow x) < \rho(\mathbf{T} \uparrow a)$ whenever $x \in \mathbf{P}$. If $\mathbf{T} \cap \mathbf{P} = \emptyset$ and $\mathbf{T}^+$ is the connected pattern with universe $\mathbf{T} \cup \mathbf{P}$ such that

- $\mathbf{T}$ and $\mathbf{P}$ are correct subpatterns of $\mathbf{T}^+$ and
- $[0, a)^{\mathbf{T}} < \mathbf{P} < a$

then $\rho(\mathbf{T}^+) = \rho(\mathbf{T})$.

To prove the claim, one can show that $\rho(\mathbf{T}^+ \uparrow x) = \rho(\mathbf{T} \uparrow x)$ for all $x \in \mathbf{T}$ by a reverse induction on the ordering of $\mathbf{T}$.

For $\mathbf{P}$ a pattern, define a function $g_{\mathbf{P}}$ from $\mathbf{P}$ into ε_0 by induction on the ordering of $\mathbf{P}$ so that

$$g_{\mathbf{P}}(a) \;=\; \begin{cases} \rho(\mathbf{P} \uparrow a) & \text{if } a = min(\mathbf{P}) \\ g_{\mathbf{P}}(x) + \rho(\mathbf{P} \uparrow a) & \text{if } x \text{ is the immediate predecessor of } a \end{cases}$$

Clearly, $g_{\mathbf{P}}$ is an order preserving map of $\mathbf{P}$ into ε_0.

Claim 5: If $\mathbf{P}$ is a correct initial segment of a pattern $\mathbf{Q}$ then $g_{\mathbf{P}}(x) = g_{\mathbf{Q}}(x)$ whenever $x \in \mathbf{P}$.

The claim can be proved by a simple induction on the ordering of $\mathbf{P}$.

Claim 6: If $\mathbf{P}$ is a pattern, $a, b \in \mathbf{P}$, $a < b$, and $(a, b]$ is correct in $\mathbf{P}$ then $g_{\mathbf{P}}(x) = g_{\mathbf{P}}(a) + g_{(a,b]}(x)$ for all x in $(a, b]$.

The proof is similar to that of claim 5.

Claim 7: If $\mathbf{P}$ is a subpattern of $\mathbf{Q}$ then $g_{\mathbf{P}}(x) \leq g_{\mathbf{Q}}(x)$ for all x in $\mathbf{P}$.

The proof is a simple induction on the ordering of $\mathbf{P}$ using claim 3.

Claim 8: If $\mathbf{P}$ generates $\mathbf{Q}$ then $g_{\mathbf{Q}}$ extends $g_{\mathbf{P}}$.

By the previous claim, we may assume that $\mathbf{P}$ exactly generates $\mathbf{Q}$ which, in turn, allows us to assume that $\mathbf{P}$ immediately generates $\mathbf{Q}$ i.e $\mathbf{Q}$ is obtained from $\mathbf{P}$ by reflection.

Let $\mathbf{Q}$ be obtained from $\mathbf{P}$ by reflecting X from b to a and let $\tilde{X} = \mathbf{Q} - \mathbf{P}$. By claim 2, $\rho(X \uparrow x) < \rho(\mathbf{P} \uparrow a)$ for all x in X. This implies that $\rho(\tilde{X} \uparrow \tilde{x}) <$

$\rho(\mathbf{P} \uparrow a)$ for all $\tilde{x}$ in $\tilde{X}$. An easy induction shows that $g_{\tilde{X}}(\tilde{x}) < \rho(\mathbf{P} \uparrow a)$ for all $\tilde{x}$ in $\tilde{X}$. One can now show by induction on y in $\mathbf{P}$ that $g_{\mathbf{P}}(y) = g_{\mathbf{Q}}(y)$. The proof uses claim 4 and, for the case $y = a$, claim 6 (note that $\tilde{X}$ is correct in $\mathbf{Q}$).

Claim 9: If $\mathbf{P}$ is an exact subpattern of $\mathbf{Q}$ then $g_{\mathbf{Q}}$ extends $g_{\mathbf{P}}$.

By claims 5 and 8.

We can now define a recursive function G from $\mathcal{P}_1/=$ into ε_0 so that

$$G(p) = g_{\mathbf{P}}(p)$$

whenever $\mathbf{P}$ is a pattern which is isominimal in $\mathcal{P}_1/=$ such that $p \in \mathbf{P}$.

Since each $g_{\mathbf{P}}$ is order preserving, so is G.

Notice that α is in the range of G iff α is in the range of some $g_{\mathbf{P}}$. To show that the range of G is $[1, \infty)$, suppose $\alpha \in [1, \infty)$. $\alpha = \alpha_1 + \cdots + \alpha_n$ for some indecomposables $\alpha_1, \ldots, \alpha, _n$ where $\alpha_1 \geq \cdots \geq \alpha_n$.

We will show that $\alpha_1 + \cdots + \alpha_i$ is in the range of G by induction on i.

The case $i = 1$ follows from claim 1.

Suppose that $\alpha_1 + \cdots + \alpha_i$ is in the range of G. This means there is a pattern $\mathbf{P}$ and a in $\mathbf{P}$ such that $g_{\mathbf{P}}(a) = \alpha_1 + \cdots + \alpha_i$. If $\alpha_1 + \cdots + \alpha_{i+1}$ is in the range of $g_{\mathbf{P}}$ we are done, so assume otherwise. By claim 1, choose a connected pattern $\mathbf{T}$ such that $\rho(\mathbf{T}) = \alpha_{i+1}$ and $\mathbf{P} \cap \mathbf{T} = \emptyset$.

Case 1: $g_{\mathbf{P}}(max(\mathbf{P})) < \alpha_1 + \cdots + \alpha_{i+1}$.

Notice that $\alpha_1 + \cdots + \alpha_i = g_{\mathbf{P}}(a) \leq g_{\mathbf{P}}(max(\mathbf{P})$. Let $\mathbf{P}'$ be the pattern with universe $\mathbf{P} \cup \mathbf{T}$ such that $\mathbf{P}$ and $\mathbf{T}$ are correct subpatterns of $\mathbf{P}'$ and $\mathbf{P} < \mathbf{T}$. By claim 5, $g_{\mathbf{P}'}(min(\mathbf{T})) = \alpha_1 + \cdots + \alpha_{i+1}$.

