# Patterns of Resemblance of Order 2

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**Abstract.** We will investigate patterns of resemblance of order 2 over a family of arithmetic structures on the ordinals. In particular, we will show that they determine a computable well ordering under appropriate assumptions.

This paper continues the investigation into the patterns formed by embeddings between initial segments of structures on the ordinals. The goals are twofold: to eventually find an ultrafinestructure for large cardinal axioms based on embeddings and to derive computable well orderings which can be used for proof theoretic analysis. The former goal appears to be some distance away. Significant progress has been made on the latter: even when restricting to embeddings which do not move ordinals the resulting analysis yields notation systems for large ordinals. Patterns of resemblance of order one, which were defined and investigated in Carlson [6,7,8], lead to an ordinal notation system for the theory $\mathsf{KP}\ell_0$ (see Wilken [13,14,15,16]). This paper develops the notatations corresponding to the next level of embedding: patterns of resemblance of order two. While traditional methods of proof theoretic analysis (see [3,4,5,9,10,11,12]) can be used to provide an analysis of $\mathsf{KP}\ell_0$, a calculus for $\beta$-logic is being developed that provides a more natural fit with notations based on embeddings.

Our study of patterns of resemblance of order 2 in this paper will be analagous to the study of patterns of resemblance of order 1 in Sections 1-6 of [8].

Consider the structure $\mathcal{R}_0 = (ORD, 0, +, \leq)$ on the collection of ordinals $ORD$, where $\leq$ is the usual ordering of ordinals, and $+$ is the usual operation of ordinal addition. Using $\mathcal{B} \preceq_{\Sigma_n} \mathcal{C}$ to indicate that $\mathcal{B}$ is a $\Sigma_n$-elementary substructure of $\mathcal{C}$, extend $\mathcal{R}_0$ to a structure $\mathcal{R}_2 = (ORD, 0, +, \leq, \leq_1, \leq_2)$ by inductively defining the binary relations $\leq_1$ and $\leq_2$ on $ORD$ so that

$$\alpha \leq_n \beta \text{ iff } (\alpha, 0, +, \leq, \leq_1, \leq_2) \preceq_{\Sigma_n} (\beta, 0, +, \leq, \leq_1, \leq_2)$$

for $n = 1, 2$ and all ordinals $\alpha$ and $\beta$ (in other words, the restriction of $\mathcal{R}_2$ to $\beta$ is defined by induction on $\beta$). To clarify this definition and the discussion below, interpret $+$ by the graph of ordinal addition rather than the function.

Let $\leq_{pw}$ be the *pointwise partial ordering* of finite sets of ordinals where $A {\leq_{pw}} B$ iff $A$ and $B$ have the same cardinality and if $\alpha_0, \ldots, \alpha_{n-1}$ enumerates the elements of $A$ in increasing order and $\beta_0, \ldots, \beta_{n-1}$ enumerates the elements of $B$ in increasing order then $\alpha_i \leq \beta_i$ for $i < n$.

Recall that an ordinal $\alpha$ is *additively indecomposable* if it is closed under addition. If $\alpha$ is not additively indecomposable we say that $\alpha$ is *additively decomposable*. A partial substructure $\mathbf{P}$ of $\mathcal{R}_2$ is *closed* if whenever $\alpha \in \mathbf{P}$ is additively decomposable there are $\xi, \eta \in \mathbf{P}$ such that $\xi, \eta < \alpha$ and $\alpha = \xi + \eta$. Notice that every finite set of ordinals is contained in a finite set of ordinals which is closed.

A finite substructure of $\mathcal{R}_2$ which is minimal in the pointwise ordering of the collection of all finite substructures of $\mathcal{R}_2$ which are isomorphic to it will be called *isominimal*. We will refer to the set of ordinals which occur in some isominimal substructure of $\mathcal{R}_2$ as the *core* of $\mathcal{R}_2$.

We will see that for a fixed finite closed substructure $\mathbf{P}$ of $\mathcal{R}_2$, there is a unique isominimal substructure $\mathbf{P}^*$ of $\mathcal{R}_2$ which is isomorphic to $\mathbf{P}$. Moreover, $\mathbf{P}^*$ is closed. This provides a way of defining notations for elements of the core: if $\alpha$ is in the core we can describe $\alpha$ by giving the isomophism type of some isominimal partial substructure which contains $\alpha$ and indicating which element of the isomorphism type corresponds to $\alpha$. These notations will be used to show that the substructure of $\mathcal{R}_2$ whose universe is the core is isomorphic to a computable structure. Moreover, a method of generating the core is established which shows that the order in which patterns of embeddings of this level occur is the same for reasonable hierarchies. Results of this nature can be established for a broad array of choices of $\mathcal{R}_0$ and the analagous construction of $\mathcal{R}_2$.

# 1 Preliminaries

$\mathsf{KP}$ will be used to denote Kripke-Platek set theory (see [1] for background) and $\mathsf{KP}\omega$ is Kripke-Platek set theory with the axiom of infinity. $\mathsf{KP}\omega$ is the base theory for the results in the paper. The theory $\mathsf{KP}\ell_0$ has an axiomatization consisting of the usual axiomatization for $\mathsf{KP}\omega$ with $\Delta_0$-comprehension removed and an additional axiom saying that every set is an element of an admissible set. $\mathsf{ZF}$ denotes Zermelo-Fraenkel set theory.

We now mention a few concepts which can be formalized in $\mathsf{KP}$. $ORD$ will denote the class of ordinals. 0 is the empty set, the least ordinal under the usual ordering $\leq$ of the ordinals. $+$ will denote the usual operation of ordinal addition. $\omega$ is the least infinite ordinal and the elements of $\omega$ are natural numbers. $\omega^+$ is the set of positive natural numbers and $\mathsf{H}(\omega)$ denotes the family of hereditarily finite sets. $\varphi$ will be used to denote the *Veblen operation* on the ordinals:

- $\varphi(0, \alpha) = \omega^\alpha$.
- $\alpha \mapsto \varphi(\xi, \alpha)$ enumerates the ordinals which are fixed points of all maps $\alpha \mapsto \varphi(\eta, \alpha)$ for $\eta < \xi$.

For the basic properties of $\varphi$ see [10]. We will use $\varphi$ in other ways at times and expect no confusion will result.

Contrary to standard practice, we will allow structures for a first-order language $\mathcal{L}$ to interpret the function symbols as partial operations on the universe and to fail to give an interpretation to some constant symbols. In other words, we use the word "structure" to refer to what are called partial structures elsewhere. We will write $|\mathbf{A}|$ for the universe of a structure $\mathbf{A}$. The definition of when a term is defined in a structure is the natural one, proceeding from bottom up, as is the definition of the value of the term in the structure. When $t$ is a term all of whose variables are among $v_1, \ldots, v_n$ and $a_1, \ldots, a_n \in |\mathbf{A}|$ we write $t(a_1, \ldots, a_n)^{\mathbf{A}}$ for the value of $t$ in $\mathbf{A}$ when $v_1, \ldots, v_n$ are interpreted as $a_1, \ldots, a_n$ respectively. See the theory of partial terms in [2] for details.

When $\mathbf{A}$ is a structure for $\mathcal{L}$ and $X \subseteq |\mathbf{A}|$, the set *generated* in $\mathbf{A}$ from $X$ is the smallest subset of $|\mathbf{A}|$ containing $X$ and closed under the interpretation of the function and constant symbols of $\mathcal{L}$ in $\mathbf{A}$.

We will allow two kinds of structures for a finite first-order language: those whose universe is a proper class and those whose universe is a set. We make the assumption that any structure whose universe is a set is itself a set e.g. the interpretation of any relation symbol must be a set and not simply a definable relation on the universe.

Suppose $\mathbf{A}$ is a structure for the language $\mathcal{L}$ and $\mathcal{S}$ is a nonempty family of structures such that for each $\mathbf{B} \in \mathcal{S}$, $|\mathbf{B}| \subseteq |\mathbf{A}|$ and the interpretation of any function or constant symbol in $\mathbf{B}$ is the restriction of the interpretation in $\mathbf{A}$ to $|\mathbf{B}|$ (so if the interpretation of a constant symbol $\mathsf{c}$ in $\mathbf{A}$ is not in $|\mathbf{B}|$ then $\mathsf{c}$ has no interpretation in $\mathbf{B}$). The *union of* $\mathcal{S}$ *with respect to* $\mathbf{A}$ is the structure $\mathbf{B}$ whose universe is the union of the universes of the structures in $\mathcal{S}$ such that the interpretation of any constant or function symbol in $\mathbf{B}$ is the restriction of the interpretation of the symbol in $\mathbf{A}$ to $|\mathbf{B}|$ and the interpretation of any predicate symbol is the union of the interpretations of the predicate symbol in the elements of $\mathcal{S}$. The *intersection of* $\mathcal{S}$ is the structure $\mathbf{B}$ whose universe is the intersection of the universes of the structures in $\mathcal{S}$ such that the interpretation of any constant or function symbol in $\mathbf{B}$ is the restriction of the interpretation of the symbol in $\mathbf{A}$ to $|\mathbf{B}|$ and the interpretation of any predicate symbol is the intersection of the interpretations of the predicate symbol in the elements of $\mathcal{S}$.

We fix a special symbol $\preceq$ which will be assumed to be a 2-place relation symbol in every language it occurs in. We will find it convenient to also use $\preceq_0$ to denote $\preceq$. Suppose $\mathbf{A}$ is a structure for the first-order language $\mathcal{L}$ which includes $\preceq$. If the interpetation of $\preceq$ in $\mathbf{A}$ is a linear ordering of $|\mathbf{A}|$ we will say that $\mathbf{A}$ is a *linearly ordered structure*. If the interpretation of $\preceq$ in $\mathbf{A}$ is a well ordering of $|\mathbf{A}|$ we will say that $\mathbf{A}$ is a *well ordered structure*.

Assume $\mathbf{A}$ is a linearly ordered structure. If $X$ is a nonempty subset of $|\mathbf{A}|$, $max(X)$ will be the largest element of $X$ and $min(X)$ will be the smallest element of $X$ if such elements exist. We will use standard interval notation e.g. for $a, b \in |\mathbf{A}|$ let $[a, b)^{\mathbf{A}}$ denote the set of all $x \in |\mathbf{A}|$ such that $a \preceq^{\mathbf{A}} x \prec^{\mathbf{A}} b$ (we write $\prec^{\mathbf{A}}$ for the strict part of the linear ordering $\preceq^{\mathbf{A}}$). We will also write $(-\infty, a)^{\mathbf{A}}$ for the set of all $x \in |\mathbf{A}|$ such that $x \prec^{\mathbf{A}} a$ and $[a, \infty)^{\mathbf{A}}$ for the set of all $x \in |\mathbf{A}|$ such that $a \preceq^{\mathbf{A}} x$. When $\mathbf{A}$ is a linearly ordered structure, $X \subseteq |\mathbf{A}|$ and $\varphi$ is a function from $X$ into $|\mathbf{A}|$ we say that $\varphi$ is *regressive* if $\varphi(x) \prec^{\mathbf{A}} x$ for all $x \in X$. A subset $I$ of $|\mathbf{A}|$ is an *initial segment* of $\mathbf{A}$ if $x \in I$ whenever $x \preceq^{\mathbf{A}} y$ for some $y \in I$. An initial segment $I$ of $\mathbf{A}$ is a *proper initial segment of* $\mathbf{A}$ if $I \neq |\mathbf{A}|$. A substructure $\mathbf{B}$ of $\mathbf{A}$ is an *initial substructure of* $\mathbf{A}$ if $|\mathbf{B}|$ is an initial segment of $\mathbf{A}$. $\mathbf{B}$ is a *proper initial substructure of* $\mathbf{A}$ if the universe of $\mathbf{B}$ is a proper initial substructure of $\mathbf{A}$. We will write $\preceq_{pw}^{\mathbf{A}}$ for the ordering of the finite subsets of $|\mathbf{A}|$ defined by $X \preceq^{\mathbf{A}}_{pw} Y$ iff $X$ and $Y$ have the same cardinality and $x_i \preceq^{\mathbf{A}} y_i$ for $i = 1, \ldots, n$ when $x_1, \ldots, x_n$ lists the elements of $X$ in increasing order and $y_1, \ldots, y_n$ lists the elements of $Y$ in increasing order.

We will need a generalization of the notion of structure. An *quasistructure* $\mathbf{A}$ for a language $\mathcal{L}$ consists of a nonempty set $|\mathbf{A}|$, the *universe* of $\mathbf{A}$ along with an interpretation of each constant, function and relation symbol of $\mathcal{L}$, as is usual, except that a constant is intepreted as a subset of the universe rather than a single element, an $n$-place function symbol is interpreted as an arbitrary $(n+1)$-ary relation on the universe and $=$ may be interpreted as any 2-ary relation on the universe.

The paper is organized as follows.

Section 1 contains background material.

Section 2 generalizes the notions of isominimal and core above.

Section 3 contains a description of the conditions on $\mathcal{R}_0$ we will need to assume: we will want $\mathcal{R}_0$ to be an Ehrenfeucht-Mostowski structure with some additional properties.

In Section 4, finite structures which are isomorphic to a finite closed substructure of $\mathcal{R}_0$ are studied.

Starting from $\mathcal{R}_0$ with the properties specified in Section 3, Section 5 provides an abstract description of which finite structures are isomorphic to some finite closed substructure of $\mathcal{R}_2$.

Rules for constructing extensions of a given pattern are investigated in Section 6.

Section 7 describes the important concept of continuity between a structure and and extension.

Section 8 delineates two basic kinds of extension: arithmetic and transcendental.

Section 9 describes a family of rules which can be used to generate the core of $\mathcal{R}_2$.

Sections 10, 11 and 12 describe methods of using given rules for extending patterns to derive other methods.

Sections 13 and 14 describe a family of rules for extending patterns which are shown to be sufficient for generating an isomophic copy of the core of $\mathcal{R}_2$. Section 14 also contains a description of the core as either the least $\kappa$ such that $\kappa \leq_1 \infty$ or, if no such $\kappa$ exists, the class of all ordinals (we have written $\kappa \leq_1 \infty$ to indicate $\kappa \leq_1 \alpha$ for all $\alpha \geq \kappa$).

The key notion of amalgamation is presented in Section 15 and used to show the core of $\mathcal{R}_2$ is isomorphic to a computable structure under suitable assumptions.

Section 16 contains concluding remarks.

## 2 Isominimality and the Core of a Structure

For this section, fix a language $\mathcal{L}$ which includes the binary relation symbol $\preceq$.

**Definition 2.1** Assume $\mathbf{A}$ is a linearly ordered structure for the language $\mathcal{L}$. An element $a$ of $|\mathbf{A}|$ is *decomposable in* $\mathbf{A}$ if $a = \mathsf{f}^{\mathbf{A}}(a_1, \ldots, a_n)$ for some n-ary function symbol $\mathsf{f}$ and $a_1, \ldots, a_n \in |\mathbf{A}|$ such that $a_1, \ldots, a_n \prec^{\mathbf{A}} a$. $a$ is *indecomposable in* $\mathbf{A}$ if it is not decomposable.

**Lemma 2.2** *Assume $\preceq^{\mathbf{A}}$ is a well ordering of $|\mathbf{A}|$. If $a \in |\mathbf{A}|$ is decomposable then there is a term $t$ and a sequence of indecomposables $a_1, \ldots, a_n$ less than $a$ such that $a = t(a_1, \ldots, a_n)^{\mathbf{A}}$.*

**Proof.** Induction on $a$. **QED**

The lemma implies that any ordinal below all indecomposables is given by a closed term when $\preceq$ is a well ordering.

**Definition 2.3** Assume $\mathbf{B}$ is a linearly ordered structure for the language $\mathcal{L}$. A substructure $\mathbf{A}$ of $\mathbf{B}$ is a *closed substructure of* $\mathbf{B}$ if every indecomposable of $\mathbf{A}$ is an indecomposable of $\mathbf{B}$. A subset of $|\mathbf{B}|$ is a *closed subset of* $\mathbf{B}$ if it is the universe of a closed substructure of $\mathbf{B}$. An embedding of a structure in $\mathbf{B}$ is a *closed embedding in* $\mathbf{B}$ if the range is the universe of a closed substructure of $\mathbf{B}$.

In circumstances when when all relevant structures are closed substructures of a given linearly ordered structure $\mathcal{R}$, the question of whether an element of $|\mathcal{R}|$ is indecomposable in one of the structures is independent of the structure. For this reason and to simplify notation, we will say that an element of $|\mathcal{R}|$ is indecomposable to mean that it is indecomposable in in $\mathcal{R}$ (or, equivalently, it is indecomposable in at least one of the structures).

The definition of closed substructure used here is more general than that used in [8].

**Lemma 2.4** *1. Assume $\mathbf{B}$ is a linearly ordered structure for $\mathcal{L}$.*

(a) *Any initial segment of* $\mathbf{B}$ *is a closed subset of* $\mathbf{B}$*.*

(b) *Any union of closed subsets of* $\mathbf{B}$ *is a closed subset of* $\mathbf{B}$*.*

2. *Assume* $\mathbf{A}$*,* $\mathbf{B}$*, and* $\mathbf{C}$ *are linearly ordered structures for* $\mathcal{L}$ *such that* $\mathbf{A}$ *is a substructure of* $\mathbf{B}$ *and* $\mathbf{B}$ *is a substructure of* $\mathbf{C}$*.*

   (a) *If* $\mathbf{A}$ *is a closed substructure of* $\mathbf{B}$ *and* $\mathbf{B}$ *is a closed substructure of* $\mathbf{C}$ *then* $\mathbf{A}$ *is a closed substructure of* $\mathbf{C}$*.*

   (b) *If* $\mathbf{A}$ *is a closed substructure of* $\mathbf{C}$ *then* $\mathbf{A}$ *is a closed substructure of* $\mathbf{B}$*.*

*Proof.* Straightforward. **QED**

**Lemma 2.5** *Assume* $\mathbf{A}$ *is a well ordered structure for* $\mathcal{L}$*. Every finite subset of* $|\mathbf{A}|$ *is contained in a finite set which is closed.*

**Proof.** By induction on the maximal element of finite subsets of $|\mathbf{A}|$. **QED**

**Definition 2.6** Assume $\mathbf{A}$ is a linearly ordered structure for $\mathcal{L}$. A finite closed substructure $\mathbf{B}$ of $\mathbf{A}$ will be called an *isominimal substructure of* $\mathbf{A}$ provided $\mathbf{B}$ is the only closed substructure $\mathbf{C}$ of $\mathbf{A}$ such that $\mathbf{C} \cong \mathbf{B}$ and $|\mathbf{C}| \preceq_{pw}^{\mathbf{A}} |\mathbf{B}|$. A finite subset of $|\mathbf{A}|$ is *isominimal with respect to* $\mathbf{A}$ if it is the universe of an isominimal substructure of $\mathbf{A}$. We will refer to the set of elements of $|\mathbf{A}|$ which occur in some isominimal substructure of $\mathbf{A}$ as the *core* of $\mathbf{A}$.

**Lemma 2.7** *Assume* $\mathbf{A}$ *is a linearly ordered structure. The union of two isominimal subsets of* $\mathbf{A}$ *is an isominimal subset of* $\mathbf{A}$*.*

**Proof.** Clear. **QED**

One might think the definition of isominimal would be more natural if the the restriction of $\mathbf{B}$ and $\mathbf{C}$ to closed substructures were omitted. However, closed sets are easier to work with and the core remains unchanged in all the cases we are interested in if we were to modify the definition of isominimal.

# 3 EM Structures

We will generalize the construction of $\mathcal{R}_2$ to those $\mathcal{R}_0$ which satisfy conditions presented in this section. Among these conditions are those which say that $\mathcal{R}_0$ is generated by a family of indiscernibles in the sense of Ehrenfeucht and Mostowski []. For this reason, we will refer to such structures as *EM-structures*. The additional conditions are reminiscent of the properties of Silver indiscernibles [].

Fix a language $\mathcal{L}$ which includes the binary relation symbol $\preceq$. Also fix a linearly ordered structure $\mathcal{R}$ for $\mathcal{L}$. We will generally be interested in the case where the universe of $\mathcal{R}$ is either an ordinal or $ORD$, $\preceq^{\mathcal{R}}$ is the usual ordering and the interpretation of any function symbol is a total operation on the universe.

Recall that $\alpha \in |\mathcal{R}|$ is decomposable if $\alpha = \mathsf{f}^{\mathcal{R}}(\alpha_1, \ldots, \alpha_n)$ where $\mathsf{f}$ is an $n$-place function symbol and $\alpha_1, \ldots, \alpha_n \prec^{\mathcal{R}} \alpha$. Also, $\alpha$ is indecomposable if it is not decomposable.

**Definition 3.1** $\mathcal{R}$ is an *EM structure* if $|\mathcal{R}|$ is generated from the set of indecomposables of $\mathcal{R}$ and the following *indiscernibility condition* holds:

> Whenever $\mathsf{A}$ is an atomic formula all of whose variables are among $v_1, \ldots, v_n$ and $\kappa_1, \ldots, \kappa_n$ and $\lambda_1, \ldots, \lambda_n$ are increasing sequences of indecomposables then
> $$\mathcal{R} \models \mathsf{A}(\kappa_1, \ldots, \kappa_n)$$
> iff
> $$\mathcal{R} \models \mathsf{A}(\lambda_1, \ldots, \lambda_n)$$

In particular

> Whenever $t_1$ and $t_2$ are terms whose variables are among $v_1, \ldots, v_n$ and $\kappa_1, \ldots, \kappa_n$ and $\lambda_1, \ldots, \lambda_n$ are increasing sequences of indecomposables then
> $$t_1(\kappa_1, \ldots, \kappa_n)^{\mathcal{R}} \preceq^{\mathcal{R}} t_2(\kappa_1, \ldots, \kappa_n)^{\mathcal{R}}$$
> iff
> $$t_1(\lambda_1, \ldots, \lambda_n)^{\mathcal{R}} \preceq^{\mathcal{R}} t_2(\lambda_1, \ldots, \lambda_n)^{\mathcal{R}}$$

When $\mathcal{R}$ is an EM structure, the *encoding of* $\mathcal{R}$ is the set of all atomic formulas $\mathsf{A}$ such that
$$\mathcal{R} \models \mathsf{A}(\kappa_1, \ldots, \kappa_n)$$

whenever the variables of $\mathsf{A}$ are among $v_1, \ldots, v_n$ and $\kappa_1, \ldots, \kappa_n$ is an increasing sequence of indecomposables in $\mathcal{R}$.