Case 2: $\alpha_1 + \cdots + \alpha_{i+1} < g_{\mathbf{P}}(max(\mathbf{P}))$.

Let b be the least element x of $\mathbf{P}$ such that $\alpha_1 + \cdots + \alpha_{i+1} < g_{\mathbf{P}}(x)$. Let c be the immediate predecessor of b in $\mathbf{P}$. Notice that $a \leq c$ which implies that $\alpha_1 + \cdots + \alpha_i \leq g_{\mathbf{P}}(c) < \alpha_1 + \cdots + \alpha_{i+1}$. Since $g_{\mathbf{P}}(b) = g_{\mathbf{P}}(c) + \rho(\mathbf{P} \uparrow b)$, we see that $\rho(\mathbf{T}) = \alpha_{i+1} < \rho(\mathbf{P} \uparrow b)$. Let $\mathbf{P}'$ be the pattern with universe $\mathbf{P} \cup \mathbf{T}$ such that $\mathbf{P}$ and $\mathbf{T}$ are correct subpatterns of $\mathbf{P}'$ and $c < \mathbf{T} < b$. The argument used to prove claim 8 can be used to show that $g_{\mathbf{P}'}$ extends $g_{\mathbf{P}}$. We see that

$$\begin{aligned} g_{\mathbf{P}'}(min(\mathbf{T})) &= g_{\mathbf{P}'}(c) + \rho(\mathbf{T}) \\ &= g_{\mathbf{P}}(c) + \alpha_{i+1} \\ &= \alpha_1 + \cdots + \alpha_{i+1} \end{aligned}$$

Therefore, $\alpha_1 + \cdots + \alpha_{i+1}$ is in the range of G. □

11 Arithmetic Structures: The Veblen Function

In this section and the next we will explain the alterations necessary in carrying out the developments of sections 3-9 if one begins with $\mathcal{R}_1 = (ORD, 0, +, \varphi, \leq)$ instead of $\mathcal{R}_1 = (ORD, 0, +, \leq)$ (recall that φ is the Veblen operation). Until section 12, $\mathcal{R}_1$ will denote $(ORD, 0, +, \varphi, \leq)$. In this section we will explain how to revise section 3, and in the next section we will discuss sections 4-9. The presentation in this section should be straightforward for those familiar with notation systems involving φ.

This section can be formalized in $I\Sigma_0(exp)$.

We will use $\mathcal{L}^\varphi$ to denote the language $\{0, +, \varphi, \leq\}$ and $\mathcal{L}_I^\varphi$ will be the extension of $\mathcal{L}^\varphi$ by new constants for each element of I. For $x \in I$, we will commit a minor transgression by making the assumption that the new constant for x in $\mathcal{L}_I^\varphi$ is x itself even though this is not strictly legitimate in all cases.

The definition of a *closed* set of ordinals now has the added condition that if $\varphi(\alpha, \beta)$ is in the set and $\alpha, \beta < \varphi(\alpha, \beta)$ then α and β are in the set.

We redefine the arithmetic part of a structure as follows. What was called the arithmetic part of a structure earlier will be called the *additive part* for the next two sections.

Definition 11.1 Assume $\mathbf{V}$ is a structure in a language extending $\{0, +, \leq\}$ and that $\mathbf{A}$ is the additive part of $\mathbf{V}$. An element a of $\mathbf{V}$ is *additively indecomposable* in $\mathbf{V}$ if a is indecomposable in $\mathbf{A}$. If $\mathbf{A}$ is an additive structure then the *additive components* of a in $\mathbf{V}$ are the components of a in $\mathbf{A}$, the *additive decomposition* of a in $\mathbf{V}$ is the decomposition of a in $\mathbf{A}$, and $mac^{\mathbf{V}}(a)$ is defined to be $mc^{\mathbf{A}}(a)$.

The analogue of the notion of additive structure is given in the next definition. It provides a characterization of the isomorphism types of finite closed substructures of $\mathcal{R}_1$.

Definition 11.2 A structure $\mathbf{V}$ for the language $\{0, +, \varphi, \leq\}$ is a *Veblen structure* provided the following conditions hold.

(1) The additive part of $\mathbf{V}$ is an additive structure.

(2) If $a, b \in \mathbf{V}$ and $\varphi(a, b)$ is defined then $\varphi(a, b)$ is additively indecomposable.

(3) If a is additively indecomposable then either $a = \varphi(a, 0)$ or $a = \varphi(x, y)$ for some $x, y \in \mathbf{V}$ where $x, y < a$.

(4) If $a, b_1, b_2 \in \mathbf{V}$ and both $\varphi(a, b_1)$ and $\varphi(a, b_2)$ are defined then $b_1 < b_2$ implies that $\varphi(a, b_1) < \varphi(a, b_2)$.

(5) If $a, b \in \mathbf{V}$ and $\varphi(a, b)$ is defined then $\varphi(x, \varphi(a, b)) = \varphi(a, b)$ whenever $x < a$.

(6) $\mathbf{A}$ is generated from the set of indecomposable elements of $\mathbf{A}$.

Lemma 11.3 *Assume* $\mathbf{V}$ *is a Veblen structure and that* a_1*,* b_1*,* a_2*, and* b_2 *are elements of* $\mathbf{V}$ *such that both* $\varphi(a_1, b_1)$ *and* $\varphi(a_2, b_2)$ *are defined.*

(1) If $a_1 = a_2$ *then* $\varphi(a_1, b_1) \leq \varphi(a_2, b_2)$ *iff* $b_1 \leq b_2$*.*

(2) If $a_1 < a_2$ then $\varphi(a_1, b_1) \le \varphi(a_2, b_2)$ iff $b_1 \le \varphi(a_2, b_2)$.

(3) If $a_2 < a_1$ then $\varphi(a_1, b_1) \le \varphi(a_2, b_2)$ iff $\varphi(a_1, b_1) \le b_2$.

Proof. Part (1) follows from condition (4) in the definition of Veblen structure.

Part (2) follows from part (1) after noting that $\varphi(a_2, b_2) = \varphi(a_1, \varphi(a_2, b_2))$ by condition (5) in the definition of Veblen structure. Part (3) follows from part (1) similarly. □

The lemma implies that if we modify statements (1), (2), and (3) by replacing $\le$ everywhere by $=$ or replacing $\le$ everywhere by $<$ then the resulting statements are also true. We will use these facts freely and without reference to the above lemma in what follows.

Lemma 11.4 *Assume* $\mathbf{V}$ *is a Veblen structure. If a is an additively indecomposable element of* $\mathbf{V}$ *then there are unique elements x and y of* $\mathbf{V}$ *such that $a = \varphi(x, y)$ and $y < a$.*

Proof. The existence of x and y follows from condition (3) in the definition of Veblen structure.

Suppose $a = \varphi(x_i, y_i)$ and $y_i < a$ for $i = 1, 2$. If $x_1 = x_2$ then $y_1 = y_2$. On the other hand, if $x_1 < x_2$ then $y_1 = \varphi(x_2, y_2)$, contradicting the assumption that $y_1 < a$. A similar argument shows that the assumption that $x_2 < x_1$ is contradictory. □

Definition 11.5 Assume that $\mathbf{V}$ is a Veblen structure and a is additively indecomposable element of $\mathbf{V}$. The *order* of a in $\mathbf{V}$, $order^{\mathbf{V}}(a)$, is the unique x such that $a = \varphi(x, y)$ for some y with $y < a$.

As usual, we will omit the superscript $\mathbf{V}$ on $order^{\mathbf{V}}(a)$ and write $order(a)$ when $\mathbf{V}$ is clear from the context.

Lemma 11.6 *Assume* $\mathbf{V}$ *is a Veblen structure and a is additively indecomposable in* $\mathbf{V}$*.*

(1) If a is indecomposable then $order(a) = a$.

(2) If a is not indecomposable then $order(a) < a$.

Proof. The lemma follows from condition (3) in the definition of Veblen structure. For part (2), notice that since a is additively indecomposable but not indecomposable, there must be $x, y < a$ with $a = \varphi(x, y)$. □

Lemma 11.7 *Assume* $\mathbf{V}$ *is a Veblen structure. An element a of* $\mathbf{V}$ *is indecomposable iff $a = \varphi(a, 0)$.*

Proof. The left to right direction of the equivalence follows from condition (3) in the definition of Veblen structure.

Assume $a = \varphi(a, 0)$. By condition (2) in the definition of Veblen structure, a is additively indecomposable. The previous lemma implies that $a \neq \varphi(x, y)$ for any $x, y < a$. □

The lemma implies that if f is an embedding of a Veblen structure $\mathbf{V}$ into a Veblen structure $\mathbf{U}$ then for any element a of $\mathbf{V}$, a is indecomposable in $\mathbf{V}$ iff $f(a)$ is indecomposable in $\mathbf{U}$. This makes the proofs of certain results concerning Veblen structures simpler than the corresponding results for additive structures in some respects.

Lemma 11.8 *Assume that* $\mathbf{V}$ *is a Veblen structure and* $a, b, c \in \mathbf{V}$*. If* $c = \varphi(a, b)$ *then either*

(1) c *is indecomposable,* $a = c$*, and* $b = 0$*,*

(2) $a, b < c$*, or*

(3) $a < order(c)$ *and* $b = c$*.*

Proof. Since c must be additively indecomposable, there are unique $x, y \in \mathbf{V}$ such that $c = \varphi(x, y)$ and $y < c$. Notice that x is the order of c.

Case 1: $x < a$.

Since $\varphi(x, y) = \varphi(a, b)$, we see that $y = \varphi(a, b) = c$, contradicting the assumption $y < c$.

Case 2: $x = a$.

We see that $y = b$. If c is indecomposable then part (1) of the conclusion holds. If c is not indecomposable then part (2) of the conclusion holds.

Case 3: $a < x$.

We see that $b = \varphi(x, y)$ in this case. Therefore, part (3) of the conclusion holds. □

Lemma 11.9 *Assume that* $\mathbf{V}$ *is a Veblen structure and* $a, b \in \mathbf{V}$*. If* $\varphi(a, b)$ *is defined in* $\mathbf{V}$ *then* $a, b \leq \varphi(a, b)$*.*

Proof. By the previous lemma, noting that $order(\varphi(a, b)) \leq \varphi(a, b)$ by lemma 11.6. □

Parts (1), (3), and (4) of lemma 3.4 concerning maximal components hold with our revised definitions.

Definition 11.10 Assume $\mathbf{V}$ is a Veblen structure and I is the set of indecomposables in $\mathbf{V}$. The set of terms which are in *normal form* over $\mathbf{V}$ is the set of closed terms of $\mathcal{L}_I^\varphi$ which is defined inductively by the following conditions.

(1) $I \subseteq X$.

(2) $0 \in X$.

(3) If s and t are closed terms of $\mathcal{L}_I^\varphi$ then $s + t$ is in normal form over $\mathbf{V}$ iff $s + t$ is defined, the value of t in $\mathbf{V}$ is additively indecomposable, s and t are in normal form over $\mathbf{V}$, and there exist terms $s_1, \dots, s_n$ such that $s = s_1 + \cdots + s_n$, the value of s_i is additively indecomposable for $i = 1, \dots, n$, $s_{i+1} \leq s_i$ whenever $1 \leq i < n$, and $t \leq s_n$.

(4) If s and t are closed terms of $\mathcal{L}_I^\varphi$ then $\varphi(s,t)$ is in normal form over $\mathbf{V}$ iff $\varphi(s,t)$ is defined, s and t are in normal form over $\mathbf{V}$, and $s, t < \varphi(s,t)$.

The set of terms which are in normal form over a Veblen structure play the role that the addition tree did for an additive structure.

Lemma 11.11 *Assume* $\mathbf{V}$ *is a Veblen structure. If* $a \in \mathbf{V}$ *then there is a unique normal form term* t *over* $\mathbf{V}$ *such that* a *is the value of* t *in* $\mathbf{V}$.

Proof. Let X be the set of elements a of $\mathbf{V}$ such that a is the value of a unique normal form term. Let I be the set of indecomposables in $\mathbf{V}$. We will use the fact that $\mathbf{V}$ is generated from I (condition (6) in the definition of Veblen structure) i.e. $\mathbf{V}$ is the smallest subset Y of $\mathbf{V}$ such that $I \subseteq Y$, $0 \in Y$, and Y is closed under the interpretations of $+$ and φ.