**Lemma 3.2** *Assume $\mathcal{R}$ is generated from its indecomposables. $\mathcal{R}$ is an EM structure iff for any sets $X$ and $Y$ of indecomposables and any order preserving bijection $h : X \to Y$ there is an isomorphism $\overline{h}$ of the substructures of $\mathcal{R}$ generated by $X$ and $Y$ respectively which extends $h$.*

**Proof.** Straightforward. **QED**

Two important examples of EM structures which were considered in [8] are $(ORD, \leq, 0, +)$ and $(ORD, \leq, 0, +, \varphi)$ where $\varphi$ is the Veblen operation.

**Definition 3.3** For sequences $\kappa_1, \ldots, \kappa_n$ and $\lambda_1, \ldots, \lambda_m$ in $|\mathcal{R}|$, define

$$\kappa_1, \ldots, \kappa_n \sim^{\mathcal{R}} \lambda_1, \ldots, \lambda_m$$

iff $n = m$ and $\kappa_i \preceq^{\mathcal{R}} \kappa_j$ iff $\lambda_i \preceq^{\mathcal{R}} \lambda_j$ for all $1 \leq i, j \leq n$.

The indiscernibility condition implies the following:

Whenever $t_1$ and $t_2$ are terms whose variables are among $v_1, \ldots, v_n$ and $v_1, \ldots, v_m$ respectively and and $\kappa_1, \ldots, \kappa_n$, $\lambda_1, \ldots, \lambda_m$, $\kappa'_1, \ldots, \kappa'_n$, and $\lambda'_1, \ldots, \lambda'_m$ are sequences of indecomposables such that

$$\kappa_1, \ldots, \kappa_n, \lambda_1, \ldots, \lambda_m \;\sim^{\mathcal{R}}\; \kappa'_1, \ldots, \kappa'_n, \lambda'_1, \ldots, \lambda'_m$$

then

$$t_1(\kappa_1, \ldots, \kappa_n)^{\mathcal{R}} \preceq^{\mathcal{R}} t_2(\lambda_1, \ldots, \lambda_m)^{\mathcal{R}} \text{ iff } t_1(\kappa'_1, \ldots, \kappa'_n)^{\mathcal{R}} \preceq^{\mathcal{R}} t_2(\lambda'_1, \ldots, \lambda'_m)^{\mathcal{R}}$$

**Lemma 3.4** *Assume $\mathcal{R}$ is a well ordered EM structure and $\mathcal{I}$ is the family of indecomposables of $\mathcal{R}$.*

1. *If $\mathcal{I}$ is unbounded then for any $\alpha \in |\mathcal{R}|$, $\alpha$ is indecomposable iff $\alpha$ is closed under the interpretations of the function symbols of $\mathcal{L}$.*

2. *If $\mathcal{I}$ is unbounded then $\mathcal{I}$ is closed i.e. $\alpha \in \mathcal{I}$ whenever $\mathcal{I}$ is unbounded below $\alpha$.*

3. *If $\mathcal{I}$ has order type at least $\omega+\omega$, the variables of a term $t$ are among $v_1,\ldots,v_n$ and $1 \le i \le n$ then the $n$-ary operation from the collection of increasing $n$-tuples from $\mathcal{I}$ into $R$ given by*

$$(\kappa_1,\ldots,\kappa_n) \mapsto t(\kappa_1,\ldots,\kappa_n)^{\mathcal{R}}$$

*is either independent of $\kappa_i$ or increasing in $\kappa_i$.*

4. *If $t$ is a term all of whose variables are among $v_1,\ldots,v_n$, $\kappa_1,\ldots,\kappa_n$ and $\lambda_1,\ldots,\lambda_n$ are increasing sequences of indecomposable ordinals and $\kappa_i \le \lambda_i$ for $i = 1,\ldots,n$ then*

$$t(\kappa_1,\ldots,\kappa_n)^{\mathcal{R}} \le t(\lambda_1,\ldots,\lambda_n)^{\mathcal{R}}$$

5. *If $\mathcal{R}|\alpha \le_{\Sigma_1} \mathcal{R}|(\alpha+1)$ then $\alpha$ is indecomposable.*

**Proof.** The reverse direction of part 1 is clear. To establish the forward direction argue by contradiction and assume $\kappa$ is indecomposable and $\kappa \preceq^{\mathcal{R}} \mathsf{f}^{\mathcal{R}}(\alpha_1,\ldots,\alpha_n)$ for some $\alpha_1,\ldots,\alpha_n \prec^{\mathcal{R}} \kappa$. By Lemma 2.2, there is a term $t$ and an increasing sequence of indecomposables $\kappa_1,\ldots,\kappa_m$ such that $t(\kappa_1,\ldots,\kappa_m)^{\mathcal{R}} = \mathsf{f}^{\mathcal{R}}(\alpha_1,\ldots,\alpha_n)$ and each $\kappa_i$ is at most as big as some $\alpha_j$, hence less than $\kappa$. We have $\kappa \preceq^{\mathcal{R}} t(\kappa_1,\ldots,\kappa_m)^{\mathcal{R}}$. The indiscernibility condition implies that $\lambda \preceq^{\mathcal{R}} t(\kappa_1,\ldots,\kappa_m)^{\mathcal{R}}$ for any indecomposable $\lambda$ which contradicts the assumption that $\mathcal{I}$ is unbounded.

Part 2 follows from part 1.

For part 3, first notice that the indiscernibility condition implies that the given operation is either independent of $\kappa_i$, increasing in $\kappa_i$, or decreasing in $\kappa_i$. The last possibility would imply there is a descending sequence of elements of the form $t(\kappa_1,\ldots,\kappa_{i-1},\lambda,\kappa_{i+1},\ldots,\kappa_n)^{\mathcal{R}}$ where $\lambda$ varies over an infinite increasing sequence of indecomposables above $\kappa_{i-1}$ (if $i \neq 1$) and below $\kappa_{i+1}$ (if $i \neq n$).

Part 4 follows from part 3.

For part 5, assume $\alpha$ is decomposable. There is a function symbol $\mathsf{f}$ and $\alpha_1,\ldots,\alpha_n \prec \alpha$ where $\mathsf{f}$ is $n$-ary such that $\alpha = \mathsf{f}(\alpha_1,\ldots,\alpha_n)$.

$$(\alpha+1) \models \exists x\, \mathsf{f}(\alpha_1,\ldots,\alpha_n) = x$$

while

$$\alpha \not\models \exists x\, \mathsf{f}(\alpha_1,\ldots,\alpha_n) = x$$

**QED**

The converse of part 5 does not generally hold for structures we are interested in. In fact, when constructing $\mathcal{R}_2$ for the examples we are interested in we will have

If $\mathcal{R}_0|\alpha \leq_{\Sigma_1} \mathcal{R}_0|(\alpha+1)$ then $\alpha$ is a limit of indecomposables.

Even if a well ordered EM structure $\mathcal{R}_0$ does not satisfy this additional condition we can extend it to another EM structure which does by adding a unary predicate for the set of indecomposables. Moreover, the new structure will have the same closed substructures as the original structure.

**Definition 3.5** For a term $t$ and variable $v_i$ which occurs in $t$, we say that $t$ is *independent of $v_i$ in $\mathcal{R}$* if

$$t(\kappa_1, \ldots, \kappa_{i-1}, \kappa_i, \kappa_{i+1}, \ldots, \kappa_n)^{\mathcal{R}} \;=\; t(\kappa_1, \ldots, \kappa_{i-1}, \kappa_i', \kappa_{i+1}, \ldots, \kappa_n)^{\mathcal{R}}$$

for any increasing sequence of indecomposables $\kappa_1, \ldots, \kappa_n$ and any indecomposible $\kappa_i'$ such that $\kappa_i \prec \kappa_i' \prec \kappa_{i+1}$. If $t$ is not independent of $v_i$ in $\mathcal{R}$ then $t$ *depends on $v_i$ in $\mathcal{R}$*.

Assuming the indecomposables of $\mathcal{R}$ have order type at least $\omega+\omega$, part 3 of the Lemma 3.4 implies that if $t$ depends on $v_i$ then

$$t(\kappa_1, \ldots, \kappa_{i-1}, \kappa_i, \kappa_{i+1}, \ldots, \kappa_n)^{\mathcal{R}} \;\prec^{\mathcal{R}}\; t(\kappa_1, \ldots, \kappa_{i-1}, \kappa_i', \kappa_{i+1}, \ldots, \kappa_n)^{\mathcal{R}}$$

whenever $\kappa_1, \ldots, \kappa_n$ is an increasing sequence of indecomposables and $\kappa_i'$ is an indecomposable such that $\kappa_i \prec^{\mathcal{R}} \kappa_i' \prec^{\mathcal{R}} \kappa_{i+1}$.

**Definition 3.6** Assume $\alpha \in |\mathcal{R}|$. An indecomposable $\kappa$ is a *component of $\alpha$ in $\mathcal{R}$* if $\kappa$ is among $\kappa_1, \ldots, \kappa_n$ whenever $\kappa_1, \ldots, \kappa_n$ are indecomposables such that there is a term $t$ with $\alpha = t(\kappa_1, \ldots, \kappa_n)^{\mathcal{R}}$.

**Lemma 3.7** *Assume $\mathcal{R}$ is a well ordered EM structure such that the collection of indecomposables is unbounded. In addition, assume $\alpha \in R$.*

1. *Assume $t$ is a term and $\kappa_1, \ldots, \kappa_n$ is an increasing sequence of indecomposables such that $\alpha = t(\kappa_1, \ldots, \kappa_n)^{\mathcal{R}}$. If $t$ depends on $v_i$ then $\kappa_i$ is a component of $\alpha$.*

2. *If $\alpha$ is indecomposable then $\alpha$ is the only component of $\alpha$.*

3. *If $\alpha$ is decomposable then every component of $\alpha$ is less than $\alpha$.*

4. *If $\kappa$ is the largest indecomposable such that $\kappa \preceq^{\mathcal{R}} \alpha$ then $\kappa$ is the largest component of $\alpha$.*

**Proof.** Fix $\alpha$.

For part 1, assume $t$ depends on $v_i$. Argue by contradiction and assume that $\kappa_i$ is not a componant of $\alpha$. There is a term $s$ and a list of indecomposables $\lambda_1, \ldots, \lambda_m$ which does not include $\kappa_i$ such that $t(\kappa_1, \ldots, \kappa_n)^{\mathcal{R}} = s(\lambda_1, \ldots, \lambda_m)^{\mathcal{R}}$. Since there are infinitely many indecomposables, there are indecomposables $\kappa'_1, \ldots, \kappa'_n, \lambda'_1, \ldots, \lambda'_m$ and $\kappa''_i$ such that $\kappa'_i \neq \kappa''_i$,

$$\kappa_1, \ldots, \kappa_n, \lambda_1, \ldots, \lambda_m \;\sim^{\mathcal{R}}\; \kappa'_1, \ldots, \kappa'_n, \lambda'_1, \ldots, \lambda'_m$$

and

$$\kappa_1, \ldots, \kappa_n, \lambda_1, \ldots, \lambda_m \;\sim^{\mathcal{R}}\; \kappa'_1, \ldots, \kappa'_{i-1}, \kappa''_i, \kappa'_{i+1}, \ldots, \kappa'_n, \lambda'_1, \ldots, \lambda'_m$$

We see that

$$t(\kappa'_1, \ldots, \kappa'_n)^{\mathcal{R}} = s(\lambda'_1, \ldots, \lambda'_m)^{\mathcal{R}} = t(\kappa'_1, \ldots, \kappa'_{i-1}, \kappa''_i, \kappa'_{i+1}, \ldots, \kappa'_n)^{\mathcal{R}}$$

implying that $t$ is independent of $v_i$.

Part 2 follows from part 1 letting $t = v_1$.

Part 3 follows from Lemma 2.2.

For part 4, assume that $\kappa$ is the largest indecomposable such that $\kappa \preceq \alpha$. By Lemma 2.2, $\alpha = t(\kappa_1, \ldots, \kappa_n)^{\mathcal{R}}$ for some term $t$ and increasing sequence of indecomposables with $\kappa_n \preceq^{\mathcal{R}} \alpha$. We may assume that $\kappa_n = \kappa$. Argue by contradiction and assume that $\kappa$ is not a component of $\alpha$. By part 1, $t$ is independent of $v_n$. Let $\lambda$ be an indecomposable with $\alpha \prec^{\mathcal{R}} \lambda$. Since $\kappa_n \preceq^{\mathcal{R}} t(\kappa_1, \ldots, \kappa_n)^{\mathcal{R}}$, $\alpha \prec^{\mathcal{R}} \lambda \preceq^{\mathcal{R}} t(\kappa_1, \ldots, \kappa_{n-1}, \lambda)^{\mathcal{R}} = \alpha$ – contradiction. **QED**

The following example shows that the converse of part 1 of the lemma can fail. Moreover, for a given $\alpha$, there may be no term $t$ such that $\alpha = t(\kappa_1, \ldots, \kappa_n)^{\mathcal{R}}$ and $\kappa_1, \ldots, \kappa_n$ lists the components of $\alpha$.

**Example 3.8** Let $\mathcal{R} = (ORD, \leq, S)$ where $S$ is the binary operation with $S(\alpha, \beta) = max\{\alpha, \beta\} + 1$ whenever $\alpha \neq \beta$ and $S(\alpha, \alpha) = \alpha$. $\mathcal{R}$ is an EM structure and the indecomposables of $\mathcal{R}$ are

$$0, 1, \omega, \omega \cdot 2, \ldots, \omega \cdot \alpha, \ldots$$

For $\alpha > 0$ and a natural number $n > 0$, the only component of $\omega \cdot \alpha + n$ is $\omega \cdot \alpha$ while there is no term $t$ with $t(\omega \cdot \alpha) = \omega \cdot \alpha + n$. 0 and 1 are the components of any natural number $n > 1$. Moreover, $S(0, 1) = 2$ and the term $S(v_1, v_2)$ is independent of $v_1$.

# 4 Arithmetic Structures

For this section, fix a well ordered EM structure $\mathcal{R}$ for a lanuguage $\mathcal{L}$ such that the indecomposables are unbounded and have order type at least $\omega + \omega$.

**Definition 4.1** A structure $\mathbf{A}$ is an *arithmetic structure* with respect to $\mathcal{R}$ if $\mathbf{A}$ is a linearly ordered structure for the language $\mathcal{L}$ such that

1. $\mathbf{A}$ is generated from its set of indecomposables.
2. For any atomic formula $\mathsf{A}$ whose variables are among $v_1, \ldots, v_n$

$$\mathbf{A} \models \mathsf{A}(a_1, \ldots, a_n)$$

iff

$$\mathsf{A} \text{ is in the encoding of } \mathcal{R}$$

for any increasing sequence of indecomposables $a_1, \ldots, a_n$ of $\mathbf{A}$.

Notice that for $\mathbf{A}$ an arithmetic strucure, every element of $|\mathbf{A}|$ has the form $t(a_1, \ldots, a_n)^{\mathbf{A}}$ for some term $t$ and increasing sequence of indecomposables $a_1, \ldots, a_n$.

When $\mathbf{A}$ is an arithmetic structure with respect to $\mathcal{R}$ and $\mathcal{R}$ is clear from the context, we will simply refer to $\mathbf{A}$ as an arithmetic structure. In particular, when we refer to a structure as an arithmetic structure in this section we mean it is an arithmetic structure with respect to our fixed $\mathcal{R}$.

**Lemma 4.2** *$\mathcal{R}$ is an arithmetic structure.*

**Proof.** Immediate. **QED**

**Lemma 4.3** *Any closed substructure of an arithmetic structure is arithmetic.*

*Proof.* Straightforward. **QED**

**Lemma 4.4** *Assume $\mathbf{A}$ and $\mathbf{B}$ are arithmetic structures and $h_1$ and $h_2$ are embeddings of $\mathbf{A}$ into $\mathbf{B}$. If $h_1(a) \preceq^{\mathbf{B}} h_2(a)$ for each indecomposable in $\mathbf{A}$ then $h_1(x) \preceq^{\mathbf{B}} h_2(x)$ for all $x \in |\mathbf{A}|$. In particular, if $h_1$ and $h_2$ agree on all indecomposables of $\mathbf{A}$ then $h_1 = h_2$.*

**Proof.** By part 4 of Lemma 3.4. **QED**

**Lemma 4.5** *Assume $\mathbf{A}$ and $\mathbf{B}$ are arithmetic structures and the interpretation of any function symbol in $\mathbf{B}$ is total. If $h$ is an order preserving map of the indecomposables of $\mathbf{A}$ into the indecomposables of $\mathbf{B}$ then there is a unique extension $h^+$ of $h$ which is a closed embedding of $\mathbf{A}$ in $\mathbf{B}$.*

**Proof.** Let $I$ be the set of indecomposables of $\mathbf{A}$ and let $h$ be an order preserving map of $I$ into the indecomposables of $\mathbf{B}$. Since $\mathbf{A}$ is an arithmetic structure, we can define a function $h^+$ with domain $|\mathbf{A}|$ which extends $h$ such that

$$h^+(t(a_1, \ldots, a_n)^{\mathbf{A}}) \;=\; t(h(a_1), \ldots, h(a_n))^{\mathbf{B}}$$

whenever $t$ is a term whose variables are among $v_1, \ldots, v_n$ and $a_1, \ldots, a_n$ are indecomposable in $\mathbf{A}$ such that $t(a_1, \ldots, a_n)^{\mathbf{A}}$ is defined.

Let $\mathbf{A}^*$ be the substructure of $\mathbf{B}$ whose universe is the range of $h^+$. To see that $\mathbf{A}^*$ is closed in $\mathbf{B}$, assume $b$ is indecomposable in $\mathbf{A}^*$. $h^{-1}(b)$ is indecomposable in $\mathbf{A}$ i.e. $h^{-1}(b) \in I$. Therefore, $b$ is in $h[I]$, a set of indecomposables of $\mathbf{B}$.

By the Lemma 4.4, $h^+$ is unique. **QED**

**Lemma 4.6** *Assume $\mathbf{A}$ is a linearly ordered structure for $\mathcal{L}$ which is generated from its set of indecomposables. $\mathbf{A}$ is an arithmetic structure iff the substructure of $\mathbf{A}$ generated by $I$ is isomorphic to a closed substructure of $\mathcal{R}$ for any finite set $I$ of indecomposables of $\mathbf{A}$.*

**Proof.**(⇒) By Lemma 4.5.

(⇐) Assume the right hand side of the lemma. Let $\mathsf{A}$ be an atomic formula all of whose free variables are among $v_1, \ldots, v_n$. Let $a_1, \ldots, a_n$ be an increasing sequence of indecomposables in $\mathbf{A}$. Choose an increasing sequence $\kappa_1, \ldots, \kappa_n$ of indecomposables of $\mathcal{R}$. By assumption, there is a closed embedding $h$ of the substructure of $\mathbf{A}$ generated by $\{a_1, \ldots, a_n\}$ into $\mathcal{R}$ such that $h(a_i) = \kappa_i$ for $i = 1, \ldots, n$. We see that

$$\begin{array}{lll} \mathbf{A} \models \mathsf{A}(a_1, \ldots, a_n) & \text{iff} & \mathcal{R} \models \mathsf{A}(\kappa_1, \ldots, \kappa_n) \\ & \text{iff} & \mathsf{A} \text{ is in the encoding of } \mathcal{R} \end{array}$$

**QED**

Notice that the lemma implies that a finite structure is arithmetic iff it is isomorphic to a closed substructure of $\mathcal{R}$.

**Lemma 4.7** *Assume* $\mathbf{B}$ *is a finite arithmetic structure,* $\mathbf{A}$ *is a closed substructure of* $\mathbf{B}$*,* $x \in |\mathbf{A}|$ *and* $y \in |\mathbf{B}|$*.*

1. *If* $y$ *is the largest indecomposable in* $|\mathbf{B}|$ *such that* $y \preceq^{\mathbf{B}} x$ *then* $y \in |\mathbf{A}|$*.*
2. *If* $y$ *is indecomposable in* $|\mathbf{B}|$ *and* $x$ *is the least element of* $|\mathbf{A}|$ *such that* $y \preceq^{\mathbf{B}} x$ *then* $x$ *is indecomposable.*

**Proof.** By the previous lemma, we may assume that $\mathbf{B}$ is a substructure of $\mathcal{R}$.

Part 1 follows from part 4 of Lemma 3.7 after noticing that all components of $x$ in $\mathbf{B}$ are components of $x$ in $\mathbf{A}$ and, hence, in $|\mathbf{A}|$.

Part 2 follows from part 1. **QED**

**Definition 4.8** Assume $\mathbf{B}$ is an arithmetic structure and $\mathbf{A}$ is a closed substructure of $\mathbf{B}$. An element $b$ of $|\mathbf{B}|$ is *arithmetic in* $\mathbf{B}$ *over* $\mathbf{A}$ if $b$ is in the substructure of $B$ generated by the elements of $|\mathbf{A}|$. $\mathbf{B}$ is an *arithmetic extension of* $\mathbf{A}$ if every element of $|\mathbf{B}|$ is arithmetic in $\mathbf{B}$ over $\mathbf{A}$.

Notice that an element $b$ of $|\mathbf{B}|$ is arithmetic in $\mathbf{B}$ over $\mathbf{A}$ iff there is a term $t$ and indecomposable elements $a_1, \ldots, a_n$ of $|\mathbf{A}|$ such that $t(a_1, \ldots, a_n)^{\mathbf{B}} = b$.

**Lemma 4.9** *Assume* $\mathbf{A}$ *is a closed substructure of the arithmetic structure* $\mathbf{B}$*.* $\mathbf{B}$ *is an arithmetic extension of* $\mathbf{A}$ *iff every indecomposable of* $\mathbf{B}$ *is in* $|\mathbf{A}|$*.*

**Proof.** ($\Longrightarrow$) Assume $b \in |\mathbf{B}| \setminus |\mathbf{A}|$. Since $\mathbf{B}$ is an arithmetic extension of $\mathbf{A}$, $b = t(a_1, \ldots, a_n)^{\mathbf{B}}$ for some term $t$ and indecomposables $a_1, \ldots, a_n$ of $\mathbf{A}$. Since $\mathbf{A}$ is a closed substructure of $\mathbf{B}$, $a_1, \ldots, a_n$ are indecomposable in $\mathbf{B}$. Since $\mathbf{B}$ is an arithmetic structure, $b$ is not indecomposable in $\mathbf{B}$.

($\Longleftarrow$) Immediate since $\mathbf{B}$ is generated from its set of of indecomposables. **QED**

**Lemma 4.10** *Assume* $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{C}$ *are arithmetic structures such that* $\mathbf{A}$ *is a closed substructure of* $\mathbf{B}$ *and* $\mathbf{B}$ *is a closed substructure of* $\mathbf{C}$. $\mathbf{C}$ *is an arithmetic extension of* $\mathbf{A}$ *iff* $\mathbf{B}$ *is an arithmetic extension of* $\mathbf{A}$ *and* $\mathbf{C}$ *is an arithmetic extension of* $\mathbf{B}$.