To see that $I \subseteq X$, notice that any element a of I is a normal form term and the value of a is a. Uniqueness is clear.

Since the value of any normal form term other than 0 is different from the interpretation of 0, 0 (or, more properly, the interpretation of 0) is in X.

To see that X is closed under $+$, assume that $a, b \in X$ and $a + b$ is defined. We may assume that $a \neq a + b$ and $b \neq a + b$. In particular, neither a or b is 0. Let s and t be terms in normal form over $\mathbf{V}$ whose values are a and b respectively. There are normal form terms $s_1, \dots, s_m$ and $t_1, \dots, t_n$ each of which has the form $\varphi(r_1, r_2)$ or is in I such that s is $s_1 + \cdots + s_m$, $s_{i+1} \leq s_i$ whenever $1 \leq i < m$, t is $t_1 + \cdots + t_n$, and $t_{j+1} \leq t_j$ whenever $1 \leq j < n$ ($i = 1$ or $j = 1$ is possible). Since $a + b \neq b$, $t_1 \leq s_1$. Let k be maximal such that $t_1 \leq s_k$. The term $s_1 + \cdots + s_k + t_1 + \cdots + t_n$ is in normal form and has the value $a + b$. That there is at most one normal form term with value $a + b$ follows from the uniqueness of additive decompositions.

To see that X is closed under the interpretation of φ, assume $a, b \in X$ and $\varphi(a,b)$ is defined. We can assume that $a \neq \varphi(a,b)$ and $b \neq \varphi(a,b)$. By lemma 11.9, we see that $a, b < \varphi(a,b)$. Let s and t be terms in normal form with values a and b respectively. $\varphi(s,t)$ is in normal form and has value $\varphi(a,b)$. The uniqueness of a normal form term with value $\varphi(a,b)$ follows from lemma 11.4. □

The previous lemma implies that for any subset X of $\mathbf{V}$, $X = \mathbf{V}$ if X satisfies the following closure conditions.

(1) Every indecomposable of $\mathbf{V}$ is in X.

(2) The interpretation of 0 is in X.

(3) If $\langle a_1, \ldots, a_{n+1} \rangle$ are elements of X which are additively indecomposable, $a_{i+1} \le a_i$ for $i = 1, \ldots, n$, and $a_1 + \cdots + a_{n+1}$ is defined in $\mathbf{V}$ then $a_1 + \ldots + a_{n+1} \in X$.

(4) If $a, b \in X$, $\varphi(a, b)$ is defined, and $a, b < \varphi(a, b)$ then $\varphi(a, b) \in X$.

Arguments using this principle will be called arguments by *normal form induction* over $\mathbf{V}$. Similarly, we will define functions on $\mathbf{V}$ by normal form induction.

The definition of closed substructure is modified as follows.

Definition 11.12 Assume $\mathbf{V}$ is a Veblen structure and $\mathbf{U}$ is a substructure of $\mathbf{V}$. $\mathbf{U}$ is a *closed* in $\mathbf{V}$ if the additive part of $\mathbf{U}$ is closed in the additive part of $\mathbf{V}$ and $a, b \in \mathbf{U}$ whenever $\varphi(a, b)$ is defined, $a, b < \varphi(a, b)$, and $\varphi(a, b) \in \mathbf{U}$.

Definition 11.13 Assume I is an arbitrary set. A set of terms T of $\mathcal{L}_I^\varphi$ is *closed* if each element of T is closed, i.e. has no occurrence of any variable, $0 \in T$, and each subterm of an element of T is in T.

If $\mathbf{V}$ is a Veblen structure then the set of terms which are in normal form over $\mathbf{V}$ is clearly closed.

Lemma 11.14 *Assume $\mathbf{V}$ is a Veblen structure and $\mathbf{U}$ is a closed substructure of $\mathbf{V}$. If $x \in \mathbf{U}$ then*

(1) x is indecomposable in $\mathbf{V}$ iff x is indecomposable in $\mathbf{U}$ and

(2) x is additively indecomposable in $\mathbf{V}$ iff x is additively indecomposable in $\mathbf{U}$.

Proof. The forward directions of both (1) and (2) hold for arbitrary substructures of $\mathbf{V}$. The reverse directions are straightforward. □

Lemma 3.8 holds with our modified definitions and with "additive structure" replaced by "Veblen structure". In fact, we have the following strengthening of part (1).

Lemma 11.15 *Assume $\mathbf{V}$ is a Veblen structure. If $\mathbf{U}$ is a substructure of $\mathbf{V}$ then the following statements are equivalent.*

(1) $\mathbf{U}$ is a Veblen structure.

(2) The set of terms t which are in normal form over $\mathbf{V}$ with the property that the value of t in $\mathbf{V}$ is in $\mathbf{U}$ is a closed set of terms.

(3) $\mathbf{U}$ *is closed.*

Proof.((1) $\Rightarrow$ (2)) By normal form induction over $\mathbf{U}$, show that for $x \in \mathbf{U}$, the normal form term for x in $\mathbf{U}$ is the same as the normal form term for x in $\mathbf{V}$. For the case when x is indecomposable in $\mathbf{U}$, notice that $x = \varphi(x, 0)$ implying that x is indecomposable in $\mathbf{V}$. Since the set of terms in normal form over $\mathbf{U}$ is closed, (2) follows.

((2) $\Rightarrow$ (3)) By lemma 11.5 and the uniqueness of additive decompositions.

((3) $\Rightarrow$ (1)) Using lemma the previous lemma, one easily checks that $\mathbf{U}$ satisfies conditions (1)-(5) in the definition of Veblen structure.

To see that $\mathbf{U}$ satisfies condition (6), let I be the set of indecomposables of $\mathbf{U}$ and use normal form induction over $\mathbf{V}$ to see that for any $x \in \mathbf{V}$, if $x \in \mathbf{U}$ then x is the value of a term of $\mathcal{L}_I^\varphi$ in $\mathbf{U}$. This implies that $\mathbf{U}$ is generated from I. □

The lemma implies that any union of Veblen substructures of a fixed Veblen structure is also a Veblen substructure.