**Proof.** By the previous lemma. **QED**

One might expect that if $\mathbf{A}$ is a closed substructure of an arithmetic structure $\mathbf{B}$ then the set $X$ of all $a \in \mathbf{B}$ which are arithmetic over $\mathbf{A}$ would be an arithmetic extension of $\mathbf{A}$. While this is true for the EM structures we are interested in, it is not true in general since it is possible that $X$ is not an arithmetic structure as the following example shows.

**Example 4.11** Consider the EM structure on the ordinals with the usual ordering, the constant 0 and the functions $f$, $g$ and $S$ such that $f(\alpha, \beta) = \max\{\alpha, \beta\} + 2$ and $g(\alpha, \beta) = \min\{\alpha, \beta\} + 2$ for $\alpha \neq \beta$, $f(\alpha, \alpha) = \alpha$, $g(\alpha, \alpha) = \alpha$ and $S(\alpha) = \alpha + 1$. The indecomposable ordinals are those of the form $\omega \cdot \xi$ where $\xi \neq 0$. Let $\mathbf{A}$ be the substructure with universe $\{\omega \cdot 2, \omega \cdot 3\}$ and let $\mathbf{B}$ be the substructure with universe $\{\omega, \omega \cdot 2, \omega \cdot 2 + 2, \omega \cdot 3\}$. The set of elements of $\mathbf{B}$ which are arithmetic over $\mathbf{A}$ is $\{\omega \cdot 2, \omega \cdot 2 + 2, \omega \cdot 3\}$ but the substructure of $\mathbf{B}$ with this universe is not an arithmetic structure.

**Lemma 4.12** *Assume* $\mathbf{A}$ *is a finite arithmetic structure. If* $t$ *is a term in the language of* $\mathcal{R}$ *whose variables are among* $v_1, \ldots, v_n$ *and* $a_1, \ldots, a_n \in |\mathbf{A}|$ *then there is a finite arithmetic extension* $\mathbf{B}$ *of* $\mathbf{A}$ *such that* $t(a_1, \ldots, a_n)^{\mathbf{B}}$ *is defined.*

**Proof.** Without loss of generality, we may assume that $\mathbf{A}$ is a closed substructure of $\mathcal{R}$. $t(a_1, \ldots, a_n)^{\mathcal{R}}$ is in the substructure $\mathbf{A}^+$ of $\mathcal{R}$ generated by $|\mathbf{A}|$. Notice that $\mathbf{A}^+$ is an arithmetic extension of $\mathbf{A}$. Considering the finite set consisting of $|\mathbf{A}|$ along with $s(a_1, \ldots, a_n)^{\mathcal{R}}$ for $s$ a subterm of $t$, the conclusion follows by showing by induction on the largest element of a finite nonempty subset $X$ of $|\mathbf{A}^+|$ that there is a finite closed substructure $\mathbf{B}$ of $\mathbf{A}^+$ such that $X \subseteq |\mathbf{B}|$. **QED**

**Definition 4.13** Assume $\mathbf{B}$ is an arithmetic structure and $\mathbf{A}$ is a closed substructure of $\mathbf{B}$. An element of $|\mathbf{B}| \setminus |\mathbf{A}|$ is *transcendental in* $\mathbf{B}$ *over* $\mathbf{A}$ if it is not arithmetic in $\mathbf{B}$ over $\mathbf{A}$. $\mathbf{B}$ is a *transcendental extension of* $\mathbf{A}$ if every element of $|\mathbf{B}| \setminus |\mathbf{A}|$ is transcendental in $\mathbf{B}$ over $\mathbf{A}$.

**Lemma 4.14** *Assume* $\mathbf{A}$ *is an arithmetic structure and* $a$ *is an indecomposable of* $\mathbf{A}$*. If* $X$ *is a subset of* $|\mathbf{A}|$ *such that* $a \preceq^{\mathbf{A}} X$ *and* $(-\infty, a)^{\mathbf{A}} \cup X$ *is a closed subset of* $\mathbf{A}$ *then the least element of* $X$ *is indecomposable in* $\mathbf{A}$*.*

**Proof.** Notice that the least element of $X$ is indecomposable in the substructure with universe $(-\infty, a)^{\mathbf{A}} \cup X$ since $(-\infty, a)^{\mathbf{A}}$ is closed under the interpretation of any function symbol by part 1 of Lemma 3.4 and Lemma 4.6. **QED**

**Lemma 4.15** *Assume* $\mathbf{A}$ *and* $\mathbf{A}^+$ *are finite arithmetic structures,* $\mathbf{A}$ *is a proper closed substructure of* $\mathbf{A}^+$*,* $a \in |\mathbf{A}|$ *is indecomposable in* $\mathbf{A}$ *and* $(-\infty, a)^{\mathbf{A}} <^{\mathbf{A}^+} x <^{\mathbf{A}^+} a$ *whenever* $x \in |\mathbf{A}^+| \setminus |\mathbf{A}|$*.* $\mathbf{A}^+$ *is a transcendental extension of* $\mathbf{A}$ *iff the least element of* $|\mathbf{A}^+| \setminus |\mathbf{A}|$ *is indecomposable.*

**Proof.** We may assume that $\mathbf{A}^+$ is a closed substructure of $\mathcal{R}$. Let $\lambda$ be the least element of $|\mathbf{A}^+| \setminus |\mathbf{A}|$.

($\Rightarrow$) If $\lambda$ were not indecomposable then $\lambda$ would be arithmetic in $\mathbf{A}^+$ over $\mathbf{A}$.

($\Leftarrow$) Suppose $\lambda$ is indecomposable. Suppose $\xi \in |\mathbf{A}^+| \setminus |\mathbf{A}|$. Let $\kappa$ be the largest indecomposable of $\mathcal{R}$ in $(-\infty, \xi]$. By Part 4 of Lemma 3.7, $\kappa$ is a component of $\xi$. Since $\lambda \leq \kappa$, $\kappa \notin |\mathbf{A}|$. If $\xi$ were arithmetic over $\mathbf{A}$, there would be a term $t$ and an increasing sequence of indecomposables $\kappa_1, \ldots, \kappa_n$ in $\mathbf{A}$ such that $\xi = t(\kappa_1, \ldots, \kappa_n)^{\mathbf{A}^+}$ contradicting the fact that $\kappa$ is a component of $\xi$. **QED**

**Lemma 4.16** *Assume* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$ *are finite arithmetic structures such that* $\mathbf{A}$ *is an initial substructure of both* $\mathbf{B}$ *and* $\mathbf{C}$*, the least element of* $|\mathbf{C}|$ *not in* $|\mathbf{A}|$ *is indecomposable in* $\mathbf{C}$ *and* $|\mathbf{C}| \setminus |\mathbf{A}|$ *is disjoint from* $|\mathbf{B}| \setminus |\mathbf{A}|$*. There is a unique arithmetic structure* $\mathbf{D}$ *such that*

1. $|\mathbf{D}| = |\mathbf{B}| \cup |\mathbf{C}|$

2. $\mathbf{B}$ *and* $\mathbf{C}$ *are closed substructures of* $\mathbf{D}$.

3. *If* $x \in |\mathbf{B}| \setminus |\mathbf{A}|$ *and* $y \in |\mathbf{C}| \setminus |\mathbf{A}|$ *then* $x \preceq^{\mathbf{D}} y$.

**Proof.** Notice that the case when $\mathcal{R}$ is simply $(ORD, \leq)$, the lemma is trivial. The general case involves arranging the indecomposables in the correct order as in this simple case and then extending the arithmetic operations.

We may assume that $\mathbf{B}$ is a closed substructure of $\mathcal{R}$. Let $h$ be an order preserving map of the indecomposables of $\mathbf{C}$ into the indecomposables of $\mathcal{R}$ such that $h(x) = x$ for $x \in |\mathbf{A}|$ and $|\mathbf{B}| < h(x)$ if $x \in |\mathbf{C}| \setminus |\mathbf{A}|$.

By Lemma 4.5, there is an extension $h^+$ of $h$ to $|\mathbf{C}|$ which is a closed embedding of $\mathbf{C}$ into $\mathcal{R}$. Since $h$ is the identity on the indecomposables in $\mathbf{A}$, $h^+$ is the identity on $|\mathbf{A}|$ by Lemma 4.4. Let $f$ be the extension of $h^+$ to $|\mathbf{B}| \cup |\mathbf{C}|$ which is the identity on $|\mathbf{B}|$. Notice that the range of $f$ is a closed subset of $|\mathcal{R}|$ since it is the union of $|\mathbf{B}|$ and the range of $h^+$ both of which are closed subsets of $\mathcal{R}$. Let $\mathbf{D}$ be the arithmetic structure with universe $|\mathbf{B}| \cup |\mathbf{C}|$ such that $f$ is an embedding of $\mathbf{D}$ into $\mathcal{R}$. $\mathbf{D}$ can be seen to be as required.

To see that $\mathbf{D}$ is unique, suppose $\mathbf{D}'$ also satisfies the conclusion of the lemma. The indecomposables of $\mathbf{D}'$ must be the union of the indecomposables in $\mathbf{C}$ and the indecomposables in $\mathbf{B}$. By Lemma 4.5, there is an embedding $f'$ of $\mathbf{D}'$ into $\mathcal{R}$ which agrees with $f$ on the indecomposables of $\mathbf{D}'$. By Lemma 4.4, $f'$ must agree with $f$ on $|\mathbf{B}|$ and, likewise, on $|\mathbf{C}|$. Therefore $f = f'$ implying $\mathbf{D}$ and $\mathbf{D}'$ are both isomorphic to the substructure of $\mathcal{R}$ whose universe is the range of $f$. **QED**

# 5 Patterns of Resemblance of Order 2

For the remainder of the paper, assume $\mathcal{R}_0$ is an EM structure for a language $\mathcal{L}_0$ including the binary predicate symbol $\preceq$ such that $\preceq^{\mathcal{R}_0}$ is the usual ordering of the ordinals, $\leq$. We also assume the indecomposables of $\mathcal{R}_0$ are unbounded and have order type at least $\omega + \omega$.

Let $\mathcal{L}_2$ be the expansion of $\mathcal{L}_0$ obtained by adding two new binary relation symbols, $\preceq_1$ and $\preceq_2$. We also define $\preceq_0$ to be $\preceq$.

**Definition 5.1** Assume $\mathcal{R}$ is a structure for $\mathcal{L}_2$. The *arithmetic part of* $\mathcal{R}$ is the restriction of $\mathcal{R}$ to $\mathcal{L}_0$.

**Definition 5.2** Assume $\mathcal{R}$ and $\mathcal{R}^+$ are structures for the language $\mathcal{L}_2$ whose arithmetic parts are arithmetic structures. A function $h : |\mathcal{R}| \to |\mathcal{R}^+|$ is a *covering of* $\mathcal{R}$ *into* $\mathcal{R}^+$ if

1. $h$ is a closed embedding of the arithmetic part of $\mathcal{R}$ into the arithmetic part of $\mathcal{R}^+$.

2. For any $x, y \in |\mathcal{R}|$ and $i = 1, 2$.

$$x \preceq_i^{\mathcal{R}} y \Longrightarrow h(x) \preceq_i^{\mathcal{R}^+} h(y)$$

$\mathcal{R}^+$ is a *covering of* $\mathcal{R}$ if there is a covering of $\mathcal{R}$ onto $\mathcal{R}^+$. $\mathcal{R}^+$ is a *cover of* $\mathcal{R}$ if the arithmetic part of $\mathcal{R}$ is the same as the arithmetic part of $\mathcal{R}^+$ and the inclusion map is a covering of $\mathcal{R}$ into $\mathcal{R}^+$. The *covering relation* is the relation $\preceq_{cov}$ on the set of structures $\mathbf{Q}$ for $\mathcal{L}_2$ such that the arithmetic part of $\mathbf{Q}$ is an arithmetic structure where

$$\mathcal{R} \preceq_{cov} \mathcal{R}^+ \quad \text{iff} \quad \text{a closed substructure of } \mathcal{R}^+ \text{ is a cover of } \mathcal{R}$$

Notice that the composition of coverings is a covering.

The structure $\mathcal{R}_0$ can be extended to a structure $\mathcal{R}_2 = (\mathcal{R}_0, \leq_1, \leq_2)$ for $\mathcal{L}_2$ as in the introduction by inductively defining the interpretations $\leq_1$ and $\leq_2$ of $\preceq_1$ and $\preceq_2$ respectively so that

$$\alpha \leq_n \beta \text{ iff } \mathcal{R}_2|\alpha \preceq_{\Sigma_n} \mathcal{R}_2|\beta$$

for $n = 1, 2$ and all ordinals $\alpha$ and $\beta$ (where we write $\mathcal{R}_2|\xi$ for the substructure of $\mathcal{R}_2$ with universe $\xi$). However, we will use an alternate definition which appears to be more natural for the study of higher levels of elementarity. The equivalence of the two definitions will be established elsewhere.

**Definition 5.3** Suppose $\mathcal{R}$ is a linearly ordered structure for a language including the binary relation symbols $\preceq$, $\preceq_1$ and $\preceq_2$. Assume $a, b \in |\mathcal{R}|$ with $a \preceq^{\mathcal{R}} b$. Define $a \preceq_1^{\infty} b$ in $\mathcal{R}$ iff

For any finite $X \subseteq (-\infty, a)^{\mathcal{R}}$ and finite $Y \subseteq [a, b)^{\mathcal{R}}$ where $X \cup Y$ is a closed subset of $\mathcal{R}$ there is a finite $\tilde{Y} \subseteq (-\infty, a)^{\mathcal{R}}$ such that

(a) $X \prec^{\mathcal{R}} \tilde{Y}$

(b) $X \cup \tilde{Y}$ is a closed subset of $\mathcal{R}$

(c) $X \cup \tilde{Y}$ is a covering of $X \cup Y$

Define $a \preceq_2^\infty b$ in $\mathcal{R}$ iff

1. For any finite $X \subseteq (-\infty, a)^{\mathcal{R}}$ and finite $Y \subseteq [a, b)^{\mathcal{R}}$ where $X \cup Y$ is a closed subset of $\mathcal{R}$ there is a finite $\tilde{Y} \subseteq (-\infty, a)^{\mathcal{R}}$ such that
   (a) $X \prec^{\mathcal{R}} \tilde{Y}$
   (b) $X \cup \tilde{Y}$ is a closed subset of $\mathcal{R}$
   (c) $X \cup \tilde{Y}$ is a covering of $X \cup Y$
   (d) For any $i < card(Y)$, if $y \preceq_1 b$ where $y$ is the $i^{\text{th}}$ element of $Y$ then $\tilde{y} \preceq_1 a$ where $\tilde{y}$ is the $i^{\text{th}}$ element of $\tilde{Y}$.

2. For any finite $X$ below $a$ and any finite structure $\mathbf{P}$, if there are cofinally many finite subsets $Y$ below $a$ such that $X \cup Y$ is closed and $X \cup Y$ is a covering of $\mathbf{P}$ then there are confinally many closed subsets $Y$ below $b$ such that $X \cup Y$ is closed and $X \cup Y$ is a covering of $\mathbf{P}$.

We will often write $a \preceq_k^\infty b$ for $a \preceq_k^\infty b$ in $\mathcal{R}$ when $\mathcal{R}$ is understood.

The relation $\preceq_k^\infty$ is in a sense a weakening of $\preceq_{\Sigma_k}$ e.g. one can show for finite languages $\mathcal{L}_0$ that using the preliminary definition of $\mathcal{R}_2$ above, for ordinals $\alpha \leq \beta$, if $\alpha \leq_k \beta$ (i.e. $\alpha \leq_{\Sigma_k} \beta$) then $\alpha \preceq_k^\infty \beta$.

**Definition 5.4** $\mathcal{R}_2$ is the expansion of $\mathcal{R}_0$ to $\mathcal{L}_2$ where the interpretations $\leq_1$ and $\leq_2$ of $\preceq_1$ and $\preceq_2$ respectively are defined so that

$$\alpha \leq_n \beta \quad \text{iff} \quad \alpha \ \preceq_n^\infty \ \beta \text{ in } \mathcal{R}_2$$

for $n = 1, 2$ and all ordinals $\alpha$ and $\beta$.

**Lemma 5.5**

1. *Assume $\alpha \leq_1 \beta$. If $X \subseteq \alpha$ and $Y \subseteq [\alpha, \beta)$ are finite and $X \cup Y$ is closed in $\mathcal{R}_2$ then there are cofinally many $\tilde{Y}$ below $\alpha$ such that*
   *(a) $X < \tilde{Y}$*
   *(b) $X \cup \tilde{Y}$ is closed in $\mathcal{R}_2$.*
   *(c) $X \cup \tilde{Y}$ is a covering of $X \cup Y$.*

2. *Assume $\alpha \leq_2 \beta$. If $X \subseteq \alpha$ and $Y \subseteq [\alpha, \beta)$ are finite and $X \cup Y$ is closed in $\mathcal{R}_2$ then there are cofinally many $\tilde{Y}$ below $\alpha$ such that*

   (a) $X < \tilde{Y}$

   (b) $X \cup \tilde{Y}$ *is closed in* $\mathcal{R}_2$.

   (c) *There is a covering $h$ of $X \cup Y$ onto $X \cup \tilde{Y}$ such that $h(\eta) \leq_1 \alpha$ whenever $\eta \in Y$ and $\eta \leq_1 \beta$*

3. $\leq_1$ *is a partial ordering.*

4. $\leq_2$ *is a partial ordering.*

5. $\leq_1$ *respects* $\leq$.

6. $\leq_2$ *respects* $\leq_1$.

7. *Assume $\alpha$ and $\beta$ are ordinals.*

   (a) *If $\alpha <_1 \beta$ then $\alpha$ is a limit of indecomposables in $\mathcal{R}_2$.*

   (b) *If $\alpha <_2 \beta$ then $\beta$ is a limit of indecomposables in $\mathcal{R}_2$.*

**Proof.** For part 1, suppose $X \subseteq \alpha$ and $Y \subseteq [\alpha, \beta)$ are finite, $X \cup Y$ is closed in $\mathcal{R}_2$ and $\alpha' < \alpha$. By Lemma 2.5, there is a finite closed $X^+ \subseteq \alpha$ such that $X \subseteq X^+$ and $\alpha' \in X^+$. Since $X^+ \cup Y$ is closed in $\mathcal{R}_2$ and $\alpha \leq_1 \beta$, there exists $\tilde{Y} \subseteq \alpha$ such that $X^+ < \tilde{Y}$, $X^+ \cup \tilde{Y}$ is closed in $\mathcal{R}_2$ and $X^+ \cup \tilde{Y}$ is a covering of $X^+ \cup Y$. It easily follows that $X < \tilde{Y}$, $X \cup \tilde{Y}$ is closed and $X \cup \tilde{Y}$ is a covering of $X \cup Y$. Moreover, $\alpha' < \tilde{Y}$.

Part 2 can be proved by an argument similar to that for part 1.

Parts 3-5 are straightforward.

For part 6, first notice that $\leq_1$ is a subset of $\leq$. Now assume that $\alpha \leq \beta \leq \gamma$, $\alpha \leq_2 \gamma$ and $\beta \leq_1 \gamma$. We will show that $\alpha \leq_2 \beta$. This is trivial if $\alpha = \beta$ or $\beta = \gamma$. So, we may assume that $\alpha < \beta < \gamma$.

To show that $\alpha \leq_2 \beta$, first suppose that $X \subseteq \alpha$ and $Y \subset [\alpha, \beta)$ are finite and $X \cup Y$ is closed in $\mathcal{R}_2$. Since $\alpha \leq_2 \gamma$, there exists $\tilde{Y} \subset \alpha$ such that $X < \tilde{Y}$, $X \cup \tilde{Y}$ is closed in $\mathcal{R}_2$, and there is a covering $h$ of $X \cup Y$ onto $X \cup \tilde{Y}$ such that $h(\eta) \leq_1 \alpha$ whenever $\eta \in Y$ and $\eta \leq_1 \gamma$. If $\eta \in Y$ and $\eta \leq_1 \beta$ then, since $\beta \leq_1 \gamma$, $\eta \leq_1 \gamma$ implying $\eta \leq_1 \alpha$.

To complete the proof that $\alpha \leq_2 \beta$, suppose that $X \subseteq \alpha$ and $\mathbf{P}$ is a pattern such that there are cofinally many $Y$ below $\alpha$ with the property that $X \cup Y$

is closed and $X \cup Y$ is a covering of $\mathbf{P}$. In particular, there exists $Y \subseteq [\beta, \gamma)$ such that $X \cup Y$ is a covering of $\mathbf{P}$. Now suppose that $\beta' < \beta$. By part 1, there exists $\tilde{Y} \subseteq [\beta', \beta)$ such that $X \cup \tilde{Y}$ is closed in $\mathcal{R}_2$ and $X \cup \tilde{Y}$ is a covering of $X \cup Y$.

For part 7(a), assume $\alpha <_1 \beta$. By part 1, there are cofinally many $\alpha'$ below $\alpha$ such that $\{\alpha'\}$ is closed and $\{\alpha'\}$ is a covering of $\{\alpha\}$. Any such $\alpha'$ must be indecomposable in $\mathcal{R}_2$.

For part 7(b), assume $\alpha <_2 \beta$. By part 6, $\alpha <_1 \beta$. There are cofinally many $\alpha'$ below $\alpha$ such that $\{\alpha'\}$ is closed and $\{\alpha'\}$ is a covering of $\{\alpha\}$. Since $\alpha <_2 \beta$, there are cofinally many $\beta'$ below $\beta$ such that $\{\beta'\}$ is closed and $\{\beta'\}$ is a covering of $\{\alpha\}$. Any such $\beta'$ must be indecomposable in $\mathcal{R}_2$. **QED**

**Definition 5.6** Assume $\mathbf{R}$ is a structure for the language $\mathcal{L}_2$. $\mathbf{R}$ is a *model of prereflection of order two with respect to* $\mathcal{R}_0$ provided

1. the restriction of $\mathbf{R}$ to $\mathcal{L}_0$ is an arithmetic structure with respect to $\mathcal{R}_0$,
2. if $a, b \in |\mathbf{R}|$ and $a \prec_1^{\mathbf{R}} b$ then $a$ is indecomposable, and
3. if $a, b \in |\mathbf{R}|$ and $a \prec_2^{\mathbf{R}} b$ then $b$ is indecomposable.

$\mathbf{R}$ is a *model of reflection of order two with respect to* $\mathcal{R}_0$ provided $\mathbf{R}$ is a model of prereflection of order two with respect to $\mathcal{R}_0$ and satisfies

4. $\preceq_1^{\mathbf{R}}$ is a partial ordering,
5. $\preceq_2^{\mathbf{R}}$ is a partial ordering,
6. $\preceq_1^{\mathbf{R}}$ respects $\preceq^{\mathbf{R}}$, and
7. $\preceq_2^{\mathbf{R}}$ respects $\preceq_1^{\mathbf{R}}$.

A finite model of reflection of order two with respect to $\mathcal{R}_0$ will be called a *pattern of resemblance of order two with respect to* $\mathcal{R}_0$.

We will sometimes refer to a model of prereflection of order two with respect to $\mathcal{R}_0$ as a model of prereflection, a model of reflection of order two with respect to $\mathcal{R}_0$ as a model of reflection, and a pattern of resemblance of order two with respect to $\mathcal{R}_0$ as a pattern when there is no possibility of confusion.

Notice that if $\mathbf{R}$ is a model of reflection then both $\preceq_1^{\mathbf{R}}$ and $\preceq_2^{\mathbf{R}}$ are forests.

**Lemma 5.7** *1. $\mathcal{R}_2$ is a model of reflection.*

*2. If $\mathbf{P}$ is isomorphic to a finite closed substructure of $\mathcal{R}_2$ then $\mathbf{P}$ is a pattern.*

*3. Assume $i \in \{1, 2\}$ and $0 < \alpha < \beta$. If for all $\alpha' < \alpha$ and $\beta' < \beta$ there are $\alpha''$ and $\beta''$ such that $\alpha' < \alpha'' \leq \alpha$, $\beta' < \beta'' \leq \beta$ and $\alpha'' \leq_i \beta''$ then $\alpha \leq_i \beta$.*

**Proof.** Part 1 follows from Lemma 5.5 and part 2 follows from part 1.

For part 3, we first consider the case when $i = 1$. Assume for all $\alpha' < \alpha$ and $\beta' < \beta$ there are $\alpha''$ and $\beta''$ such that $\alpha' < \alpha'' \leq \alpha$, $\beta' < \beta'' \leq \beta$ and $\alpha''\leq_1\beta''$. Suppose $X \subseteq \alpha$ and $Y \subseteq [\alpha, \beta)$ are finite such that $X \cup Y$ is closed in $\mathcal{R}_2$. Since the case $Y = \emptyset$ is trivial, we may assume $Y$ is nonempty. Choose $\alpha' < \alpha$ and $\beta' < \beta$ such that $X \subseteq [0, \alpha']$ and $Y \subseteq [\alpha, \beta']$. By our assumption, there are $\alpha''$ and $\beta''$ such that $\alpha' < \alpha'' \leq \alpha$, $\beta' < \beta'' \leq \beta$ and $\alpha''\leq_1\beta''$. There exists $\tilde{Y} \subseteq \alpha''$ such that $X < \tilde{Y}$, $X \cup \tilde{Y}$ is closed and $X \cup \tilde{Y}$ is a covering of $X \cup Y$. Since $\alpha'' \leq \alpha$, $\tilde{Y} \subseteq \alpha$.

Now suppose $i = 2$ in part 3 and assume for all $\alpha' < \alpha$ and $\beta' < \beta$ there are $\alpha''$ and $\beta''$ such that $\alpha' < \alpha'' \leq \alpha$, $\beta' < \beta'' \leq \beta$ and $\alpha''\leq_2\beta''$. We will show that $\alpha\leq_2\beta$. Since this is clear if $\alpha = \beta$, we may assume that $\alpha < \beta$.

First suppose $X \subseteq \alpha$ and $Y \subseteq [\alpha, \beta)$ are finite such that $X \cup Y$ is closed in $\mathcal{R}_2$. Since the case $Y = \emptyset$ is trivial, we may assume $Y$ is nonempty. Choose $\alpha' < \alpha$ and $\beta' < \beta$ such that $X \subseteq [0, \alpha']$ and $Y \subseteq [\alpha, \beta']$. By our assumption, there are $\alpha''$ and $\beta''$ such that $\alpha' < \alpha'' \leq \alpha$, $\beta' < \beta'' \leq \beta$ and $\alpha''\leq_2\beta''$. There exists $\tilde{Y} \subseteq \alpha''$ such that $X < \tilde{Y}$, $X \cup \tilde{Y}$ is closed in $\mathcal{R}_2$ and there is a covering $h$ of $X \cup Y$ onto $X \cup \tilde{Y}$ such that $h(\eta)\leq_1\alpha''$ whenever $\eta \in Y$ and $\eta\leq_1\beta''$. Since $\alpha'' \leq \alpha$, $\tilde{Y} \subseteq \alpha$. Since $\alpha'' \leq \alpha \leq \beta''$, $\alpha''\leq_1\alpha$. Hence, if $\eta \in Y$ and $\eta\leq_1\beta$ then $\eta\leq_1\beta''$ implying $h(\eta)\leq_1\alpha''$ which in turn implies that $h(\eta)\leq_1\alpha$.

Now suppose that $X \subseteq \alpha$ is finite and $\mathbf{P}$ is a pattern such that there are cofinally many finite $Y \subseteq \alpha$ such that $X \cup Y$ is closed and $X \cup Y$ is a covering of $\mathbf{P}$. Also suppose $\beta' < \beta$. Without loss of generality, $\alpha \leq \beta'$. By assumption, there are $\alpha''$ and $\beta''$ such that $X < \alpha'' \leq \alpha$, $\beta' < \beta'' \leq \beta$ and $\alpha''\leq_2\beta''$. Since $\alpha'' \leq \alpha \leq \beta''$, $\alpha''\leq_1\alpha$. This implies that there are cofinally many $Y$ below $\alpha''$ such that $X \cup Y$ is closed and $X \cup Y$ is a covering of $\mathbf{P}$. Therefore, there are cofinally many such $Y$ below $\beta''$. In particular, there exists such $Y$ with $Y \subseteq [\beta', \beta'')$. **QED**

We will establish the converse of the second part of the lemma under certain assumptions.

**Definition 5.8** Assume $\mathbf{A}$ is an arithmetic structure with respect to $\mathcal{R}_0$ and $\mathcal{F}$ is a collection of structures for $\mathcal{L}_2$ such that the arithmetic part of each element of $\mathcal{F}$ is a closed substructure of $\mathbf{A}$. For $\mathcal{F}$ nonempty, the *intersection of* $\mathcal{F}$ is the structure for $\mathcal{L}_2$ whose arithmetic part is the substructure of $\mathbf{A}$ whose universe is the intersection of the universes of the elements of $\mathcal{F}$ such that the interpretation of $\preceq_k$ is the intersection of the interpretations of $\preceq_k$ in the elements of $\mathcal{F}$ for $k = 1, 2$. The *union of* $\mathcal{F}$ is the structure for $\mathcal{L}_2$ whose arithmetic part is the substructure of $\mathbf{A}$ whose universe is the union of the universes of the elements of $\mathcal{F}$ such that the interpretation of $\preceq_k$ is the union of the interpretations of $\preceq_k$ in the elements of $\mathcal{F}$ for $k = 1, 2$.

Notice that the union of $\mathcal{F}$ generally depends on $\mathbf{A}$ while the intersection does not.

**Lemma 5.9** *Let* $\mathbf{A}$ *be an arithmetic structure and let* $\mathcal{P}$ *be the family of structures* $\mathbf{R}$ *for* $\mathcal{L}_2$ *such that the arithmetic part of* $\mathbf{R}$ *is a closed substructure of* $\mathbf{A}$.

1. $\preceq_{cov}$ *is a partial ordering of* $\mathcal{P}$.

2. *There is a largest element* $\mathbf{L}$ *in the set of models of prereflection with arithmetic part* $\mathbf{A}$. *Moreover,* $\mathbf{L}$ *is a model of reflection.*

3. *There is a smallest element* $\mathbf{S}$ *of in the set of models of prereflection with arithmetic part* $\mathbf{A}$. *Moreover,* $\mathbf{S}$ *is a model of reflection.*

4. *If* $\mathbf{R}_1, \mathbf{R}_2 \in \mathcal{P}$, $\mathbf{R}_2$ *is a model of prereflection and* $\mathbf{R}_1 \preceq_{cov} \mathbf{R}_2$ *then* $\mathbf{R}_1$ *is a model of prereflection.*

5. *Assume* $\mathcal{F}$ *is a set of elements of* $\mathcal{P}$.

   (a) *If each element of* $\mathcal{F}$ *is a model of prereflection then so is the union of* $\mathcal{F}$.

   (b) *For* $\mathcal{F}$ *nonempty, if each element of* $\mathcal{F}$ *is a model of reflection then so is the intersection of* $\mathcal{F}$.

**Proof.** Parts 1, 4 and 5 are straightforward.

For part 2, let $\mathbf{L}$ be the structure for $\mathcal{L}_2$ with arithmetic part $\mathbf{A}$ such that for all $x, y \in |\mathbf{A}|$ with $x \preceq_{\mathbf{A}} y$

$$x \preceq_1^{\mathbf{L}} y \quad \text{iff} \quad \text{either } x = y \text{ or } x \text{ is indecomposable in } \mathbf{A}$$

and

$$x \preceq_2^{\mathbf{P}} y \quad \text{iff} \quad \text{either } x = y \text{ or both } x \text{ and } y \text{ are indecomposable in } \mathbf{A}$$

Clearly, $\mathbf{L}$ is the largest element of $\mathcal{P}$ which is a model of prereflection and $\mathbf{L}$ is a model of reflection.

For part 3, notice that the structure for $\mathcal{L}_2$ with arithmetic part $\mathbf{A}$ in which both $\preceq_1$ and $\preceq_2$ have empty interpretation is a model of reflection. **QED**

**Lemma 5.10** *If* $\mathbf{R}$ *is a model of prereflection then there is a smallest model of reflection which is a cover of* $\mathbf{R}$ *and has the same arithmetic part as* $\mathbf{R}$.

**Proof.** Let $\mathbf{A}$ be the arithmetic part of $\mathbf{R}$. By part 2 of the previous lemma, there is model of reflection which is a cover of $\mathbf{R}$. The intersection of all models of reflection which are covers of $\mathbf{R}$ is the required structure. **QED**

# 6 Valid Rules

**Definition 6.1** Assume $\mathbf{P}$ and $\mathbf{P}^+$ are patterns and $\mathbf{P}$ is a closed substructure of $\mathbf{P}^+$. The rule $\mathbf{P}|\mathbf{P}^+$ is *valid* if for every covering $h$ of $\mathbf{P}$ in $\mathcal{R}_2$, there is a covering $h^+$ of $\mathbf{P}^+$ into $\mathcal{R}_2$ which extends $h$.

We will have need for an apparently stronger version of this notion which requires the following preliminary definition.

**Definition 6.2** Assume $\mathbf{P}$ is a pattern, $h$ is a covering of $\mathbf{P}$ in $\mathcal{R}_2$ and $\varphi$ is a regressive function on the nonzero indecomposable ordinals in the range of $h$. Suppose also that $\mathbf{P}^+$ is a pattern and $\mathbf{P}$ is a closed substructure of $\mathbf{P}^+$. A covering $h^+$ of $\mathbf{P}^+$ in $\mathcal{R}_2$ *extends* $h$ *above* $\varphi$ if $h^+$ extends $h$ and

$$\varphi(h(a)) < h^+(b)$$

for any indecomposable $b$ in $\mathbf{P}^+$ and any indecomposable $a$ in $\mathbf{P}$ such that $(-\infty, a)^{\mathbf{P}} \prec^{\mathbf{P}^+} b \prec^{\mathbf{P}^+} a$.

**Definition 6.3** Assume $\mathbf{P}$ and $\mathbf{P}^+$ are patterns and $\mathbf{P}$ is a closed substructure of $\mathbf{P}^+$. The rule $\mathbf{P}|\mathbf{P}^+$ is *cofinally valid* if for every covering $h$ of $\mathbf{P}$ in $\mathcal{R}_2$ and every regressive function $\varphi$ on the nonzero indecomposable ordinals in the range of $h$ there is a covering $h^+$ of $\mathbf{P}^+$ into $\mathcal{R}_2$ which extends $h$ above $\varphi$.

We will show later that under certain assumptions every valid rule is cofinally valid.

**Lemma 6.4** *Assume* $\mathbf{P}_1$*,* $\mathbf{P}_2$ *and* $\mathbf{P}_3$ *are patterns such that* $\mathbf{P}_1$ *is a closed subpattern of* $\mathbf{P}_2$ *and* $\mathbf{P}_2$ *is a closed subpattern of* $\mathbf{P}_3$*.*

1. *If* $\mathbf{P}_1|\mathbf{P}_2$ *and* $\mathbf{P}_2|\mathbf{P}_3$ *are valid then* $\mathbf{P}_1|\mathbf{P}_3$ *is valid.*
2. *If* $\mathbf{P}_1|\mathbf{P}_2$ *and* $\mathbf{P}_2|\mathbf{P}_3$ *are cofinally valid then* $\mathbf{P}_1|\mathbf{P}_3$ *is cofinally valid.*
3. *If* $\mathbf{P}_1|\mathbf{P}_3$ *is valid then* $\mathbf{P}_1|\mathbf{P}_2$ *is valid.*
4. *If* $\mathbf{P}_1|\mathbf{P}_3$ *is cofinally valid then* $\mathbf{P}_1|\mathbf{P}_2$ *is cofinally valid.*

**Proof.** Parts 1, 3 and 4 are straightforward.

For part 2, assume $\mathbf{P}_1|\mathbf{P}_2$ and $\mathbf{P}_2|\mathbf{P}_3$ are cofinally valid. To show that $\mathbf{P}_1|\mathbf{P}_3$ is cofinally valid, assume $h_1$ is a covering of $\mathbf{P}_1$ in $\mathcal{R}_2$ and $\varphi_1$ is a regressive function on the indecomposable ordinals in the range of $h_1$. Since $\mathbf{P}_1|\mathbf{P}_2$ is cofinally valid, there is a covering $h_2$ of $\mathbf{P}_2$ in $\mathcal{R}_2$ extending $h_1$ above $\varphi_1$. Define a regressive function on the nonzero indecomposables in the range of $h_2$ so that $\varphi_2(h_2(a)) = \varphi_1(h_1(b))$ where $b$ is the least indecomposable in $|\mathbf{P}_1|$ such that $a \preceq^{\mathbf{P}_2} b$. Since $\mathbf{P}_2|\mathbf{P}_3$ is cofinally valid, there is a covering $h_3$ of $\mathbf{P}_3$ in $\mathcal{R}_2$ extending $h_2$ above $\varphi_2$. Clearly, $h_3$ extends $h_1$ above $\varphi_1$. **QED**

The following example shows that under the assumptions of the lemma, we cannot generally conclude that if $\mathbf{P}_1|\mathbf{P}_3$ is cofinally valid then $\mathbf{P}_2|\mathbf{P}_3$ is cofinally valid, or even valid.

**Example 6.5** Assume $\mathcal{R}_0$ is the EM structure on the ordinals without functions or constants and whose only relation is the usual ordering $\leq$ on the ordinals. Let $\mathbf{P}_3$ be a pattern with universe $\{a, b, c, d, e\}$ where $a \preceq^{\mathbf{P}_3} b \preceq^{\mathbf{P}_3} c \preceq^{\mathbf{P}_3} d \preceq^{\mathbf{P}_3} e$, $\preceq_1^{\mathbf{P}_3}$ is $\{(a, b), (c, d), (c, e)\}$, and $\preceq_2^{\mathbf{P}_3}$ is empty. Let $\mathbf{P}_1$ be the substructure of $\mathbf{P}_3$ with universe $\{c, d, e\}$ and let $\mathbf{P}_2$ be the substructure of $\mathbf{P}_3$ with universe $\{a, c, d, e\}$.

# 7 Continuity

In this section, we introduce a key property called *continuity* which all valid rules have.

Notice that if $\mathbf{P}$ is a closed substructure of a pattern $\mathbf{P}^+$, $x \in |\mathbf{P}^+|$ and $p \in |\mathbf{P}|$ then $(-\infty, p)^{\mathbf{P}} \prec^{\mathbf{P}^+} x \prec^{\mathbf{P}^+} p$ is equivalent to $p$ being the least element of $|\mathbf{P}|$ such that $x \preceq^{\mathbf{P}^+} p$.

**Definition 7.1** Assume $\mathbf{P}^+$ is a well ordered structure for $\mathcal{L}_2$ and $\mathbf{P}$ is a substructure of $\mathbf{P}^+$. For $k = 1, 2$, $\mathbf{P}^+$ is a *$k$-correct extension of* $\mathbf{P}$ if for any $p \in |\mathbf{P}|$ and any $x, y \in |\mathbf{P}^+|$ with $(-\infty, p)^{\mathbf{P}} \prec^{\mathbf{P}^+} x \preceq^{\mathbf{P}^+} p \prec^{\mathbf{P}^+} y$ and $x \preceq_k^{\mathbf{P}^+} y$ there exists $a \in |\mathbf{P}|$ such that $y \preceq^{\mathbf{P}^+} a$ and if $p'$ is the least $a \in |\mathbf{P}|$ with $y \preceq^{\mathbf{P}^+} a$ then $p \preceq_k^{\mathbf{P}} p'$. $\mathbf{P}$ is *$k$-correct in* $\mathbf{P}^+$ if $\mathbf{P}^+$ is a $k$-correct extension of $\mathbf{P}$. If $h$ is an embedding of $\mathbf{P}$ into $\mathbf{P}^+$ then $h$ is a *$k$-correct embedding of* $\mathbf{P}$ *into* $\mathbf{P}^+$ if $\mathbf{P}^+$ is a $k$-correct extension of the substructure whose universe is the range of $h$.

Generally, we will be interested in whether $\mathbf{P}^+$ is a $k$-correct extension of $\mathbf{P}$ only in the case when both are finite structures. The only other case of interest is $k$-correctness when $\mathbf{P}$ is an initial substructure of $\mathbf{P}^+$ and both are models of reflection of order two. In this case, $\mathbf{P}^+$ being a $k$-correct extension of $\mathbf{P}$ is equivalent to

$$x \not\preceq_k^{\mathbf{P}^+} y \quad \text{for all} \quad x \in |\mathbf{P}| \quad \text{and} \quad y \in |\mathbf{P}^+| \setminus |\mathbf{P}|$$

for both $k = 1, 2$.

**Lemma 7.2** *Assume that* $\mathbf{P}^+$ *is a pattern,* $\mathbf{P}$ *is a closed substructure of* $\mathbf{P}^+$*,* $k \in \{1, 2\}$*,* $\mathbf{P}^+$ *is a $k$-correct extension of* $\mathbf{P}$*,* $x, y \in |\mathbf{P}^+|$*,* $p, p' \in |\mathbf{P}|$*,* $x \preceq^{\mathbf{P}^+} p \preceq^{\mathbf{P}^+} y \preceq^{\mathbf{P}^+} p'$*,* $p$ *is the least element of* $|\mathbf{P}|$ *such that* $x \preceq^{\mathbf{P}^+} p$*, and* $p'$ *is the least element of* $|\mathbf{P}|$ *such that* $y \preceq^{\mathbf{P}^+} p'$*. If* $x \preceq_k^{\mathbf{P}^+} y$ *then* $x \preceq^{\mathbf{P}^+} p \preceq^{\mathbf{P}^+} p'$*.*

**Proof.** Assume $x \preceq_k^{\mathbf{P}^+} y$. By definition, $p \preceq_k^{\mathbf{P}} p'$. This implies that $p \preceq_{k-1}^{\mathbf{P}^+} y$. Therefore, $x \preceq_k^{\mathbf{P}^+} p$. **QED**

**Lemma 7.3** *Assume* $\mathbf{P}_1$*,* $\mathbf{P}_2$ *and* $\mathbf{P}_3$ *are linearly ordered structures for* $\mathcal{L}_2$ *such that* $\mathbf{P}_1$ *is a substructure of* $\mathbf{P}_2$ *and* $\mathbf{P}_2$ *is a substructure of* $\mathbf{P}_3$*.*

1. *If* $\mathbf{P}_3$ *is a $k$-correct extension of* $\mathbf{P}_2$ *and* $\mathbf{P}_2$ *is a $k$-correct extension of* $\mathbf{P}_1$ *then* $\mathbf{P}_3$ *is a $k$-correct extension of* $\mathbf{P}_1$*.*

2. *If $\mathbf{P}_3$ is a $k$-correct extension of $\mathbf{P}_1$ then $\mathbf{P}_2$ is a $k$-correct extension of $\mathbf{P}_1$.*

**Proof.** Straightforward. **QED**

**Definition 7.4** Assume $\mathbf{P}^+$ and $\mathbf{P}$ are linearly ordered structures for $\mathcal{L}_2$ and $\mathbf{P}$ is a substructure of $\mathbf{P}^+$. $\mathbf{P}^+$ is a *continuous extension of* $\mathbf{P}$ if the following conditions hold.

1. Every indecomposable of $\mathbf{P}^+$ is bounded above by an indecomposable of $\mathbf{P}$.

2. For $k = 1, 2$, $\mathbf{P}^+$ is a $k$-correct extension of $\mathbf{P}$.

$\mathbf{P}$ is *continuous in* $\mathbf{P}^+$ if $\mathbf{P}^+$ is a continuous extension of $\mathbf{P}$. If $h$ is an embedding of $\mathbf{P}$ into $\mathbf{P}^+$ then $h$ is a *continuous embedding of* $\mathbf{P}$ *into* $\mathbf{P}^+$ if $\mathbf{P}^+$ is a continuous extension of the substructure whose universe is the range of $h$.

**Lemma 7.5** *Assume $\mathbf{P}^+$ is a continuous extension of $\mathbf{P}$. If $x, y \in |\mathbf{P}^+|$ and $x \prec_k^{\mathbf{P}^+} y$ for either $k = 1$ or $k = 2$ then there is an element $p$ of $|\mathbf{P}|$ such that $y \preceq^{\mathbf{P}^+} p$.*

**Proof.** Assume $x, y \in |\mathbf{P}^+|$ and $x \prec_k^{\mathbf{P}^+} y$ where $k = 1$ or $k = 2$. Since $x \neq y$, $x$ is indecomposable. Therefore, there is an element $a$ of $|\mathbf{P}|$ such that $x \preceq^{\mathbf{P}^+} a$. Choosing $a$ to be minimal, the existence of $p$ follows immediately from the definition of $k$-correctness. **QED**

We remark that if $\mathbf{P}|\mathbf{P}^+$ is valid and there is an embedding of $\mathbf{P}$ in $\mathcal{R}_2$ then $\mathbf{P}^+$ is a continuous extension of $\mathbf{P}$. This will follow from results to be established later.