The analogues of lemmas 3.11-3.13 and 3.18 are either false or unnecessary.

Definition 11.16 Assume $\mathbf{V}$ is a Veblen structure and $\langle a_1, \ldots, a_{n+1} \rangle$ is a descending sequence of elements of $\mathbf{V}$ which are additively indecomposable such that $a_1 + \cdots + a_n$ is defined but $a_1 + \cdots + a_{n+1}$ is not defined. A Veblen structure $\mathbf{V}^+$ is an *extension* of $\mathbf{V}$ to $a_1 + \cdots + a_{n+1}$ if $\mathbf{V}$ is a substructure of $\mathbf{V}^+$, $a_1 + \cdots + a_{n+1}$ is defined in $\mathbf{V}^+$, and the value of $a_1 + \cdots + a_{n+1}$ is the unique element of $\mathbf{V}^+ - \mathbf{V}$.

Notice that if $\mathbf{V}$ and $\langle a_1, \ldots, a_{n+1} \rangle$ are as in the definition and $\mathbf{V}$ is a substructure of a Veblen structure $\mathbf{U}$ in which $a_1 + \cdots + a_{n+1}$ is defined and has value a then $\mathbf{V} \cup \{a\}$ is an extension of $\mathbf{V}$ to $a_1 + \cdots + a_{n+1}$.

Lemma 11.17 *Assume* $\mathbf{V}$ *is a Veblen structure and* $\langle a_1, \ldots, a_{n+1} \rangle$ *is a descending sequence of additive indecomposables of* $\mathbf{V}$*. If* $a_1 + \cdots + a_n$ *is defined in* $\mathbf{V}$ *but* $a_1 + \cdots + a_{n+1}$ *is not defined in* $\mathbf{V}$ *then there exists an extension of* $\mathbf{V}$ *to* $a_1 + \cdots + a_{n+1}$*. Moreover, any two extensions of* $\mathbf{V}$ *to* $a_1 + \cdots + a_{n+1}$ *are isomorphic over* $\mathbf{V}$*.*

Proof. Let $\mathbf{A}$ be the additive part of $\mathbf{V}$. Let $\mathbf{A}^+$ be an extension of $\mathbf{A}$ to $a_1 + \cdots + a_{n+1}$ and let a be the value of $a_1 + \cdots + a_{n+1}$ in $\mathbf{A}^+$. Extend $\mathbf{A}^+$ to a structure $\mathbf{V}^+$ for the language $\mathcal{L}^\varphi$ such that for all $x, y, z \in \mathbf{V}^+$, $\varphi(x, y) = z$ in $\mathbf{V}^+$ iff either

- $x, y, z \in \mathbf{V}$ and $\varphi(x, y) = z$ in $\mathbf{V}$ or
- z is an additively indecomposable element of $\mathbf{V}$, $y = z$, $x = a$, and $a < order^{\mathbf{V}}(z)$.

The proof that $\mathbf{V}^+$ is a Veblen structure is straightforward.

To see that any two extensions of $\mathbf{V}$ to $\langle a_1, \ldots, a_{n+1} \rangle$ are isomorphic over $\mathbf{V}$, first note that the additive parts are isomorphic over $\mathbf{V}$. This isomorphism of the additive parts is an isomorphism of the original Veblen structures. The details are left to the reader. □

Definition 11.18 Assume $\mathbf{V}$ is a Veblen structure, $a, b \in \mathbf{V}$, and $\varphi(a, b)$ is not defined. A Veblen structure $\mathbf{V}^+$ is an *extension* of $\mathbf{V}$ to $\varphi(a, b)$ if $\mathbf{V}$ is a substructure of $\mathbf{V}^+$, $\varphi(a, b)$ is defined in $\mathbf{V}^+$, and the value of $\varphi(a, b)$ is the unique element of $\mathbf{V}^+ - \mathbf{V}$.

Notice that if $\mathbf{V}$, a, and b are as in the definition and $\mathbf{V}$ is a substructure of a Veblen structure $\mathbf{U}$ in which $\varphi(a, b)$ is defined and has value c then $\mathbf{V} \cup \{c\}$ is an extension of $\mathbf{V}$ to $\varphi(a, b)$.

Lemma 11.19 *Assume* $\mathbf{V}$ *is a Veblen structure and* $a, b \in \mathbf{V}$*. If* $\varphi(a, b)$ *is not defined in* $\mathbf{V}$ *then there exists an extension of* $\mathbf{V}$ *to* $\varphi(a, b)$*. Moreover, any two extensions of* $\mathbf{V}$ *to* $\varphi(a, b)$ *are isomorphic over* $\mathbf{V}$*.*

Proof. Let $\mathbf{A}$ be the additive part of $\mathbf{V}$ and fix some a^+ such that $a^+ \notin \mathbf{V}$. a^+ will be $\varphi(a, b)$.

We begin by determining which elements of $\mathbf{V}$ will be less than $\varphi(a, b)$. Define X by normal form induction to be the unique subset of $\mathbf{V}$ satisfying the following conditions.

- If x is indecomposable then $x \in X$ iff either $x \leq a$ or $x < b$.
- $0 \in X$.
- If $\langle x_1, \ldots, x_{n+1} \rangle$ is a descending sequence of elements of $\mathbf{V}$ which are additively indecomposable and $x_1 + \cdots + x_{n+1}$ is defined then $x_1 + \cdots + x_{n+1} \in X$ iff $x_1 \in X$.
- If $x, y \in \mathbf{V}$, $\varphi(x, y)$ is defined, and $x, y < \varphi(x, y)$ then $\varphi(x, y) \in X$ iff either
  - $x = a$ and $y < b$,
  - $x < a$ and $y \in X$, or
  - $a < x$ and $\varphi(x, y) < b$.

Claim: If $x \leq a$ or $x \leq b$ then $x \in X$.

The claim can be proved by normal form induction over $\mathbf{V}$. The tedious but straightforward proof is left to the reader.