**Lemma 7.6** *Assume $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ are linearly ordered structures for $\mathcal{L}_2$ such that $\mathbf{P}_1$ is a substructure of $\mathbf{P}_2$ and $\mathbf{P}_2$ is a substructure of $\mathbf{P}_3$.*

1. *If $\mathbf{P}_3$ is a continuous extension of $\mathbf{P}_2$ and $\mathbf{P}_2$ is a continuous extension of $\mathbf{P}_1$ then $\mathbf{P}_3$ is a continuous extension of $\mathbf{P}_1$.*

2. *If $\mathbf{P}_3$ is a continuous extension of $\mathbf{P}_1$ then $\mathbf{P}_2$ is a continuous extension of $\mathbf{P}_1$.*

**Proof.** By Lemma 7.3. **QED**

**Definition 7.7** Assume $\mathbf{P}$ and $\mathbf{P}^+$ are patterns and $\mathbf{P}$ is a closed substructure of $\mathbf{P}^+$. $\mathbf{P}|\mathbf{P}^+$ is a *continuous rule* provided $\mathbf{P}^+$ is a continuous extension of $\mathbf{P}$.

**Lemma 7.8** *Assume $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ are patterns such that $\mathbf{P}_1$ is a substructure of $\mathbf{P}_2$ and $\mathbf{P}_2$ is a substructure of $\mathbf{P}_3$.*

1. *If $\mathbf{P}_1|\mathbf{P}_2$ and $\mathbf{P}_2|\mathbf{P}_3$ are continuous rules then so is $\mathbf{P}_1|\mathbf{P}_3$.*
2. *If $\mathbf{P}_1|\mathbf{P}_3$ is a continuous rule then so is $\mathbf{P}_1|\mathbf{P}_2$.*

**Proof.** By Lemma 7.6 and part 2 of Lemma 2.4. **QED**

# 8 Arithmetic and Transcendental Extensions

**Definition 8.1** Assume $\mathbf{P}$ is a substructure of $\mathbf{P}^+$. $\mathbf{P}^+$ is an *arithmetic extension of* $\mathbf{P}$ if the arithmetic part of $\mathbf{P}^+$ is an arithmetic extension of the arithmetic part of $\mathbf{P}$.

**Lemma 8.2** *Assume $\mathbf{P}$ is a pattern, $\mathbf{A}$ is the arithmetic part of $\mathbf{P}$ and $\mathbf{A}^+$ is an arithmetic extension of $\mathbf{A}$. There is a unique arithmetic extension $\mathbf{P}^+$ of $\mathbf{P}$ such that $\mathbf{P}^+$ has arithmetic part $\mathbf{A}^+$ and $\mathbf{P}^+$ is a 1-correct extension of $\mathbf{P}$.*

**Proof.** Define $\mathbf{P}^+$ to be the $\mathcal{L}_2$ structure with arithmetic part $\mathbf{A}^+$ such that for all $x, y \in |\mathbf{P}^+|$

$$x \preceq_1^{\mathbf{P}^+} y \quad \text{iff} \quad \begin{array}{l} x = y \text{ or } x \in |\mathbf{P}| \text{ and there exists } a \in |\mathbf{P}| \text{ such that} \\ \qquad\qquad x \preceq^{\mathbf{P}^+} y \preceq^{\mathbf{P}^+} a \text{ and } x \preceq_1^{\mathbf{P}} a \end{array}$$

and $\preceq_2^{\mathbf{P}^+}$ is the same as $\preceq_2^{\mathbf{P}}$. Since Lemma 4.9 implies that any indecomposable in $\mathbf{A}^+$ is in $|\mathbf{A}|$. It is routine to check that $\mathbf{P}^+$ is the unique 1-correct extension of $\mathbf{P}$ with arithmetic part $\mathbf{A}^+$. **QED**

**Lemma 8.3** *If $\mathbf{P}^+$ is a 1-correct arithmetic extension of $\mathbf{P}$ then $\mathbf{P}^+$ is a continuous extension of $\mathbf{P}$.*

**Proof.** Trivial. **QED**

**Lemma 8.4** *Assume* $\mathbf{P}^+$ *is a 1-correct arithmetic extension of* $\mathbf{P}$*. The rule* $\mathbf{P}|\mathbf{P}^+$ *is cofinally valid.*

**Proof.** By Lemma 4.9, there are no new indecomposable elements in $\mathbf{P}^+$. Therefore, it suffices to show that $\mathbf{P}|\mathbf{P}^+$ is valid.

Assume $h$ is a covering of $\mathbf{P}$ in $\mathcal{R}_0$. By Lemma 4.5, there is a closed embedding $h^+$ of the arithmetic part of $\mathbf{P}^+$ into $\mathcal{R}_0$ which extends $h$. We claim that $h^+$ is a covering of $\mathbf{P}^+$ into $\mathcal{R}_2$.

Suppose $x \preceq_1^{\mathbf{P}^+} y$. We will show that $h^+(x) \leq_1 h^+(y)$. If $x = y$ this is immediate, so we may assume that $x \prec^{\mathbf{P}^+} y$. This implies $x$ is indecomposable in $\mathbf{P}^+$ which implies that $x \in |\mathbf{P}|$. Since $\mathbf{P}^+$ is a 1-correct extension of $\mathbf{P}$, there exists $z \in |\mathbf{P}|$ such that $y \preceq^{\mathbf{P}^+} z$ and $x \preceq_1^{\mathbf{P}} z$. Hence, $h^+(x) \leq h^+(y) \leq h^+(z)$ and, since $h^+$ extends $h$, $h^+(x) \leq_1 h^+(z)$. Therefore, $h^+(x) \leq_1 h^+(y)$.

Now suppose $x \preceq_2^{\mathbf{P}^+} y$. We will show that $h^+(x) \leq_2 h^+(y)$. If $x = y$ this is immediate, so we may assume that $x \prec^{\mathbf{P}^+} y$. This implies $x$ and $y$ are indecomposable in $\mathbf{P}^+$. Hence, $x, y \in |\mathbf{P}|$. Since $h^+$ extends $h$, $h^+(x) \leq_2 h^+(y)$. **QED**

**Lemma 8.5** *Assume* $\mathbf{P}$ *is a pattern. If* $t$ *is a term in the language of* $\mathcal{R}_0$ *whose variables are among* $v_1, \ldots, v_n$ *and* $a_1, \ldots, a_n \in |\mathbf{P}|$ *then there is a 1-correct arithmetic extension* $\mathbf{P}^+$ *of* $\mathbf{P}$ *such that* $t(a_1, \ldots, a_n)^{\mathbf{P}^+}$ *is defined.*

**Proof.** By Lemmas 4.12 and 8.2. **QED**

**Definition 8.6** Assume $\mathbf{P}$ and $\mathbf{P}^+$ are patterns and $\mathbf{P}$ is a closed substructure of $\mathbf{P}^+$. $\mathbf{P}^+$ is a *transcendental extension of* $\mathbf{P}$ if the arithmetic part of $\mathbf{P}^+$ is a transcendental extension of the arithmetic part of $\mathbf{P}$. For $a$ an indecomposable of $\mathbf{P}$, $\mathbf{P}^+$ is an *extension of* $\mathbf{P}$ *at* $a$ if $\mathbf{P}^+$ is a transcendental extension of $\mathbf{P}$, $|\mathbf{P}^+| \setminus |\mathbf{P}|$ is nonempty and $(-\infty, a)^{\mathbf{P}} <^{\mathbf{P}^+} x <^{\mathbf{P}^+} a$ whenever $x \in |\mathbf{P}^+| \setminus |\mathbf{P}|$.

# 9 Downward Reflection

**Definition 9.1** Assume $\mathbf{P}$ is a pattern, $a \prec_1^{\mathbf{P}} b$, $X \subseteq [a, b)^{\mathbf{P}}$ is nonempty and $(-\infty, a)^{\mathbf{P}} \cup X$ is a closed subset of $\mathbf{P}$. A structure $\mathbf{P}^+$ is *obtained from* $\mathbf{P}$ *by* 1*-reflecting* $X$ *downward from* $b$ *to* $a$ provided $\mathbf{P}$ is a substructure of $\mathbf{P}^+$ and, letting $X^+$ be $|\mathbf{P}|^+ \setminus |\mathbf{P}|$, the following conditions hold.

1. The arithmetic part of $\mathbf{P}^+$ is an arithmetic structure.

2. $\mathbf{P}$ is a closed substructure of $\mathbf{P}^+$.

3. $(-\infty, a)^{\mathbf{P}} \prec^{\mathbf{P}^+} X^+ \prec^{\mathbf{P}^+} a$

4. The substructure of $\mathbf{P}$ with universe $(-\infty, a)^{\mathbf{P}} \cup X$ and the substructure of $\mathbf{P}^+$ with universe $(-\infty, a)^{\mathbf{P}} \cup X^+$ are isomorphic.

5. If $\tilde{x} \in X^+$, $a \preceq^{\mathbf{P}^+} y$ then

$$\tilde{x} \not\preceq_1^{\mathbf{P}^+} y \text{ and } \tilde{x} \not\preceq_2^{\mathbf{P}^+} y$$

**Lemma 9.2** *Assume* $\mathbf{P}$ *is a pattern,* $a \prec_1^{\mathbf{P}} b$*,* $X \subseteq [a, b)^{\mathbf{P}}$ *is nonempty and* $(-\infty, a)^{\mathbf{P}} \cup X$ *is the universe of a closed substructure of* $\mathbf{P}$*.*

1. *The family of structures which are obtained from* $\mathbf{P}$ *by* 1*-reflecting* $X$ *downward from* $b$ *to* $a$ *is nonempty and unique up to isomorphism over* $\mathbf{P}$*.*

2. *If* $\mathbf{P}^+$ *is obtained from* $\mathbf{P}$ *by* 1*-reflecting* $X$ *downward from* $b$ *to* $a$ *then* $\mathbf{P}^+$ *is a continuous extension of* $\mathbf{P}$ *at* $a$*.*

**Proof.** Let $\mathbf{A}$ be the arithmetic part of $\mathbf{P}$ and let $\mathcal{A}^+$ be the collection of all arithmetic structures $\mathbf{A}^+$ such that, setting $X^+ = |\mathbf{A}^+| \setminus |\mathbf{A}|$,

1. $\mathbf{A}$ is a closed substructure of $\mathbf{A}^+$.

2. $(-\infty, a)^{\mathbf{A}} \prec^{\mathbf{A}} X^+ \prec^{\mathbf{A}} a$

3. The substructure of $\mathbf{A}^+$ with universe $(-\infty, a)^{\mathbf{A}} \cup X^+$ is isomorphic to the substructure of $\mathbf{A}$ with universe $(-\infty, a)^{\mathbf{A}} \cup X$.

By Lemma 4.16, the family $\mathcal{A}^+$ is nonempty and unique up to isomorphism over $\mathbf{A}$. Clearly, for $\mathbf{A}^+ \in \mathcal{A}^+$, there is a unique structure $\mathbf{P}^+$ with arithmetic part $\mathbf{A}^+$ which is obtained from $\mathbf{P}$ by 1-reflecting $X$ downward from $b$ to $a$. The uniqueness of the family of structures which are obtained from $\mathbf{P}$ by 1-reflecting $X$ downward from $b$ to $a$ up to isomorphism over $\mathbf{P}$ follows.

For part 2, assume $\mathbf{P}^+$ is obtained from $\mathbf{P}$ by 1-reflecting $X$ downward from $b$ to $a$. Let $X^+$ be $|\mathbf{P}^+| \setminus |\mathbf{P}|$ and let $h$ be the isomorphism of the substructure of $\mathbf{P}^+$ with universe $(-\infty, a)^{\mathbf{P}} \cup X^+$ and the substructure of $\mathbf{P}$ with universe $(-\infty, a)^{\mathbf{P}} \cup X$. Notice that $(-\infty, a)^{\mathbf{P}} \cup X^+$ is a closed subset of $\mathbf{P}^+$ since it is the initial substructure with universe $(-\infty, a)^{\mathbf{P}^+}$.

We begin by showing that the continuity conditions 1 and 2 of Definition 7.4 hold.

To establish condition 1, notice that $[a, \infty)^{\mathbf{P}^+} = [a, \infty)^{\mathbf{P}}$ and $a$ is indecomposable in $\mathbf{P}^+$. This implies that the largest indecomposable of $\mathbf{P}^+$ is in $\mathbf{P}$.

For condition 2, assume $k \in \{1, 2\}$. We must show that $\mathbf{P}^+$ is a $k$-correct extension of $\mathbf{P}$. Suppose

$$(-\infty, c)^{\mathbf{P}^+} \prec^{\mathbf{P}^+} x \preceq^{\mathbf{P}^+} c \prec^{\mathbf{P}^+} y$$

and $x \preceq_k^{\mathbf{P}^+} y$ where $c \in |\mathbf{P}|$. Let $d$ be the least element of $|\mathbf{P}|$ such that $y \preceq^{\mathbf{P}^+} d$. We must show $c \preceq_k^{\mathbf{P}} d$.

If $x, y \in |\mathbf{P}|$ then $c = x$ and $d = y$. So we may assume either $x \notin |\mathbf{P}|$ or $y \notin |\mathbf{P}|$. In the first case, $x \in X^+$, $c = a$ and $a \prec^{\mathbf{P}} y$. Since $x \preceq_k^{\mathbf{P}} y$, this contradicts the definition of $\mathbf{P}^+$. So we may assume $y \notin |\mathbf{P}|$. In this case, $y \in X^+$, $d = a$ and $c < a$ which implies $x = c$. So, we must show that $x \preceq_k^{\mathbf{P}} a$. Since $x \preceq_k^{\mathbf{P}^+} y$, $x \preceq_k^{\mathbf{P}} h(y)$. When $k = 1$, this implies $x \preceq_1^{\mathbf{P}} a$ since $x \preceq^{\mathbf{P}} a \preceq^{\mathbf{P}} h(y)$. Now assume $k = 2$. Notice that $a \preceq^{\mathbf{P}} h(y) \preceq^{\mathbf{P}} b$ implying $a \preceq_1^{\mathbf{P}} h(y)$. Combined with $x \preceq_2^{\mathbf{P}} h(y)$, this implies that $x \preceq_2^{\mathbf{P}} a$.

To show $\mathbf{P}^+$ is a pattern, we will show that conditions 1-7 of Definition 5.6 hold.

Condition 1 is part 1 of Definition 9.1.

To verify conditions 2 and 3, assume $u, v \in \mathbf{P}^+$. We must verify

$$u \prec_1^{\mathbf{P}^+} v \implies u \text{ is indecomposable}$$

and

$$u \prec_2^{\mathbf{P}^+} v \implies v \text{ is indecomposable}$$

Since the restrictions of $\mathbf{P}^+$ to $(-\infty, a)^{\mathbf{P}} \cup X^+$ and $|\mathbf{P}|$ are both patterns, we may assume that $\{u, v\}$ is not a subset of either set. Under this assumption we must have $u \in X^+$ and $a \preceq^{\mathbf{P}^+} v$. By condition 5 of Definition 9.1, $u \not\preceq_1^{\mathbf{P}^+} v$ and $u \not\preceq_2^{\mathbf{P}^+} v$. Therefore, both implications above are vacuously true.

To verify conditions 4-7, first notice that $\preceq_1^{\mathbf{P}^+}$ is contained in $\preceq^{\mathbf{P}^+}$ and $\preceq_2^{\mathbf{P}^+}$ is contained in $\preceq_1^{\mathbf{P}^+}$. Now assume that $u \preceq^{\mathbf{P}^+} v \preceq^{\mathbf{P}^+} w$. We will show that

$$u \preceq_1^{\mathbf{P}^+} v \ \preceq_1^{\mathbf{P}^+} w \implies u \preceq_1^{\mathbf{P}^+} w$$

$$u \preceq_2^{\mathbf{P}^+} v \preceq_2^{\mathbf{P}^+} w \implies u \preceq_2^{\mathbf{P}^+} w$$

$$u \preceq_1^{\mathbf{P}^+} w \implies u \preceq_1^{\mathbf{P}^+} v$$

$$u \preceq_2^{\mathbf{P}^+} w \text{ and } v \preceq_1^{\mathbf{P}^+} w \implies u \preceq_2^{\mathbf{P}^+} v$$

Since the restrictions of $\mathbf{P}^+$ to $(-\infty, a)^{\mathbf{P}} \cup X^+$ and $|\mathbf{P}|$ are both patterns, we may assume, as in the proof of conditions 2 and 3, that $\{u, v, w\}$ is not a subset of either set. Therefore, $\{u, v, w\}$ intersects both $X^+$ and $[a, \infty)^{\mathbf{P}}$. Hence, $a \preceq^{\mathbf{P}^+} w$ and either $u \in X^+$ or $v \in X^+$.

By condition 5 of Definition 9.1, the only nontrivial part of the conditions to be verified which are listed above is

$$u \preceq_1^{\mathbf{P}^+} w \implies u \preceq_1^{\mathbf{P}^+} v$$

in the case when $u \in (-\infty, a)^{\mathbf{P}}$ and $v \in X^+$ (recall we also have $a \preceq^{\mathbf{P}} w$).

Suppose $u \in (-\infty, a)^{\mathbf{P}}$ and $v \in X^+$ as well as $u \preceq_1^{\mathbf{P}^+} w$. Since $a \preceq w$, this implies that $u \preceq_1 a$. Since $a \preceq h(v) \prec b$ and $a \preceq_1 b$, $a \preceq_1 h(v)$. Therefore, $u \preceq_1 h(v)$. Since $h$ is an isomorphism of $(-\infty, a)^{\mathbf{P}} \cup X$ and $(-\infty, a)^{\mathbf{P}} \cup X^+$, $u \preceq_1 v$.

Lemma 4.14 implies that the least element of $X$ is indecomposable in $\mathbf{P}$. By part 4 of Definition 9.1, the least element of $X^+$ is indecomposable in $(-\infty, a) \cup X^+$ implying it is indecomposable in $\mathbf{P}^+$. By Lemma 4.15, $\mathbf{P}^+$ is a transcendental extension of $\mathbf{P}$. **QED**

Assume $\mathbf{P}^+$ is obtained from $\mathbf{P}$ by 1-reflecting $X$ downward from $b$ to $a$. Let $\mathbf{A}^+$ be the arithmetic part of $\mathbf{P}^+$ and let $\mathbf{Q}$ be the substructure of $\mathbf{P}^+$ with universe $(-\infty, a) \cup X^+$ (which equals $(-\infty, a)^{\mathbf{P}^+}$). By the lemma, $\mathbf{P}^+$ is a pattern. By Definition 9.1, this easily implies that $\mathbf{P}^+$ is the pattern generated by $\mathbf{P}$ and $\mathbf{Q}$ over $\mathbf{A}^+$.

**Lemma 9.3** *If $\mathbf{P}^+$ is obtained from $\mathbf{P}$ by 1-reflecting $Y$ downward from $b$ to $a$ then the rule $\mathbf{P}|\mathbf{P}^+$ is a cofinally valid continuous rule.*

**Proof.** Assume $\mathbf{P}^+$ is obtained from $\mathbf{P}$ by 1-reflecting $Y$ downward from $b$ to $a$. Let $\tilde{Y} = |\mathbf{P}^+| \setminus |\mathbf{P}|$ and let $g$ be an isomorphism of $(-\infty, a)^{\mathbf{P}} \cup \tilde{Y}$ and $(-\infty, a)^{\mathbf{P}} \cup Y$.

$\mathbf{P}|\mathbf{P}^+$ is a continuous rule by part 2 of Lemma 9.2.

Suppose $h$ is a covering of $\mathbf{P}$ in $\mathcal{R}_2$ and $\varphi$ is a regressive function on the nonzero indecomposables in $h[|\mathbf{P}|]$. By Lemma 2.5, there is a finite closed subset $X$ of $h(a)$ such that $\varphi(h(a)) \in X$ and $h[(-\infty, a)^{\mathbf{P}}] \subseteq X$. Since $a \preceq_1^{\mathbf{P}} b$ and $h$ is a covering, $h(a) \leq_1 h(b)$. Therefore, there is a finite set $\tilde{Y}^* \subseteq h(a)$ such that $X < \tilde{Y}^*$, $X \cup \tilde{Y}^*$ is a closed subset of $\mathcal{R}_2$ and there is a covering $f$ of $X \cup h[Y]$ onto $X \cup \tilde{Y}^*$. Notice that $f \circ h \circ g$ is a covering of $(-\infty, a)^{\mathbf{P}} \cup \tilde{Y}$ onto $(-\infty, a)^{\mathbf{P}} \cup \tilde{Y}^*$ which agrees with $h$ on $(-\infty, a)^{\mathbf{P}}$ and maps $\tilde{Y}$ onto $\tilde{Y}^*$.

Extend $h$ to a function $h^+$ from $|\mathbf{P}^+|$ into the ordinals so that $h^+(\tilde{y}) = f(h(g(\tilde{y})))$ for $\tilde{y} \in \tilde{Y}$. Since $f \circ h \circ g$ agrees with $h$ on $(-\infty, a)^{\mathbf{P}}$, $h^+$ extends $f \circ h \circ g$.

By Lemma 4.5, there is a closed embedding $h^*$ of the arithmetic part of $\mathbf{P}^+$ into $\mathcal{R}_0$ which agrees with $h^+$ on the indecomposables of $\mathbf{P}^+$. By Lemma 4.4, $h^*$ must extend $h$ and $f \circ h \circ g$. Therefore, $h^* = h^+$ implying $h^+$ is a closed embedding of the arithmetic part of $\mathbf{P}^+$ into $\mathcal{R}_0$.

To see that $h^+$ is a covering, suppose $u, v \in |\mathbf{P}^+|$ and $u \preceq_k^{\mathbf{P}^+} v$ where $k \in \{1, 2\}$. We need to show $h^+(u) \leq_k h^+(v)$. By Definition 9.1, the case that $u \in \tilde{Y}$ and $v \in [a, \infty)^{\mathbf{P}}$ is impossible. Hence, either $u, v \in |\mathbf{P}|$ or $u, v \in (-\infty, a)^{\mathbf{P}} \cup \tilde{Y}$. In the first case, $h^+(u) \leq_k h^+(v)$ since $h^+$ extends $h$ and $h$ is a covering of $\mathbf{P}$ in $\mathcal{R}_2$. In the second case, $h^+(u) \leq_k h^+(v)$ since $h^+$ extends $f \circ h \circ g$ which is a covering of $(-\infty, a)^{\mathbf{P}} \cup \tilde{Y}$ into $\mathcal{R}_2$.

Since $(-\infty, a)^{\mathbf{P}} \prec^{\mathbf{P}^+} \tilde{Y} \prec^{\mathbf{P}^+} a$, to see that $h^+$ extends $h$ above $\varphi$ it suffices to show that $\varphi(h(a)) < h^+[\tilde{Y}]$. This follows from the fact that $h^+$ maps $\tilde{Y}$ onto $\tilde{Y}^*$ and $\varphi(h(a)) \in X < \tilde{Y}^*$. **QED**

**Definition 9.4** Assume $\mathbf{P}$ is a pattern, $a \prec_2^{\mathbf{P}} b$, $X \subseteq [a, b)^{\mathbf{P}}$ is nonempty and $(-\infty, a)^{\mathbf{P}} \cup X$ is the universe of a closed substructure of $\mathbf{P}$. A structure $\mathbf{P}^+$ is *obtained from* $\mathbf{P}$ *by* 2*-reflecting* $X$ *downward from* $b$ *to* $a$ provided that $\mathbf{P}$ is a substructure of $\mathbf{P}^+$ and, letting $X^+$ be $|\mathbf{P}^+| \setminus |\mathbf{P}|$, the following conditions hold.

1. The arithmetic part of $\mathbf{P}^+$ is an arithmetic structure.

2. $\mathbf{P}$ is a closed substructure of $\mathbf{P}^+$.

3. $(-\infty, a)^{\mathbf{P}} \prec^{\mathbf{P}^+} X^+ \prec^{\mathbf{P}^+} a$

4. The substructure of $\mathbf{P}$ with universe $(-\infty, a)^{\mathbf{P}} \cup X$ is isomorphic to the substructure of $\mathbf{P}^+$ with universe $(-\infty, a)^{\mathbf{P}} \cup X^+$.

5. If $\tilde{x} \in X^+$, $a \preceq^{\mathbf{P}^+} y$ then

$$\tilde{x} \preceq_1^{\mathbf{P}^+} y \quad \text{iff} \quad x \preceq_1^{\mathbf{P}} b \text{ and } a \preceq_1^{\mathbf{P}} y$$

where $x$ corresponds to $\tilde{x}$ under the isomorphism of the substructures of $\mathbf{P}^+$ with universes $(-\infty, a)^{\mathbf{P}} \cup X$ and $(-\infty, a)^{\mathbf{P}} \cup X^+$ respectively and

$$\tilde{x} \not\preceq_2^{\mathbf{P}^+} y$$

**Lemma 9.5** *Assume* $\mathbf{P}$ *is a pattern,* $a \prec_2^{\mathbf{P}} b$*,* $X \subseteq [a, b)^{\mathbf{P}}$ *is nonempty and* $(-\infty, a)^{\mathbf{P}} \cup X$ *is the universe of a closed substructure of* $\mathbf{P}$*.*

1. *The family of structures which are obtained from* $\mathbf{P}$ *by* 2*-reflecting* $X$ *downward from* $b$ *to* $a$ *is nonempty and unique up to isomorphism over* $\mathbf{P}$*.*

2. *If* $\mathbf{P}^+$ *is obtained from* $\mathbf{P}$ *by* 2*-reflecting* $X$ *downward from* $b$ *to* $a$ *then* $\mathbf{P}^+$ *is a continuous extension of* $\mathbf{P}$ *at* $a$*.*

**Proof.** The proof of part 1 is analogous to the proof of part 1 for Lemma 9.2.

For part 2, assume $\mathbf{P}^+$ is obtained from $\mathbf{P}$ by 2-reflecting $X$ downward from $b$ to $a$. Let $X^+$ be $|\mathbf{P}^+| \setminus |\mathbf{P}|$ and let $h$ be the isomorphism of the substructure of $\mathbf{P}^+$ with universe $(-\infty, a)^{\mathbf{P}} \cup X^+$ and the substructure of $\mathbf{P}$ with universe $(-\infty, a)^{\mathbf{P}} \cup X$. Notice $(-\infty, a)^{\mathbf{P}} \cup X^+$ is a closed subset of $\mathbf{P}^+$ since it is an initial segment of $\mathbf{P}^+$.

We begin by showing that the continuity conditions 1 and 2 of Definition 7.4 hold. The argument is very close to the corresponding part of the proof of Lemma 9.2.

To establish condition 1, notice that $[a, \infty)^{\mathbf{P}^+} = [a, \infty)^{\mathbf{P}}$ and $a$ is indecomposable in $\mathbf{P}^+$. This implies that the largest indecomposable of $\mathbf{P}^+$ is in $\mathbf{P}$.

For condition 2, assume $k \in \{1, 2\}$. We must show that $\mathbf{P}^+$ is a $k$-correct extension of $\mathbf{P}$. Suppose

$$(-\infty, c)^{\mathbf{P}} \prec^{\mathbf{P}^+} x \preceq^{\mathbf{P}^+} c \prec^{\mathbf{P}^+} y$$

and $x \preceq_k^{\mathbf{P}^+} y$ where $c \in |\mathbf{P}|$. Let $d$ be the least element of $|\mathbf{P}|$ such that $y \preceq^{\mathbf{P}^+} d$. We must show $c \preceq_k^{\mathbf{P}} d$.

If $x, y \in |\mathbf{P}|$ then $c = x$ and $d = y$. So we may assume either $x \notin |\mathbf{P}|$ or $y \notin |\mathbf{P}|$.

First suppose $x \notin |\mathbf{P}|$. In this case, $x \in X^+$, $c = a$, $c \prec^{\mathbf{P}} y$ and $d = y$. So, we must show that $a \preceq_k^{\mathbf{P}} y$. Since $x \preceq_k^{\mathbf{P}} y$, we must have $k = 1$ by the definition of $\mathbf{P}^+$. By the definition of $\mathbf{P}^+$, $a \preceq_1^{\mathbf{P}} y$.

Now assume $y \notin |\mathbf{P}|$. In this case, $y \in X^+$, $d = a$ and $c \prec^{\mathbf{P}} a$ which implies $x = c$. So, we must show that $x \preceq_k^{\mathbf{P}} a$. Since $x \preceq_k^{\mathbf{P}^+} y$, $c \preceq_k^{\mathbf{P}} h(y)$. When $k = 1$, this implies $x \preceq_1^{\mathbf{P}} a$ since $x \preceq^{\mathbf{P}} a \preceq^{\mathbf{P}} h(y)$. Now assume $k = 2$. Notice that $a \preceq^{\mathbf{P}} h(y) \preceq^{\mathbf{P}} b$ implying $a \preceq_1^{\mathbf{P}} h(y)$. Combined with $x \preceq_2^{\mathbf{P}} h(y)$, this implies that $x \preceq_2^{\mathbf{P}} a$.

To establish that $\mathbf{P}^+$ is a pattern, we will show that conditions 1-7 of Definition 5.6 hold.

Condition 1 is part 1 of Definition 9.4.

To verify conditions 2 and 3, assume $u, v \in \mathbf{P}^+$. We will show

$$u \prec_1^{\mathbf{P}^+} v \implies u \text{ is indecomposable in } \mathbf{P}^+$$

and

$$u \prec_2^{\mathbf{P}^+} v \implies u \text{ and } v \text{ are indecomposable in } \mathbf{P}^+$$

Since the restrictions of $\mathbf{P}^+$ to $(-\infty, a)^{\mathbf{P}} \cup X^+$ and $|\mathbf{P}|$ are both patterns, we may assume that $\{u, v\}$ is not a subset of either set. Under this assumption we must have $u \in X^+$ and $a \preceq^{\mathbf{P}^+} v$. Since $u \in X^+$, we cannot have $u \preceq_2^{\mathbf{P}^+} v$ by part 5 of Definition 9.4. Therefore, the second implication above is vacuously true.

For the first implication, assume that $u \prec_1^{\mathbf{P}^+} v$. This implies that $h(u) \prec_1^{\mathbf{P}} b$. Therefore, $h(u)$ is indecomposable in $\mathbf{P}$. Since $(-\infty, a)^{\mathbf{P}} \cup X$ is a closed subset of $\mathbf{P}$, $h(u)$ is indecomposable in the substructure of $\mathbf{P}^+$ with universe $(-\infty, a)^{\mathbf{P}} \cup X$. Since $h$ is an isomorphism, $u$ is indecomposable in the substructure of $\mathbf{P}^+$ with universe $(-\infty, a)^{\mathbf{P}} \cup X^+$. Since $(-\infty, a)^{\mathbf{P}} \cup X^+$ is a closed subset of $\mathbf{P}^+$, $u$ is indecomposable in $\mathbf{P}^+$.

To verify conditions 4-7, first notice that $\preceq_1^{\mathbf{P}^+}$ is contained in $\preceq^{\mathbf{P}^+}$ and $\preceq_2^{\mathbf{P}^+}$ is contained in $\preceq_1^{\mathbf{P}^+}$. Now assume $u \preceq^{\mathbf{P}^+} v \preceq^{\mathbf{P}^+} w$. We will establish the following implications.

$$u \preceq_1^{\mathbf{P}^+} v \preceq_1^{\mathbf{P}^+} w \implies u \preceq_1^{\mathbf{P}^+} w$$

$$u \preceq_2^{\mathbf{P}^+} v \preceq_2^{\mathbf{P}^+} w \implies u \preceq_2^{\mathbf{P}^+} w$$

$$u \preceq_1^{\mathbf{P}^+} w \implies u \preceq_1^{\mathbf{P}^+} v$$

$$u \preceq_2^{\mathbf{P}^+} w \text{ and } v \preceq_1^{\mathbf{P}^+} w \implies u \preceq_2^{\mathbf{P}^+} v$$

Since the restrictions of $\mathbf{P}^+$ to $(-\infty, a)^{\mathbf{P}} \cup X^+$ and $|\mathbf{P}|$ are both patterns, we may assume, as in the proof of conditions 2 and 3, that $\{u, v, w\}$ is not a subset of either set. Hence, $a \preceq^{\mathbf{P}^+} w$ and either $u \in X^+$ or $v \in X^+$.

**Case 1:** $u \in (-\infty, a)^{\mathbf{P}}$.

In this case, we must have $v \in X^+$.

To show the first implication, assume $u \preceq_1^{\mathbf{P}^+} v \preceq_1^{\mathbf{P}^+} w$. Since we have shown $\mathbf{P}^+$ is a 1-correct extension of $\mathbf{P}$, we have $u \preceq_1^{\mathbf{P}} a \preceq_1^{\mathbf{P}} w$. Therefore, $u \preceq_1^{\mathbf{P}} w$ implying $u \preceq_1^{\mathbf{P}^+} w$.

The second implication is vacuous in this case since $v \not\preceq_2^{\mathbf{P}^+} w$ by Definition 9.4.

To show the third implication, assume $u \preceq_1^{\mathbf{P}^+} w$. This implies $u \preceq_1^{\mathbf{P}} w$. Since $u \preceq^{\mathbf{P}} a \preceq^{\mathbf{P}} w$, $u \preceq_1^{\mathbf{P}} a$. Since $a \preceq^{\mathbf{P}} h(v) \preceq^{\mathbf{P}} b$, $a \preceq_1^{\mathbf{P}} h(v)$. Therefore, $u \preceq_1^{\mathbf{P}} h(v)$. By part 4 of Definition 9.4, $u \preceq_1^{\mathbf{P}^+} v$.

To show the fourth implication, assume $u \preceq_2^{\mathbf{P}^+} w$ and $v \preceq_1^{\mathbf{P}^+} w$. By Definition 9.4, we have

(i) $u \preceq_2^{\mathbf{P}} w$

(ii) $h(v) \preceq_1^{\mathbf{P}} b$

(iii) $a \preceq_1^{\mathbf{P}} w$

By (i) and (iii), $u \preceq_2^{\mathbf{P}} a$. Since $a \preceq_2^{\mathbf{P}} b$, $u \preceq_2^{\mathbf{P}} b$. Using (ii), we have $u \preceq_2^{\mathbf{P}} h(v)$. Therefore, $u \preceq_2^{\mathbf{P}^+} v$.

**Case 2:** $u \notin (-\infty, a)^{\mathbf{P}}$.

In this case, we must have $u \in X^+$.

**Subcase 1 of Case 2:** $v \in X^+$.

To show the first implication, assume $u \preceq_1^{\mathbf{P}^+} v \preceq_1^{\mathbf{P}^+} w$. Using Definition 9.4, we have

(i) $h(u) \preceq_1^{\mathbf{P}} h(v)$

(ii) $h(v) \preceq_1^{\mathbf{P}} b$

(iii) $a \preceq_1^{\mathbf{P}} w$

By (i) and (ii), $h(u) \preceq_1^{\mathbf{P}} b$. By (iii) and part 5 of Definition 9.4, this implies $u \preceq_1^{\mathbf{P}^+} w$.

The second implication is vacuously true since $v \not\preceq_2^{\mathbf{P}^+} w$ in this subcase.

To show the third implication, assume $u \preceq_1^{\mathbf{P}^+} w$. By part 5 of Definition 9.4, we have $h(u) \preceq_1^{\mathbf{P}} b$. Since $h(u) \preceq^{\mathbf{P}} h(v) \preceq^{\mathbf{P}} b$, $h(u) \preceq_1^{\mathbf{P}} h(v)$. Therefore, $u \preceq_1^{\mathbf{P}^+} v$.

The fourth implication is vacuously true since $u \not\preceq_2^{\mathbf{P}^+} w$.

**Subcase 2 of Case 2:** $v \notin X^+$.

In this subcase, we must have $a \preceq^{\mathbf{P}^+} v$.

To show the first implication, assume $u \preceq_1^{\mathbf{P}^+} v \preceq_1^{\mathbf{P}^+} w$. Using Definition 9.4, we have

(i) $h(u) \preceq_1^{\mathbf{P}} b$

(ii) $a \preceq_1^{\mathbf{P}} v$

(iii) $v \preceq_1^{\mathbf{P}} w$

By (ii) and (iii), $a \preceq_1^{\mathbf{P}} w$. Along with (i), this implies that $u \preceq_1^{\mathbf{P}^+} w$.

The second implication is vacuously true since $u \not\preceq_2^{\mathbf{P}^+} v$.

To show the third implication, assume $u \preceq_1^{\mathbf{P}^+} w$. By Definition 9.4, we have

(i) $h(u) \preceq_1^{\mathbf{P}} b$

(ii) $a \preceq_1^{\mathbf{P}} w$

Since $a \preceq^{\mathbf{P}} v \preceq^{\mathbf{P}} w$, (ii) implies that $a \preceq_1^{\mathbf{P}} v$. Along with (i), this implies that $u \preceq_1^{\mathbf{P}^+} v$.

The fourth implication is vacuously true since $u \not\preceq_2^{\mathbf{P}^+} w$ in this subcase.

The argument that $\mathbf{P}^+$ is a transcendental extension of $\mathbf{P}$ is the same as that used in the proof of Lemma 9.2. **QED**

By the lemma, if $\mathbf{P}^+$ is obtained from $\mathbf{P}$ by 2-reflecting $X$ downward from $b$ to $a$, $\mathbf{A}^+$ is the arithmetic part of $\mathbf{P}^+$ and $X^+ = |\mathbf{P}^+| \setminus |\mathbf{P}|$ then $\mathbf{P}^+$ is the smallest pattern whose arithmetic part is $\mathbf{A}^+$ which satisfies conditions 2-4 of Definition 9.4 and such that

$$\tilde{x} \preceq_1^{\mathbf{P}^+} a \text{ iff } x \preceq_1^{\mathbf{P}} b$$

whenever $\tilde{x} \in X^+$ and $x$ corresponds to $\tilde{x}$ under the isomorphism of $(-\infty, a)^{\mathbf{P}} \cup X^+$ and $(-\infty, a)^{\mathbf{P}} \cup X$.

**Lemma 9.6** *If $\mathbf{P}^+$ is obtained from $\mathbf{P}$ by 2-reflecting $Y$ downward from $b$ to $a$ then the rule $\mathbf{P}|\mathbf{P}^+$ is a cofinally valid continuous rule.*

**Proof.** The proof is similar to that of Lemma 9.2.

Assume $\mathbf{P}^+$ is obtained from $\mathbf{P}$ by 2-reflecting $Y$ downward from $b$ to $a$. Let $\tilde{Y} = |\mathbf{P}^+| \setminus |\mathbf{P}|$ and let $g$ be an isomorphism of $(-\infty, a)^{\mathbf{P}} \cup \tilde{Y}$ and $(-\infty, a)^{\mathbf{P}} \cup Y$.

$\mathbf{P}|\mathbf{P}^+$ is a continuous rule by part 2 of Lemma 9.5.

Suppose $h$ is a covering of $\mathbf{P}$ in $\mathcal{R}_2$ and $\varphi$ is a regressive function on the nonzero indecomposables in $h[|\mathbf{P}|]$. There is a finite closed subset $X$ of $h(a)$ such that $\varphi(h(a)) \in X$ and $h[(-\infty, a)^{\mathbf{P}}] \subseteq X$. Since $a \preceq_2^{\mathbf{P}} b$ and $h$ is a covering, $h(a) \leq_2 h(b)$. Therefore, there is a finite set $\tilde{Y}^* \subseteq h(a)$ such that $X < \tilde{Y}^*$, $X \cup \tilde{Y}^*$ is closed and there is a covering $f$ of $X \cup h[Y]$ onto $X \cup \tilde{Y}^*$ with the property that $f(\xi) \leq_1 h(a)$ whenever $\xi \in h[Y]$ and $\xi \leq_1 h(b)$. Notice that $f$ is the identity on $X$ and, hence, $f \circ h \circ g$ is a covering of $(-\infty, a)^{\mathbf{P}} \cup \tilde{Y}$ onto $h[(-\infty, a)^{\mathbf{P}}] \cup \tilde{Y}^*$ which agrees with $h$ on $(-\infty, a)^{\mathbf{P}}$ and maps $\tilde{Y}$ onto $\tilde{Y}^*$.

Extend $h$ to a function $h^+$ from $|\mathbf{P}^+|$ into the ordinals so that $h^+(\tilde{y}) = f(h(g(\tilde{y})))$ for $\tilde{y} \in \tilde{Y}$. Since $f \circ h \circ g$ agrees with $h$ on $(-\infty, a)^{\mathbf{P}}$, $h^+$ extends $f \circ h \circ g$.

By Lemma 4.5, there is a closed embedding $h^*$ of the arithmetic part of $\mathbf{P}^+$ into $\mathcal{R}_0$ which agrees with $h^+$ on the indecomposables of $\mathbf{P}^+$. By Lemma 4.4, $h^*$ must extend $h$ and $f \circ h \circ g$. Therefore, $h^* = h^+$ implying $h^+$ is a closed embedding of $\mathbf{P}^+$ into $\mathcal{R}_2$.

To see that $h^+$ is a covering, suppose $u, v \in |\mathbf{P}^+|$ and $u \preceq_k^{\mathbf{P}^+} v$ where $k \in \{1, 2\}$. We need to show $h^+(u) \leq_k h^+(v)$.

**Case 1:** Assume $k = 1$.

Since $h^+$ extends $h$ and $h$ is a covering of $\mathbf{P}$ in $\mathcal{R}_2$, $h^+(u) \leq_1 h^+(v)$ whenever $u, v \in |\mathbf{P}|$. Also, since $h^+$ extends $f \circ h \circ g$ and $f \circ h \circ g$ is a covering of $(-\infty, a)^{\mathbf{P}} \cup \tilde{Y}$ into $\mathcal{R}_2$, $h^+(u) \leq_1 h^+(v)$ whenever $u, v \in (-\infty, a)^{\mathbf{P}} \cup \tilde{Y}$. Therefore, we may assume that $u \in \tilde{Y}$ and $v \in [a, \infty)^{\mathbf{P}}$. By Definition 9.4, $g(u) \preceq_1^{\mathbf{P}} b$ and $a \preceq_1^{\mathbf{P}} v$. Therefore, $h(g(u)) \leq_1 h(b)$ and $h(a) \leq_1 h(v)$. Since $f$ is a covering, $f(h(g(u))) \leq_1 h(a)$. Since $f(h(g(u))) = h^+(u)$, $h^+(u) \leq_1 h(v)$.

**Case 2:** Assume $k = 2$.

By Definition 9.4, the case that $u \in \tilde{Y}$ and $v \in [a, \infty)^{\mathbf{P}}$ is impossible. Hence, either $u, v \in |\mathbf{P}|$ or $u, v \in (-\infty, a)^{\mathbf{P}} \cup \tilde{Y}$. By an argument similar to that in case 1, $h^+(u) \leq_2 h^+(v)$.

Since $(-\infty, a)^{\mathbf{P}} \prec^{\mathbf{P}^+} \tilde{Y} \prec^{\mathbf{P}^+} a$, to see that $h^+$ extends $h$ above $\varphi$ it suffices to show that $\varphi(h(a)) < h^+[\tilde{Y}]$. This follows from the fact that $h^+$ maps $\tilde{Y}$ onto $\tilde{Y}^*$ and $\varphi(h(a)) \in X < \tilde{Y}^*$. **QED**

# 10 Upward Reflection

**Definition 10.1** Assume $\mathbf{P}^+$ is a continuous extension of $\mathbf{P}$ at $a,\ \ b \in |\mathbf{P}|$ and $a \prec_2^{\mathbf{P}} b$. The rule $\mathbf{P}|\mathbf{P}^*$ is *obtained by 2-reflecting* $\mathbf{P}|\mathbf{P}^+$ *upward from* $a$ *to* $b$ provided $\mathbf{P}$ is a closed substructure of $\mathbf{P}^*$ and, letting $X^+ = |\mathbf{P}^+| \setminus |\mathbf{P}|$ and $X^* = |\mathbf{P}^*| \setminus |\mathbf{P}|$,

1. The arithmetic part of $\mathbf{P}^*$ is an arithmetic structure.
2. $(-\infty, b)^{\mathbf{P}} \prec^{\mathbf{P}^*} X^* \prec^{\mathbf{P}^*} b$
3. $(-\infty, a)^{\mathbf{P}} \cup X^*$ is a closed subset of $\mathbf{P}^*$.
4. The substructure of $\mathbf{P}^*$ with universe $(-\infty, a)^{\mathbf{P}} \cup X^*$ is isomorphic to the substructure of $\mathbf{P}^+$ with universe $(-\infty, a)^{\mathbf{P}} \cup X^+$.
5. If $x^* \in X^*$ and $b \preceq^{\mathbf{P}^*} y$ then
$$x^* \not\preceq_1^{\mathbf{P}^*} y \text{ and } x^* \not\preceq_2^{\mathbf{P}^*} y$$
6. If $y \in [a, b)^{\mathbf{P}}$ and $x^* \in X^*$ then
$$y \preceq_1^{\mathbf{P}^*} x^* \quad \text{iff} \quad y \preceq_1^{\mathbf{P}} b$$
and
$$y \not\preceq_2^{\mathbf{P}^*} x^*$$

Notice that we have not required $\mathbf{P}|\mathbf{P}^+$ to be continuous in the above definition. Ultimately, we will only be interested in the case when $\mathbf{P}|\mathbf{P}^+$ is valid which, as we will see later, implies continuity.