Let T be the addition tree of $\mathbf{A}$. $\mathbf{A}$ is a closed substructure of an additive structure $\mathbf{A}^+$ such that the addition tree of $\mathbf{A}^+$ is $T \cup \{\langle a^+ \rangle\}$ and for any indecomposable x of $\mathbf{A}$, $x < a^+$ iff $x \in X$. Since a^+ is indecomposable in $\mathbf{A}^+$, we see that $x \in X$ iff $x < a^+$ for any x in $\mathbf{A}^+$.

Extend $\mathbf{A}^+$ to a structure $\mathbf{V}^+$ for the language $\mathcal{L}^\varphi$ by interpreting φ as follows. For $x, y, z \in \mathbf{A}^+$, define $\varphi(x, y) = z$ in $\mathbf{V}^+$ iff either

- $x, y, z \in \mathbf{V}$ and $\varphi(x, y) = z$ in $\mathbf{V}$,
- $x = a$, $y = b$, and $z = a^+$,
- $y \in \mathbf{V}$, $x = a^+ < \mathit{order}^{\mathbf{V}}(y)$, and $y = z$, or
- $y = z = a^+$ and $x < a$.

Simple arguments show that $\mathbf{V}^+$ satisfies conditions (1)-(3) and (6) of the definition of Veblen structure. The verification of conditions (4) and (5) is straightforward but tedious, involving many cases. We will only sketch the proofs.

For condition (4), assume $x, y_1, y_2 \in \mathbf{V}^+$, $y_1 < y_2$, and both $\varphi(x, y_1)$ and $\varphi(x, y_2)$ are defined in $\mathbf{V}^+$. Consider the cases $x = a^+$, $a < x \neq a^+$, $a = x$, and $x < a$ (these cases are exhaustive by the claim above which implies $a, b < \varphi(a, b)$).

For condition (5), assume $x, y, z \in \mathbf{V}^+$, $\varphi(y, z)$ is defined in $\mathbf{V}^+$, and $x < y$. To establish that $\varphi(x, \varphi(y, z)) = \varphi(y, z)$, consider the cases $\varphi(y, z) = a^+$, and $\varphi(y, z) \neq a^+$. For the latter case, notice that $y \leq \mathit{order}^{\mathbf{V}}(\varphi(y, z))$ by the definition of the interpretation of φ. This implies that $x < \mathit{order}^{\mathbf{V}}(\varphi(y, z))$. Considering the subcases $z = a^+$ and $z \neq a^+$, we see that $\varphi(x, \varphi(y, z)) = \varphi(y, z)$.

To show uniqueness up to isomorphism over $\mathbf{V}$, assume $\mathbf{U}$ is an extension of $\mathbf{V}$ to $\varphi(a, b)$. Without loss of generality, we may assume that a^+ is the unique element of $\mathbf{U} - \mathbf{V}$ and, consequently, is the value of $\varphi(a, b)$ in $\mathbf{U}$. Under this assumption, we are reduced to showing that $\mathbf{U} = \mathbf{V}^+$. First notice that $[0, a^+)^{\mathbf{U}}$ satisfies the defining conditions for X. Therefore, $X = [0, a^+)^{\mathbf{U}}$. This implies that the additive part of $\mathbf{U}$ is the same as the additive part of $\mathbf{V}^+$. The proof that the interpretation of φ is the same in both structures is left to the reader. $\square$

If t is a term of some language and h is a function then $\overline{h}(t)$ will denote the term obtained by replacing each constant c in the domain of h by $h(c)$ throughout t.

Lemma 11.20 *Assume* $\mathbf{V}$ *and* $\mathbf{U}$ *are Veblen structures. If* h *is an order preserving function from the indecomposables of* $\mathbf{V}$ *to the indecomposables of* $\mathbf{U}$ *then* h *extends to an embedding of* $\mathbf{V}$ *into* $\mathbf{U}$ *iff* $\overline{h}(t)$ *is defined in* $\mathbf{U}$ *for every normal form term* t *over* $\mathbf{V}$.

Proof. The forward direction of the conclusion is straightforward.

Suppose h is an order preserving function from the indecomposables of $\mathbf{V}$ into the set of indecomposables of $\mathbf{U}$ such that $\overline{h}(t)$ is defined in $\mathbf{U}$ for each term t which is in normal form over $\mathbf{V}$. Define h^+ from $\mathbf{V}$ into $\mathbf{U}$ so that $h^+(x)$ is the value in $\mathbf{U}$ of the normal form term for x in $\mathbf{V}$. Notice that h^+ extends h.

Claim: Assume $\mathbf{V}_0$ is a finite Veblen substructure of $\mathbf{V}$ such that the restriction of h^+ to $\mathbf{V}_0$ is an embedding of $\mathbf{V}_0$ into $\mathbf{U}$.

(1) If $\langle a_1, \ldots, a_{n+1} \rangle$ is a descending sequence of elements of $\mathbf{V}_0$ which are additively indecomposable such that $a_1 + \cdots + a_n$ is defined in $\mathbf{V}_0$, $a_1 + \cdots + a_{n+1}$ is not defined in $\mathbf{V}_0$, and $a_1 + \cdots + a_{n+1}$ is defined in $\mathbf{V}$ then the restriction of h^+ to $\mathbf{V}_0 \cup \{a_1 + \cdots + a_{n+1}\}$ is an embedding.

(2) If $a, b \in \mathbf{V}_0$, $\varphi(a, b)$ is not defined in $\mathbf{V}_0$, and $\varphi(a, b)$ is defined in $\mathbf{V}$ then the restriction of h^+ to $\mathbf{V}_0 \cup \{\varphi(a, b)\}$ is an embedding.

The claim can be proved easily using lemmas 11.17 and 11.19.

The claim implies that the restriction of h^+ to any finite Veblen substructure of $\mathbf{V}$ is an embedding of that substructure into $\mathbf{U}$. Therefore, h^+ is an embedding of $\mathbf{V}$ into $\mathbf{U}$. □

The following lemma says that whether a defined term is in normal form or not depends only on the ordering of the indecomposables.