**Lemma 10.2** *Assume* $\mathbf{P}^+$ *is a continuous extension of* $\mathbf{P}$ *at* $a,\ \ b \in |\mathbf{P}|$ *and* $a \prec_2^{\mathbf{P}} b$.

1. *The family of structures* $\mathbf{P}^*$ *such that* $\mathbf{P}|\mathbf{P}^*$ *is obtained by* 2*-reflecting* $\mathbf{P}|\mathbf{P}^+$ *upward from* $a$ *to* $b$ *is nonempty and unique up to isomorphism over* $\mathbf{P}$.
2. *If* $\mathbf{P}|\mathbf{P}^*$ *is obtained by* 2*-reflecting* $\mathbf{P}|\mathbf{P}^+$ *upward from* $a$ *to* $b$ *then* $\mathbf{P}^*$ *is a continuous extension of* $\mathbf{P}$ *at* $a$.

**Proof.** The proof of part 1 is analogous to the proof of part 1 for Lemma 9.2 where the construction of the arithmetic part of a structure obtained by 2-reflecting $\mathbf{P}|\mathbf{P}^+$ up from $a$ to $b$ uses Lemma 4.16 twice.

For part 2, assume $\mathbf{P}|\mathbf{P}^*$ is obtained by 2-reflecting $\mathbf{P}|\mathbf{P}^+$ upward from $a$ to $b$ . Let $X^+$ be $|\mathbf{P}^+| \setminus |\mathbf{P}|$, $X^*$ be $|\mathbf{P}^*| \setminus |\mathbf{P}|$ and let $h$ be the isomorphism of the substructure of $\mathbf{P}^*$ with universe $(-\infty, a)^{\mathbf{P}} \cup X^*$ and the substructure of $\mathbf{P}^+$ with universe $(-\infty, a)^{\mathbf{P}} \cup X^+$. Notice that since $(-\infty, a)^{\mathbf{P}} \cup X^+$ is an initial segment of $\mathbf{P}^+$, it is a closed subset of $\mathbf{P}^+$.

We begin by showing that the continuity conditions 1 and 2 of Definition 7.4 hold.

To establish condition 1, notice that $[b, \infty)^{\mathbf{P}^*} = [b, \infty)^{\mathbf{P}}$ and $b$ is indecomposable in $\mathbf{P}^*$. This implies that the largest indecomposable of $\mathbf{P}^*$ is in $\mathbf{P}$.

For condition 2, assume $k \in \{1, 2\}$. We must show that $\mathbf{P}^*$ is a $k$-correct extension of $\mathbf{P}$. Suppose

$$(-\infty, c)^{\mathbf{P}} \prec^{\mathbf{P}^*} x \preceq^{\mathbf{P}^*} c \prec^{\mathbf{P}^*} y$$

and $x \preceq_k^{\mathbf{P}^*} y$ where $c \in |\mathbf{P}|$. Let $d$ be the least element of $|\mathbf{P}|$ such that $y \preceq^{\mathbf{P}^*} d$. We must show $c \preceq_k^{\mathbf{P}} d$.

If $x, y \in |\mathbf{P}|$ then $c = x$ and $d = y$. So we may assume either $x \notin |\mathbf{P}|$ or $y \notin |\mathbf{P}|$. If $x \notin |\mathbf{P}|$ then $x \in X^*$, $c = b$, $b \prec^{\mathbf{P}} y$ and $x \preceq_k^{\mathbf{P}} y$ – contradicting the definition of $\mathbf{P}^+$. So we may assume $y \notin |\mathbf{P}|$. In this case, $y \in X^*$, $d = b$ and $c < b$ which implies $x = c$. So, we must show that $x \preceq_k^{\mathbf{P}^*} b$.

First suppose $x \in (-\infty, a)^{\mathbf{P}}$. In this case, $x \preceq_k^{\mathbf{P}^+} h(y)$. Since $\mathbf{P}^+$ is a continuous extension of $\mathbf{P}$, $x \preceq_k^{\mathbf{P}} a$. Since $a \preceq_2^{\mathbf{P}} b$, $x \preceq_k^{\mathbf{P}} b$.

Now suppose $x \in [a, b)^{\mathbf{P}}$. By definition of $\mathbf{P}^*$, $k = 1$ and $x \preceq_1^{\mathbf{P}} b$.

To establish that $\mathbf{P}^*$ is a pattern, we will show that conditions 1-7 of Definition 5.6 hold.

Condition 1 is immediate from Definition 10.1.

To verify conditions 2 and 3, assume $u, v \in \mathbf{P}^*$. We will show

$$u \prec_1^{\mathbf{P}^+} v \implies u \text{ is indecomposable in } \mathbf{P}^*$$

and

$$u \prec_2^{\mathbf{P}^+} v \implies u \text{ and } v \text{ are indecomposable in } \mathbf{P}^*$$

Since the restrictions of $\mathbf{P}^*$ to $(-\infty, a)^{\mathbf{P}} \cup X^*$ and $|\mathbf{P}|$ are both patterns, the cases where $\{u, v\}$ is a subset of one of these sets follows immediately.

The remaining cases, both $u \in [a,b)^{\mathbf{P}}$ and $v \in X^*$ or both $u \in X^*$ and $v \in [b,\infty)^{\mathbf{P}}$, are straightforward.

To verify conditions 4-7, first notice that $\preceq_1^{\mathbf{P}^*}$ is contained in $\preceq^{\mathbf{P}^*}$ and $\preceq_2^{\mathbf{P}^*}$ is contained in $\preceq_1^{\mathbf{P}^*}$. Now assume $u \preceq^{\mathbf{P}^*} v \preceq^{\mathbf{P}^*} w$. We will establish the following implications.

$$u \preceq_1^{\mathbf{P}^*} v \preceq_1^{\mathbf{P}^*} w \quad \Longrightarrow \quad u \preceq_1^{\mathbf{P}^*} w$$

$$u \preceq_2^{\mathbf{P}^*} v \preceq_2^{\mathbf{P}^*} w \quad \Longrightarrow \quad u \preceq_2^{\mathbf{P}^*} w$$

$$u \preceq_1^{\mathbf{P}^*} w \quad \Longrightarrow \quad u \preceq_1^{\mathbf{P}^*} v$$

$$u \preceq_2^{\mathbf{P}^*} w \;\text{ and }\; v \preceq_1^{\mathbf{P}^*} w \quad \Longrightarrow \quad u \preceq_2^{\mathbf{P}^*} v$$

We will assume that $u \prec^{\mathbf{P}^*} v \prec^{\mathbf{P}^*} w$ since the other cases are trivial.

Since the restrictions of $\mathbf{P}^*$ to $(-\infty,a)^{\mathbf{P}} \cup X^*$ and $|\mathbf{P}|$ are both patterns, we may assume, as in the proof of conditions 2 and 3, that $\{u,v,w\}$ is not a subset of either set. We now consider the remaining cases depending on which of $(-\infty,a)^{\mathbf{P}}$, $[a,b)^{\mathbf{P}}$, $X^*$ and $[b,\infty)^{\mathbf{P}}$ each of $u$, $v$ and $w$ are in.

**Case 1:** Assume $u \in (-\infty,a)^{\mathbf{P}}$, $v \in [a,b)^{\mathbf{P}}$ and $w \in X^*$.

For the first implication, assume $u \preceq_1^{\mathbf{P}^*} v \preceq_1^{\mathbf{P}^*} w$. Since $u \preceq^{\mathbf{P}^+} h(w) \preceq^{\mathbf{P}^+} a \preceq^{\mathbf{P}^+} v$, $u \preceq_1^{\mathbf{P}^+} h(w)$. By choice of $h$, $u \preceq_1^{\mathbf{P}^*} w$.

The second implication is vacuous since $v \not\preceq_2^{\mathbf{P}^*} w$.

For the third implication, assume $u \preceq_1^{\mathbf{P}^*} w$. By continuity, $u \preceq_1^{\mathbf{P}} b$. Since $u \preceq^{\mathbf{P}} v \preceq^{\mathbf{P}} b$, $u \preceq_1^{\mathbf{P}} v$ implying $u \preceq_1^{\mathbf{P}^*} v$.

For the fourth implication assume $u \preceq_2^{\mathbf{P}^*} w$ and $v \preceq_1^{\mathbf{P}^*} w$. By continuity, $u \preceq_2^{\mathbf{P}} b$ and $v \preceq_1^{\mathbf{P}} b$. Therefore, $u \preceq_2^{\mathbf{P}} v$ implying $u \preceq_2^{\mathbf{P}^*} v$

**Case 2:** Assume $u \in (-\infty,a)^{\mathbf{P}}$, $v \in X^*$ and $w \in [b,\infty)^{\mathbf{P}}$.

In this case, $v \not\preceq_k^{\mathbf{P}^*} w$ for $k = 1,2$. Therefore, the first, second and fourth implications are vacuous.

For the third implication, assume $u \preceq_1^{\mathbf{P}^*} w$. Since $u,w \in |\mathbf{P}|$, $u \preceq_1^{\mathbf{P}} w$ implying $u \preceq_1^{\mathbf{P}^+} w$. Since $u \preceq^{\mathbf{P}^+} h(v) \preceq^{\mathbf{P}^+} a \preceq^{\mathbf{P}^+} w$, $u \preceq_1^{\mathbf{P}^+} h(v)$ implying $u \preceq_1^{\mathbf{P}^*} v$.

**Case 3:** Assume $u, v \in [a,b)^{\mathbf{P}}$ and $w \in X^*$.

In this case, $u \not\preceq_2^{\mathbf{P}^*} w$ and $v \not\preceq_2^{\mathbf{P}^*} w$. Therefore, the second and fourth implications are vacuous.

For the first implication, assume $u \preceq_1^{\mathbf{P}^*} v \preceq_1^{\mathbf{P}^*} w$. Since $u, v \in |\mathbf{P}|$, $u \preceq_1^{\mathbf{P}} v$. By the definition of $\mathbf{P}^*$, $v \preceq_1^{\mathbf{P}} b$. Therefore, $u \preceq_1^{\mathbf{P}} b$. By definition of $\mathbf{P}^*$, $u \preceq_1^{\mathbf{P}^*} w$.

For the third implication, assume $u \preceq_1^{\mathbf{P}^*} w$. By the definition of $\mathbf{P}^*$, $u \preceq_1^{\mathbf{P}} b$. Therefore, $u \preceq_1^{\mathbf{P}} v$ implying $u \preceq_1^{\mathbf{P}^*} v$.

**Case 4:** Assume $u \in [a,b)^{\mathbf{P}}$ and $v,w \in X^*$.

In this case, $u \not\preceq_2^{\mathbf{P}^*} v$ and $u \not\preceq_2^{\mathbf{P}^*} w$. Therefore, the second and fourth implications are vacuous.

For the first implication, assume $u \preceq_1^{\mathbf{P}^*} v \preceq_1^{\mathbf{P}^*} w$. By the definition of $\mathbf{P}^*$, $u \preceq_1^{\mathbf{P}} b$. Again by the definition of $\mathbf{P}^*$, $u \preceq_1^{\mathbf{P}^*} w$.

For the third implication, assume $u \preceq_1^{\mathbf{P}^*} w$. By the definition of $\mathbf{P}^*$, $u \preceq_1^{\mathbf{P}} b$. Again by the definition of $\mathbf{P}^*$, $u \preceq_1^{\mathbf{P}^*} v$.

**Case 5:** Assume $u \in [a,b)^{\mathbf{P}}$, $v \in X^*$ and $w \in [b,\infty)^{\mathbf{P}}$.

In this case, $v \not\preceq_k^{\mathbf{P}^*} w$ for $k = 1,2$. Therefore, the first, second and fourth implications are vacuous.

For the third implication, assume $u \preceq_1^{\mathbf{P}^*} w$. Since $u,w \in |\mathbf{P}|$, $u \preceq_1^{\mathbf{P}} w$. Since $u \preceq^{\mathbf{P}} b \preceq^{\mathbf{P}} w$, $u \preceq_1^{\mathbf{P}} b$. By definition of $\mathbf{P}^*$, $u \preceq_1^{\mathbf{P}^*} v$.

**Case 6:** Assume $u,v \in X^*$ and $w \in [b,\infty)^{\mathbf{P}}$.

In this case, $u \not\preceq_k^{\mathbf{P}^*} w$ and $v \not\preceq_k^{\mathbf{P}^*} w$ for $k = 1,2$. Therefore, all four implications are vacuous.

**Case 7:** Assume $u \in X^*$ and $v,w \in [b,\infty)^{\mathbf{P}}$.

In this case, $u \not\preceq_k^{\mathbf{P}^*} v$ and $u \not\preceq_k^{\mathbf{P}^*} w$ for $k = 1,2$. Therefore, all four implications are vacuous.

The argument that $\mathbf{P}^*$ is a transcendental extension of $\mathbf{P}$ is similar to that used in the proof of Lemma 9.2. **QED**

**Lemma 10.3** *Assume* $\mathbf{P}|\mathbf{P}^*$ *is obtained by 2-reflecting* $\mathbf{P}|\mathbf{P}^+$ *upward from* $a$ *to* $b$*. If* $\mathbf{P}|\mathbf{P}^+$ *is cofinally valid and continuous then* $\mathbf{P}|\mathbf{P}^*$ *is a cofinally valid and continuous.*

**Proof.** Assume $\mathbf{P}|\mathbf{P}^*$ is obtained by 2-reflecting $\mathbf{P}|\mathbf{P}^+$ upward from $a$ to $b$. Let $Y^+ = |\mathbf{P}^+| \setminus |\mathbf{P}|$, let $Y^* = |\mathbf{P}^*| \setminus |\mathbf{P}|$ and let $g$ be an isomorphism of $(-\infty,a)^{\mathbf{P}} \cup Y^*$ and $(-\infty,a)^{\mathbf{P}} \cup Y^+$. Notice that $g$ is the identity on $(-\infty,a)^{\mathbf{P}}$ and maps $Y^*$ onto $Y^+$.

$\mathbf{P}|\mathbf{P}^*$ is a continuous rule by part 2 of Lemma 10.2.

Suppose $h$ is a covering of $\mathbf{P}$ in $\mathcal{R}_2$ and $\varphi$ is a regressive function on the nonzero indecomposables in $h[|\mathbf{P}|]$. Since $a \preceq_2^{\mathbf{P}} b$, $h(a) \leq_2 h(b)$. Also, since $\mathbf{P}|\mathbf{P}^+$ is cofinally valid, there are cofinally many $\tilde{Y}^+$ below $h(a)$ such that $h[|\mathbf{P}|] \cup \tilde{Y}^+$ is closed and is a covering of $\mathbf{P}^+$. In particular, there are cofinally many $\tilde{Y}^+$ below $h(a)$ such that $h[(-\infty,a)^{\mathbf{P}}] \cup \tilde{Y}^+$ is closed and is a covering of $(-\infty,a)^{\mathbf{P}} \cup Y^+$. Therefore, there are cofinally many $\tilde{Y}^*$ below $h(b)$ such that $h[(-\infty,a)^{\mathbf{P}}] \cup \tilde{Y}^*$ is closed and is a covering of $(-\infty,a)^{\mathbf{P}} \cup Y^+$. Fix such a $\tilde{Y}^*$ with the property that $h[(-\infty, b)] < \tilde{Y}^*$ and $\varphi(h(b)) < \tilde{Y}^*$ and let $f$ be a covering of $(-\infty, a)^{\mathbf{P}} \cup Y^+$ onto $h[(-\infty, a)^{\mathbf{P}}] \cup \tilde{Y}^*$. Notice that $f$ agrees with $h$ on $(-\infty, a)^{\mathbf{P}}$ and maps $Y^+$ onto $\tilde{Y}^*$. Hence, $f \circ g$ is a covering of $(-\infty, a)^{\mathbf{P}} \cup Y^*$ onto $h[(-\infty, a)^{\mathbf{P}}] \cup \tilde{Y}^*$ which agrees with $h$ on $(-\infty, a)^{\mathbf{P}}$ and maps $Y^*$ onto $\tilde{Y}^*$.

Extend $h$ to a function $h^+$ from $|\mathbf{P}^*|$ into the ordinals so that $h^+(y^*) = f(g(y^*))$ for $y^* \in Y^*$. Since $f \circ g$ agrees with $h$ on $(-\infty, a)^{\mathbf{P}}$, $h^+$ extends $f \circ g$.

By Lemma 4.5, there is a closed embedding $h^*$ of the arithmetic part of $\mathbf{P}^*$ into $\mathcal{R}_0$ which agrees with $h^+$ on the indecomposables of $\mathbf{P}^*$. By Lemma 4.4, $h^*$ must extend $h$ and $f \circ g$. Therefore, $h^* = h^+$ implying $h^+$ is a closed embedding of $\mathbf{P}^+$ into $\mathcal{R}_2$.

To see that $h^+$ is a covering, suppose $u, v \in |\mathbf{P}^*|$ and $u \preceq_k^{\mathbf{P}^*} v$ where $k \in \{1, 2\}$. We need to show $h^+(u) \leq_k h^+(v)$. By Definition 10.1, the case that $u \in Y^*$ and $v \in [b, \infty)^{\mathbf{P}}$ is impossible. Since $h^+$ extends $h$ and $f \circ g$ and both are coverings, the desired conclusion follows if either $u, v \in |\mathbf{P}|$ or $u, v \in (-\infty, a)^{\mathbf{P}} \cup Y^*$. The only remaining possibility is $u \in [a, b)^{\mathbf{P}}$ and $v \in Y^*$. In this case, we must have $k = 1$ and $u \preceq_1^{\mathbf{P}} b$ by Definition 10.1. Therefore, $h(u) \leq_1 h(b)$. Since $f(g(v)) \in \tilde{Y}^*$ and $h(u) < \tilde{Y}^* < h(b)$, $h(u) \leq_1 f(g(v))$ or, equivalently, $h^+(u) \leq_1 h^+(v)$.

Since $(-\infty, b)^{\mathbf{P}} \prec^{\mathbf{P}^*} Y^* \prec^{\mathbf{P}^*} b$, to see that $h^+$ extends $h$ above $\varphi$ it suffices to show that $\varphi(h(b)) < h^+[Y^*]$. This follows from the fact that $h^+$ maps $Y^*$ onto $\tilde{Y}^*$ and $\varphi(h(b)) < \tilde{Y}^*$. **QED**

# 11 Arithmetic Liftings

We will need a generalization of Lemma 4.16.

To motivate the following definition, assume $\mathcal{R}_0$ is $(ORD, \leq)$ so that arithmetic structures are linear orderings, every element is indecomposable and every substructure is closed. Assume $\mathbf{A}$ is a finite linear ordering and $\mathbf{A}^+$ is a finite linear ordering extending $\mathbf{A}$. Also, suppose that $\mathbf{B}$ is a finite linear ordering which extends $\mathbf{A}$. We would like to define an extension of $\mathbf{B}$ which extends $\mathbf{B}$ in the way that $\mathbf{A}^+$ extends $\mathbf{A}$. For simplicity, also assume $|\mathbf{A}^+| \setminus |\mathbf{A}|$ and $|\mathbf{B}| \setminus |\mathbf{A}|$ are disjoint. We can take the desired structure to have $|\mathbf{B}| \cup |\mathbf{A}^+|$ as its universe. The main issue is to determine where to place the elements of $|\mathbf{A}^+| \setminus |\mathbf{A}|$ in relation to those of $|\mathbf{B}| \setminus |\mathbf{A}|$ (we already know where they must be in relation to those of $|\mathbf{A}|$). The following definition requires that elements of $|\mathbf{A}^+|\backslash|\mathbf{A}|$ be placed above those of $|\mathbf{B}|\backslash|\mathbf{A}|$ whenever possible. Clearly, this process is not symmetric in $\mathbf{A}^+$ and $\mathbf{B}$. In the general case, we order the indecomposables as above which then determines the remainder of the structure.

**Definition 11.1** Assume $\mathbf{A}$, $\mathbf{A}^+$ and $\mathbf{B}$ are finite arithmetic structures and $h$ is a closed embedding of $\mathbf{A}$ in $\mathbf{B}$. $(\mathbf{A}^+, \mathbf{A}, h, \mathbf{B}, \mathbf{B}^+)$ is a *possible arithmetic lifting* if $\mathbf{A}$ is a closed substructure of $\mathbf{A}^+$, every indecomposable in $\mathbf{A}^+$ is bounded above by an indecomposable in $\mathbf{A}$ and $\mathbf{B}$ is a closed substructure of $\mathbf{B}^+$. A closed embedding $h^+$ of $\mathbf{A}^+$ into $\mathbf{B}^+$ is a *lifting map for* $(\mathbf{A}^+, \mathbf{A}, h, \mathbf{B}, \mathbf{B}^+)$ if

1. $h^+$ extends $h$.
2. $|\mathbf{B}^+| \; = \; |\mathbf{B}| \cup h^+[|\mathbf{A}^+|]$
3. If $a \in |\mathbf{A}|$ is indecomposable in $\mathbf{A}$, $a^+ \in |\mathbf{A}^+|$ is indecomposable in $\mathbf{A}^+$ and
$$(-\infty, a)^{\mathbf{A}} \prec^{\mathbf{A}^+} a^+ \prec^{\mathbf{A}^+} a$$
then
$$(-\infty, h(a))^{\mathbf{B}^+} \prec^{\mathbf{B}^+} h^+(a^+) \prec^{\mathbf{B}^+} h(a)$$

$(\mathbf{A}^+, \mathbf{A}, h, \mathbf{B}, \mathbf{B}^+)$ is a *lifting* if it has a lifting map. If $(\mathbf{A}^+, \mathbf{A}, h, \mathbf{B}, \mathbf{B}^+)$ is a lifting then we say that $\mathbf{B}^+$ is a *lifting of* $\mathbf{A}|\mathbf{A}^+$ *to* $\mathbf{B}$ *with respect to* $h$.

When $\mathbf{A}$ is a closed substructure of $\mathbf{B}$ and $\mathbf{B}^+$ is a lifting of $\mathbf{A}|\mathbf{A}^+$ with respect to the inclusion map of $|\mathbf{A}|$ in $|\mathbf{B}|$ we will simply say $\mathbf{B}^+$ is a lifting of $\mathbf{A}|\mathbf{A}^+$ to $\mathbf{B}$.

**Lemma 11.2** *Assume* $(\mathbf{A}^+, \mathbf{A}, h, \mathbf{B}, \mathbf{B}^+)$ *is a lifting. There is a unique lifting map for* $(\mathbf{A}^+, \mathbf{A}, h, \mathbf{B}, \mathbf{B}^+)$.