Lemma 11.21 *Assume that $\mathbf{V}$ and $\mathbf{U}$ are Veblen structures such that every indecomposable of $\mathbf{V}$ is an indecomposable of $\mathbf{U}$ and the ordering on the indecomposables of $\mathbf{V}$ agrees with the ordering of $\mathbf{U}$. If t is a closed term in the language $\mathcal{L}_I^\varphi$ where I is the set of indecomposables of $\mathbf{V}$ and t is defined in both $\mathbf{V}$ and $\mathbf{U}$ then t is in normal form over $\mathbf{V}$ iff t is in normal form over $\mathbf{U}$.*

Proof. By the previous lemma. □

Definition 11.22 Assume $\mathbf{I}$ is a linear ordering with universe I. A term t of $\mathcal{L}_I^\varphi$ is in *normal form* over $\mathbf{I}$ iff there is some Veblen structure $\mathbf{V}$ such that I is the set of indecomposables of $\mathbf{V}$, the ordering of I in $\mathbf{I}$ and $\mathbf{V}$ agree, and t is a normal form term over $\mathbf{V}$.

Lemma 11.23 *Assume $\mathbf{I}$ is a linear ordering with universe I. If T is a closed set of terms of $\mathcal{L}_I^\varphi$ which are in normal form over $\mathbf{I}$ then there is a Veblen structure $\mathbf{V}$ such that I is the set of indecomposables of $\mathbf{V}$, the ordering of I given by $\mathbf{I}$ and $\mathbf{V}$ agree, and the set of normal form terms over $\mathbf{V}$ is T. Moreover, any two such structures are isomorphic over I.*

Proof. The uniqueness of $\mathbf{V}$ up to isomorphism over $\mathbf{I}$ follows from lemma 11.21. The existence of $\mathbf{V}$ follows from lemmas 11.17 and 11.19 if $T - I$ is finite. The general case follows (e.g. construct $\mathbf{V}$ so that the universe of $\mathbf{V}$ is T and the value of an element t of T in $\mathbf{V}$ is t). □

If T is the set of all terms which are in normal form over $\mathbf{I}$ in the lemma, we see there is a unique (up to isomorphism over $\mathbf{I}$) Veblen structure $\mathbf{V}$ whose indecomposables are I ordered according to $\mathbf{I}$ and which is maximal in the sense that any other such Veblen structure can be embedded into $\mathbf{V}$ by an embedding which is the identity on I. If we work in a strong enough theory, like $KP + Infinity$, and $\mathbf{I}$ is an ordinal α with the usual ordering, then this structure is isomorphic to the ordinal Γ_α with the standard interpretations of 0, $+$, φ, and $\leq$. Under this isomorphism, an ordinal $\xi \in \alpha$ corresponds to Γ_ξ.

Lemma 3.15 holds with "additive structures" replaced by "Veblen structures". The proof is similar.

The definition of reflection stands essentially as before with "additive structure" replaced by "Veblen structure":

Definition 11.24 Assume $\mathbf{V}$ is a Veblen structure, a is an indecomposable element of $\mathbf{V}$, and X is a subset of $\mathbf{V}$ such that $a \leq X$ and $[0,a)^{\mathbf{V}} \cup X$ is a Veblen substructure of $\mathbf{V}$. A Veblen structure $\mathbf{V}^+$ is *obtained* from $\mathbf{V}$ by reflecting X below a provided $\mathbf{V}$ is a substructure of $\mathbf{V}^+$ and the universe of $\mathbf{V}^+$ is $\mathbf{V} \cup \tilde{X}$ for some $\tilde{X}$ such that

(1) $[0,a)^{\mathbf{V}} < \tilde{X} < a$ and

(2) $[0,a)^{\mathbf{V}} \cup \tilde{X} \cong [0,a)^{\mathbf{V}} \cup X$.

Lemma 11.25 *If $\mathbf{A}$, a, and X are as in the assumption of the definition then there exists a structure which is obtained from $\mathbf{A}$ by reflecting X below a. Moreover, any two additive structures which are obtained from $\mathbf{A}$ by reflecting X from b to a are isomorphic over $\mathbf{A}$.*

Proof. By the revised version of lemma 3.15. □

12 Veblen Patterns of Order One

We continue the discussion begun in the previous section concerning the modifications needed in sections 3-9 if $\mathcal{R}_0$ is redefined to be $(ORD, 0, +, \varphi, \leq)$.

The *arithmetic part* of a structure for the language $\{0, +, \varphi, \leq, \leq_1\}$ is defined to be the restriction of the structure to $\{0, +, \varphi, \leq\}$.

The analogue of additive pattern of order one is Veblen pattern of order one:

Definition 12.1 A finite structure $\mathbf{P}$ for the language $\{0, +, \varphi, \leq, \leq_1\}$ is a *Veblen pattern of resemblance of order one* provided

(1) the arithmetic part of $\mathbf{P}$ is a Veblen structure,

(2) $\leq_1$ is a forest respecting $\leq$, and

(3) if $a, b \in \mathbf{P}$ and $a <_1 b$ then a is indecomposable.

For the rest of this section we will use the word pattern to refer to a Veblen pattern of order one.

Suppose $\mathcal{R}$ is a structure which satisfies conditions (1)-(3) of the previous definition. Lemma 11.16 implies that if $\mathbf{P}$ is a finite substructure of $\mathcal{R}$ then the following are equivalent.

- $\mathbf{P}$ is closed in $\mathcal{R}$.
- $\mathbf{P}$ is a pattern.
- The arithmetic part of $\mathbf{P}$ is a Veblen structure.

This makes the use of the adjective "closed" superfluous, though harmless, in many places. We leave the reader to make such changes.

The two methods of constructing immediate extensions of additive patterns, simple additive extensions and extensions by reflection, are defined for Veblen patterns as in definitions 4.4 and 4.7 except that in definition 4.4 "indecomposables" should be replaced by "additive indecomposables". Similarly, in lemma 4.5, replace "indecomposables" by "additive indecomposables". Lemma 4.8 carries over as stated.

For Veblen patterns, there is a third way of constructing immediate extensions:

Definition 12.2 Assume $\mathbf{P}$ is a pattern, $a, b \in \mathbf{P}$, and $\varphi(a, b)$ is not defined in $\mathbf{P}$. A pattern $\mathbf{P}^+$ extending $\mathbf{P}$ is an *extension* of $\mathbf{P}$ to $\varphi(a, b)$ provided the arithmetic part of $\mathbf{P}^+$ is an extension of the arithmetic part of $\mathbf{P}$ to $\varphi(a, b)$ and for all $x \in \mathbf{P}$, $x \leq_1 \varphi(a, b)$ iff there is $y \in \mathbf{P}$ such that $x \leq \varphi(a, b) \leq y$ and $x \leq_1 y$. $\mathbf{P}^+$ is a *simple Veblen extension* of $\mathbf{P}$ if $\mathbf{P}^+$ is an extension of $\mathbf{V}$ to $\varphi(a, b)$ for some a and b.

Notice that if $\mathbf{P}^+$ is an extension of $\mathbf{P}$ to $\varphi(a, b)$ then $\varphi(a, b)$ is not indecomposable in $\mathbf{P}^+$.

Lemma 12.3 *Assume* $\mathbf{P}$ *is a pattern. If* $a, b \in \mathbf{P}$ *and* $\varphi(a, b)$ *is undefined in* $\mathbf{P}$ *then there is an extension of* $\mathbf{P}$ *to* $\varphi(a, b)$*. Moreover, any two extensions of* $\mathbf{P}$ *to* $\varphi(a, b)$ *are isomorphic over* $\mathbf{P}$*.*

Proof. Follows easily from lemma 11.19. □

The definition of immediate extension is now extended to include this third possibility. The definitions of "exactly generated from" and "generated from" are as before.

The material after definition 4.9 up to and including lemma 4.13 remains unchanged.

Lemma 4.14 should be modified by replacing trees by closed sets of normal form terms:

Lemma 12.4 *Assume* $\mathbf{P}$ *is a pattern and let* $T^{\mathbf{P}}$ *be the set of terms which are in normal form over* $\mathbf{P}$. *If* T *is a closed set of terms which are in normal form over the set of indecomposables of* $\mathbf{P}$ *with the ordering induced by* $\mathbf{P}$ *such that* $T^{\mathbf{P}} \subseteq T$ *then* $\mathbf{P}$ *exactly generates some* $\mathbf{P}^+$ *such that the set of terms which are in normal form over* $\mathbf{P}^+$ *is* T.

Proof. By lemma 11.23. □

In the definition of covering (definition 5.1) the language $\{0, +, \leq, \leq_1\}$ should be replaced by $\{0, +, \varphi, \leq, \leq_1\}$.

The only other change required in section 5 is in the proof of lemma 5.3. An additional case must be considered for when $\mathbf{P}^+$ is a simple Veblen extension of $\mathbf{P}$. The argument is similar to that given for the case when $\mathbf{P}^+$ is a simple additive extension of $\mathbf{P}$. The reader should notice that the proof of case 1 is simplified by the fact that the image of an indecomposable under a covering is indecomposable for Veblen patterns.

In section 6, the proof of lemma 6.4 must be extended to include a third case for when $\mathbf{Q}$ is a simple Veblen extension of $\mathbf{P}$.

The following changes should be made in section 7:

- Lemma 7.1 is unnecessary and trivial by the absoluteness of indecomposables between Veblen structure.
- A third case must be added in the proof of lemma 7.3 for when $\mathbf{P}^+$ is a simple Veblen extension of $\mathbf{P}$.
- A third subcase must be added for case 2 in the proof of lemma 7.4 for when $\mathbf{Q}_{i+1}$ is a simple Veblen extension of $\mathbf{Q}_i$.
- In the proof of lemma 7.7, the appeal to lemma 7.1 is unnecessary since the range of h is already closed, being a subpattern of $\mathbf{Q}$.
- The language $\{0, +, \leq, \leq_1\}$ should be replaced by $\{0, +, \varphi, \leq, \leq_1\}$ in definition 7.14 and the obvious interpretation should be given to φ.
- In the proof of lemma 7.16, the interpretation of φ needs to be considered.
- Part (3) of theorem 7.20 should be extended to say that the interpretation of φ is total.

In section 8, the proof of lemma 8.3 must be extended to show that Δ preserves φ. This is accomplished by simply adding the preservation of φ to the inductive argument.

The only change required in section 9 correspond to the addition of φ to the languages involved. In theorem 9.1 and corollary 9.2 one needs also the assumption that $\mathcal{B}$ interprets φ in the standard way i.e. as the restriction of the Veblen operation. Also, in corollary 9.2, change $(\lambda, 0, +, \leq, \preceq)$ to $(\lambda, 0, +, \varphi, \leq, \preceq)$.

13 Concluding Remarks

Using the relation $\preceq_{\Sigma_1}$, we have constructed the notation system $\mathcal{P}_1/=$ which will be shown elsewhere to represent the ordinal of $KP\ell_0$. Larger ordinals can be generated by defining relations $\leq_n$ for $n \in \omega$ inductively so as to generate a structure $\mathcal{R}_\omega$ where

$$\alpha \leq_n \beta \text{ iff } \mathcal{R}_\omega|\alpha \preceq_{\Sigma_n} \mathcal{R}_\omega|\beta$$

Notice that $\leq_0$ is the usual ordering $\leq$. We expect the core of $\mathcal{R}_\omega$ to yield a notation system for the ordinal of formal second order number theory (this paper may be viewed as the first step in validating a proof of this proposition).

The reader may notice some similarity between the structures $\mathcal{R}_1$ and $\mathcal{R}_\omega$ with core models in set theory. This is not accidental. The idea in both constructions is to add information level by level concerning various embeddings. However, the embeddings we have considered here are weaker than the kinds of embeddings that show up in set-theoretic core model constructions. We intend to generalize $\mathcal{R}_\omega$ to structures with embeddings which move ordinals. Perhaps the two sorts of constructions, proof-theoretic and set-theoretic, will find common ground someday.

Though the kind of ordinal analysis begun here will be seen to owe a great deal to traditional methods (see [3], [4], [7], [9], [10], [13] and [14] and, for work at the forefront of the area, [11] and [12]), the exact relationship has yet to be investigated.

Our final comments concern generalizations of the constructions given here in another direction. These constructions have properties like dilators (and can be considered dilators under minor modifications). One example arises by extending $\mathcal{R}_0$ by adding constants for all ordinals less than some fixed ordinal α which is additively indecomposable. We can then consider the core of the modified version of $\mathcal{R}_1$ and develop a notation system for it. Working in a model of $KP + Infinity$, the degree of well-foundedness of the notation system based on the class ORD says a great deal about the theory of the model.

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