**Proof.** Assume $h^+$ is a lifting map for $(\mathbf{A}^+, \mathbf{A}, h, \mathbf{B}, \mathbf{B}^+)$. Notice that $h^+$ must agree with $h$ on the indecomposables of $\mathbf{A}$ and the indecomposables of $\mathbf{A}^+$ which are in $|\mathbf{A}^+| \setminus |\mathbf{A}|$ must be mapped onto those in $|\mathbf{B}^+| \setminus |\mathbf{B}|$ in increasing order. Since the values of $h^+$ on the indecomposables of $\mathbf{A}^+$ completely determine $h^+$ by Lemma 4.4, $h^+$ is unique. **QED**

Suppose $h$ is a closed embedding of $\mathbf{A}$ into $\mathbf{B}$ and $\mathbf{B}^+$ is a lifting of $\mathbf{A}|\mathbf{A}^+$ to $\mathbf{B}$ with lifting map $h^+$. One might expect that every element of

$|\mathbf{B}| \cap h^+[|\mathbf{A}^+|]$ would be arithmetic over $\mathbf{A}$. While this will be true for the examples we are interested in, it is not true for arbitrary choices of $\mathcal{R}_0$ (in fact, this fails for $\mathcal{R}_0$ as in Example 3.8).

**Lemma 11.3** *Assume* $(\mathbf{A}_k^+, \mathbf{A}_k, h_k, \mathbf{B}_k, \mathbf{B}_k^+)$ *is a possible lifting for* $k = 1, 2$. *If* $f$ *is an isomorphism of* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *which extends to an isomorphism* $f^+$ *of* $\mathbf{A}_1^+$ *and* $\mathbf{A}_2^+$, $g$ *is an isomorphism of* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *which extends to an isomorphism* $g^+$ *of* $\mathbf{B}_1^+$ *and* $\mathbf{B}_2^+$ *such that*

$$g \circ h_1 = h_2 \circ f$$

*and* $h_2^+$ *is a lifting map for* $(\mathbf{A}_2^+, \mathbf{A}_2, h_2, \mathbf{B}_2, \mathbf{B}_2^+)$ *then* $(g^+)^{-1} \circ h_2^+ \circ f^+$ *is a lifting map for* $(\mathbf{A}_1^+, \mathbf{A}_1, h_1, \mathbf{B}_1, \mathbf{B}_1^+)$.

**Proof.** Straightforward. **QED**

One consequence of the lemma is that the collection of liftings of $\mathbf{A}|\mathbf{A}^+$ to $\mathbf{B}$ with respect to $h$ is closed under isomorphism over $\mathbf{B}$.

**Lemma 11.4** *Assume* $\mathbf{A}$, $\mathbf{A}^+$ *and* $\mathbf{B}$ *are finite arithmetic structures such that* $\mathbf{A}$ *is a closed substructure of* $\mathbf{A}^+$ *and let* $h$ *be a closed embedding of* $\mathbf{A}$ *into* $\mathbf{B}$. *The family of arithmetic liftings of* $\mathbf{A}|\mathbf{A}^+$ *to* $\mathbf{B}$ *with respect to* $h$ *is nonempty and unique up to isomorphism over* $\mathbf{B}$.

**Proof.** Let $\mathcal{B}^+$ be the collection of arithmetic liftings of $\mathbf{A}|\mathbf{A}^+$ to $\mathbf{B}$ with respect to $h$. By the previous lemma, we may assume that $\mathbf{B}$ is a closed substructure of $\mathcal{R}_0$, $\mathbf{A}$ is a closed substructure of $\mathbf{B}$ and $h$ is the inclusion map. Moreover, we may assume that the indecomposables in $\mathbf{B}$ are sufficiently spread out. For example, it is enough to assume that there are at least $card(|\mathbf{A}^+|)$ many indecomposables of $\mathcal{R}_0$ between $(-\infty, \beta)^{\mathbf{B}}$ and $\beta$ for any indecomposable $\beta$ of $\mathbf{B}$ as well as above all indecomposables in $\mathbf{B}$.

Let $\alpha_1, \ldots, \alpha_n$ list the indecomposables of $\mathbf{A}$ in increasing order. For $i = 1, \ldots, n$ let $I_i$ consist of all indecomposables $a$ of $\mathbf{A}^+$ such that

$$(-\infty, \alpha_i)^{\mathbf{A}} \prec^{\mathbf{A}^+} a \prec^{\mathbf{A}^+} \alpha_i$$

and let $I_\infty$ consist of all indecomposables of $\mathbf{A}^+$ which are above all indecomposables of $\mathbf{A}$.

Let $f$ be an order preserving function mapping the indecomposables of $\mathbf{A}^+$ into those of $\mathcal{R}_0$ such that

- For $i = 1, \ldots, n$, $f(\alpha_i) = \alpha_i$.
- For $i = 1, \ldots, n$, $f$ maps $I_i$ between $(-\infty, \alpha_i)^{\mathbf{B}}$ and $\alpha_i$.
- $f$ maps each element of $I_\infty$ above all indecomposables of $\mathbf{B}$.

By Lemma 4.5, there is a unique extension $h^+$ of $f$ to $\mathbf{A}^+$. Since $h^+$ fixes the indecomposables of $\mathbf{A}$, $h^+$ is the identity on $|\mathbf{A}|$ by Lemma 4.4. Let $\mathbf{B}^+$ be the substructure of $\mathcal{R}_0$ whose universe is the union of the range of $h^+$ and $|\mathbf{B}|$. Checking that $h^+$ is a lifting map for $(\mathbf{A}^+, \mathbf{A}, h, \mathbf{B}, \mathbf{B}^+)$ is straightforward.

To verify the uniqueness of $\mathcal{B}^+$ up to isomorphism over $\mathbf{B}$, assume $\mathbf{B}' \in \mathcal{B}^+$. Let $h'$ be a lifting map for $(\mathbf{A}^+, \mathbf{A}, h, \mathbf{B}, \mathbf{B}')$. Define a function $g$ from the indecomposables of $\mathbf{B}'$ into those of $\mathbf{B}^+$ so that

$$g(b) = b$$

if $b \in |\mathbf{B}|$ and

$$g(h'(a)) = h^+(a)$$

if $a$ is an indecomposable of $\mathbf{A}^+$. To see that this definition is proper, notice that if $b$ is an indecomposable of $\mathbf{B}$ and $a$ is an indecomposable of $\mathbf{A}^+$ with $b = h'(a)$ then $a \in |\mathbf{A}|$ implying $b = h'(a) = a = h^+(a)$. It is not difficult to verify that $g$ is an order preserving map onto the indecomposables of $\mathbf{B}^+$. By Lemma 4.5, there is a closed embedding $g'$ of $\mathbf{B}'$ into $\mathcal{R}_0$ which extends $g$. Since $g$ is the identity on the indecomposables of $\mathbf{B}$, $g'$ is the identity on $|\mathbf{B}|$. Since $g' \circ h'$ agrees with $h^+$ on the indecomposables of $\mathbf{A}^+$, $g' \circ h' = h^+$. Hence, $g'$ maps $h'[|\mathbf{A}^+|]$ onto $h^+[|\mathbf{A}^+|]$. We conclude that the range of $g'$ is the universe of $\mathbf{B}^+$. Hence, $g'$ is an isomorphism of $\mathbf{B}'$ and $\mathbf{B}^+$ over $\mathbf{B}$. **QED**

# 12 Liftings for Patterns

**Definition 12.1** Assume $\mathbf{P}$, $\mathbf{P}^+$, $\mathbf{Q}$ and $\mathbf{Q}^+$ are patterns and $h$ is a closed embedding of $\mathbf{P}$ in $\mathbf{Q}$. $(\mathbf{P}^+, \mathbf{P}, h, \mathbf{Q}, \mathbf{Q}^+)$ is a *possible lifting* if $\mathbf{P}$ is a closed substructure of $\mathbf{P}^+$, every indecomposable of $\mathbf{P}^+$ is bounded above by an indecomposable of $\mathbf{P}$ and $\mathbf{Q}$ is a closed substructure of $\mathbf{Q}^+$. Let $\mathbf{A}$, $\mathbf{A}^+$, $\mathbf{B}$ and $\mathbf{B}^+$ be the arithmetic parts of $\mathbf{P}$, $\mathbf{P}^+$, $\mathbf{Q}$ and $\mathbf{Q}^+$ respectively. If $(\mathbf{P}^+, \mathbf{P}, h, \mathbf{Q}, \mathbf{Q}^+)$ is a possible lifting then a closed embedding $h^+$ of $\mathbf{P}^+$ into $\mathbf{Q}^+$ is a *lifting map for* $(\mathbf{P}^+, \mathbf{P}, h, \mathbf{Q}, \mathbf{Q}^+)$ if $h^+$ is a lifting map for

$(\mathbf{A}^+, \mathbf{A}, h, \mathbf{B}, \mathbf{B}^+)$. $(\mathbf{P}^+, \mathbf{P}, h, \mathbf{Q}, \mathbf{Q}^+)$ is a *lifting* if it has a lifting map. If $(\mathbf{P}^+, \mathbf{P}, h, \mathbf{Q}, \mathbf{Q}^+)$ is a lifting we say that $\mathbf{Q}^+$ is a lifting of $\mathbf{P}|\mathbf{P}^+$ to $\mathbf{Q}$ with respect to $h$.

When $\mathbf{P}$ is a closed substructure of $\mathbf{P}^+$ and $\mathbf{Q}^+$ is a lifting of $\mathbf{P}|\mathbf{P}^+$ with respect to the inclusion map of $|\mathbf{P}|$ in $|\mathbf{Q}|$ we will simply say $\mathbf{Q}^+$ is a lifting of $\mathbf{P}|\mathbf{P}^+$ to $\mathbf{Q}$.

**Example 12.2** Even if the assumptions of the definition hold, liftings may fail to exist. Assume $\mathcal{L}_0$ is the language consisting only of the relation symbol $\preceq$. Let $\mathbf{P}^+$ have universe $\{0, 2\}$ where $0 \preceq_1^{\mathbf{P}^+} 2$ and $0 \not\preceq_2^{\mathbf{P}^+} 2$. Let $\mathbf{Q}$ be the pattern with universe $\{0, 1\}$ where $0 \preceq^{\mathbf{Q}} 1$ and $0 \not\preceq_1^{\mathbf{Q}} 1$. Let $\mathbf{P}$ be the substructure of $\mathbf{Q}$ (and $\mathbf{P}^+$) with universe $\{0\}$. Assume $\mathbf{Q}^+$ is a lifting of $\mathbf{P}^+$ to $\mathbf{Q}$ with respect to $h$. We may assume $\mathbf{Q}^+$ has universe $\{0, 1, 2\}$ and $h^+$ is the identity map. We must have $0 \preceq^{\mathbf{Q}^+} 1 \preceq^{\mathbf{Q}^+} 2$, $0 \preceq_1^{\mathbf{Q}^+} 2$ and $0 \not\preceq_1^{\mathbf{Q}^+} 1$ which is impossible.

**Lemma 12.3** *Assume* $(\mathbf{P}_k^+, \mathbf{P}_k, h_k, \mathbf{Q}_k, \mathbf{Q}_k^+)$ *is a possible lifting for* $k = 1, 2$. *If* $f$ *is an isomorphism of* $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *which extends to an isomorphism of* $\mathbf{P}_1^+$ *and* $\mathbf{P}_2^+$ *and* $g$ *is an isomorphism of* $\mathbf{Q}_1$ *and* $\mathbf{Q}_2$ *which extends to an isomorphism of of* $\mathbf{Q}_1^+$ *and* $\mathbf{Q}_2^+$ *and has the property that*

$$g \circ h_1 = h_2 \circ f$$

*then*

$$(\mathbf{P}_1^+, \mathbf{P}_1, h_1, \mathbf{Q}_1, \mathbf{Q}_1^+) \textit{ is a lifting} \quad \textit{iff} \quad (\mathbf{P}_2^+, \mathbf{P}_2, h_2, \mathbf{Q}_2, \mathbf{Q}_2^+) \textit{ is a lifting}$$

**Proof.** By Lemma 11.3. **QED**

**Definition 12.4** Assume $(\mathbf{P}^+, \mathbf{P}, h, \mathbf{Q}, \mathbf{Q}^+)$ is a lifting. $\mathbf{Q}^+$ is a *minimal lifting of* $\mathbf{P}|\mathbf{P}^+$ *to* $\mathbf{Q}$ *with respect to* $h$ if $\mathbf{Q}^*$ is a cover of $\mathbf{Q}^+$ whenever $\mathbf{Q}^*$ is a lifting of $\mathbf{P}|\mathbf{P}^+$ to $\mathbf{Q}$ with respect to $h$ which has the same arithmetic part as $\mathbf{Q}^+$.

**Lemma 12.5** *Assume* $(\mathbf{P}^+, \mathbf{P}, h, \mathbf{Q}, \mathbf{Q}^+)$ *is a possible lifting. If there is a lifting* $\mathbf{Q}^+$ *of* $\mathbf{P}|\mathbf{P}^+$ *to* $\mathbf{Q}$ *with respect to* $h$ *then there is a minimal lifting of* $\mathbf{P}|\mathbf{P}^+$ *to* $\mathbf{Q}$ *with respect to* $h$ *which has the same arithmetic part as* $\mathbf{Q}^+$. *Moreover, minimal liftings of* $\mathbf{P}|\mathbf{P}^+$ *to* $\mathbf{Q}$ *with respect to* $h$ *are unique up to isomorphism over* $\mathbf{Q}$.

**Proof.** If there is a lifting of $\mathbf{P}|\mathbf{P}^+$ to $\mathbf{Q}$ with respect to $h$ which has arithmetic part $\mathbf{B}^+$ then the greatest lower bound (with respect to the covering relation) of the family of all such liftings is easily seen to be a minimal lifting of $\mathbf{P}|\mathbf{P}^+$ to $\mathbf{Q}$ with respect to $h$.

Uniqueness up to isomorphism over $\mathbf{Q}$ follows from the previous lemma and Lemma 11.4. **QED**

**Lemma 12.6** *Assume $\mathbf{P}^+$ is a continuous extension of $\mathbf{P}$ and $h$ is a continuous embedding of $\mathbf{P}$ in $\mathbf{Q}$.*

1. *There is a lifting of $\mathbf{P}|\mathbf{P}^+$ to $\mathbf{Q}$ with respect to $h$.*
2. *Any minimal lifting of $\mathbf{P}|\mathbf{P}^+$ to $\mathbf{Q}$ with respect to $h$ is a continuous extension of $\mathbf{Q}$.*
3. *If $\mathbf{P}^+$ is an extension of $\mathbf{P}$ at $a$ and $\mathbf{Q}^+$ is a lifting of $\mathbf{P}|\mathbf{P}^+$ to $\mathbf{Q}$ with respect to $h$ then $\mathbf{Q}^+$ is an extension of $\mathbf{Q}$ at $a$.*
4. *If $\mathbf{P}^+$is an arithmetic extension of $\mathbf{P}$ and $\mathbf{Q}^+$ is a lifting of $\mathbf{P}|\mathbf{P}^+$ to $\mathbf{Q}$ with respect to $h$ then $\mathbf{Q}^+$ is an arithmetic extension of $\mathbf{Q}$.*

**Proof.** Let $\mathbf{A}$, $\mathbf{A}^+$ and $\mathbf{B}$ be the arithmetic parts of $\mathbf{P}$, $\mathbf{P}^+$ and $\mathbf{Q}$ respectively. Let $\mathbf{B}^+$ be a lifting of $\mathbf{A}|\mathbf{A}^+$ to $\mathbf{B}$ with respect to $h$ and let $h^+$ be the lifting map for $(\mathbf{A}^+, \mathbf{A}, h, \mathbf{B}, \mathbf{B}^+)$.

Let $\overline{\mathbf{P}}$ be the pattern which is isomorphic to $\mathbf{P}$ whose arithmetic part is the substructure of $\mathbf{B}^+$ with universe $h[|\mathbf{P}|]$ and let $\overline{\mathbf{P}^+}$ be the pattern which is isomorphic to $\mathbf{P}^+$ whose arithmetic part is the substructure of $\mathbf{B}^+$ with universe $h^+[|\mathbf{P}^+|]$. By Lemma 12.3, we may assume that $\mathbf{P} = \overline{\mathbf{P}}$, $\mathbf{P}^+ = \overline{\mathbf{P}^+}$ and $h$ is the inclusion map of $\overline{\mathbf{P}}$ in $\mathbf{Q}$.

Notice that $|\mathbf{B}^+| = |\mathbf{A}^+| \cup |\mathbf{B}|$.

We collect some useful facts in the following claim.

**Claim 1:**

1. Any indecomposable in $|\mathbf{A}^+| \cap |\mathbf{B}|$ is in $|\mathbf{A}|$.
2. For any indecomposable $x$ in $|\mathbf{A}^+|$ there is an indecomposable $a$ in $|\mathbf{A}|$ such that $(-\infty, a)^{\mathbf{B}} \prec^{\mathbf{B}^+} x \preceq^{\mathbf{B}^+} a$ i.e. the least element of $|\mathbf{B}|$ which is an upper bound for $x$ is an indecomposable in $|\mathbf{A}|$.

3. If $x \in |\mathbf{A}^+| \cap |\mathbf{B}|$ and $a$ is the largest indecomposable in $|\mathbf{B}^+|$ with $a \preceq x$ then $a \in |\mathbf{A}|$.

Parts 1 and 2 of the claim follows from the choice of $\mathbf{B}^+$. Part 3 follows from part 1 and Lemma 4.7.

**Claim 2:** Assume $u, v \in |\mathbf{Q}| \cap |\mathbf{P}^+|$. For $k = 1, 2$, $u \preceq_k^{\mathbf{Q}} v$ iff $u \preceq_k^{\mathbf{P}^+} v$.

If $u = v$ then each side of both equivalences are true. So, we may assume that $u \neq v$.

Let $k = 1$. If $u$ is not indecomposable then both sides of the equivalence are false. Hence, we may assume that $u$ is indecomposable impling that $u \in |\mathbf{P}|$.

Assume $u \preceq_1^{\mathbf{Q}} v$. Since $\mathbf{Q}$ is a continuous extension of $\mathbf{P}$, there is $p \in |\mathbf{P}|$ such that $u \preceq^{\mathbf{Q}} v \preceq^{\mathbf{Q}} p$ and $u \preceq_1^{\mathbf{Q}} p$. Since $u, p \in |\mathbf{P}|$, $u \preceq_1^{\mathbf{P}} p$ implying $u \preceq_1^{\mathbf{P}^+} p$.

A similar argument using the fact that $\mathbf{P}^+$ is a continuous extension of $\mathbf{P}$ shows that $u \preceq_1^{\mathbf{P}^+} v$ implies $u \preceq_1^{\mathbf{Q}} v$.

Now let $k = 2$. If either $u$ or $v$ is not indecomposable then both sides of the equivalence are false. So, we may assume that both $u$ and $v$ are indecomposable. This implies that $u, v \in |\mathbf{P}|$. Therefore,

$$\begin{aligned} u \preceq_2^{\mathbf{Q}} v \quad &\text{iff} \quad u \preceq_2^{\mathbf{P}} v \\ &\text{iff} \quad u \preceq_2^{\mathbf{P}^+} v \end{aligned}$$

For $k = 1, 2$, $u \in |\mathbf{Q}|$ and $v \in |\mathbf{P}^+|$, define $u \lll_k v$ iff $u \preceq^{\mathbf{Q}^+} v$ and there is $p \in |\mathbf{P}|$ such that $v \preceq_{k-1}^{\mathbf{P}^+} p$ and $u \preceq_k^{\mathbf{Q}} p$.

Claim 2 allows us to define a structure $\mathbf{Q}^+$ whose arithmetic part is $\mathbf{B}^+$ so that

1. $\mathbf{P}^+$ is a substructure of $\mathbf{Q}^+$.
2. $\mathbf{Q}$ is a substructure of $\mathbf{Q}^+$.
3. For all $u \in |\mathbf{Q}^+|$, $u \preceq_k^{\mathbf{Q}^+} u$ for $k = 1, 2$.
4. Assume $x \in |\mathbf{P}^+| \setminus |\mathbf{Q}|$ and $y \in |\mathbf{Q}| \setminus |\mathbf{P}^+|$.
   (a) For $k = 1, 2$, $y \preceq_k^{\mathbf{Q}^+} x$ iff
   $$y \lll_k a \quad \text{and} \quad a \preceq_k^{\mathbf{P}^+} x$$
   for some $a \in |\mathbf{P}^+|$.

(b) For $k = 1, 2$, $x \preceq_k^{\mathbf{Q}^+} y$ iff

$$x \preceq_k^{\mathbf{P}^+} p \quad \text{and} \quad p \preceq_k^{\mathbf{Q}} y$$

for some $p \in |\mathbf{P}|$.

We will show that $\mathbf{Q}^+$ is the minimal lifting of $\mathbf{P}|\mathbf{P}^+$ to $\mathbf{Q}$ with respect to the inclusion of $\mathbf{P}$ in $\mathbf{Q}$ with arithmetic part $\mathbf{B}^+$ and that $\mathbf{Q}^+$ is a continuous extension of $\mathbf{Q}$. Notice that once we establish that $\mathbf{Q}^+$ is a pattern, it follows immediately that $\mathbf{Q}^+$ is the minimal lifting of $\mathbf{P}|\mathbf{P}^+$ to $\mathbf{Q}$ with arithmetic part $\mathbf{B}^+$ (the lifting map is the inclusion of $|\mathbf{P}^+|$ in $|\mathbf{Q}^+|$).

We abbreviate $u \preceq_k^{\mathbf{Q}^+} v$ as $u \preceq_k v$ for the remainder of the proof.

Since both $\mathbf{P}^+$ and $\mathbf{Q}$ are continuous extensions of $\mathbf{P}$, Lemma 7.5 easily implies that if $x, y \in |\mathbf{Q}^+|$ and $x \prec_k y$ for either $k = 1$ or $k = 2$ then there is an element $p$ of $|\mathbf{P}|$ such that $y \preceq p$.

To establish that $\mathbf{Q}^+$ is a pattern, we will show that conditions 1-7 of Definition 5.6 hold.

Condition 1, that the arithmetic part of $\mathbf{Q}^+$ is an arithmetic structure, follows from the choice of $\mathbf{B}^+$.

To verify conditions 2 and 3, assume $u, v \in |\mathbf{Q}^+|$. We claim

$$u \prec_1 v \implies u \text{ is indecomposable}$$

and

$$u \prec_2 v \implies v \text{ is indecomposable}$$

Since both $\mathbf{P}^+$ and $\mathbf{Q}$ are patterns and closed substructures of $\mathbf{Q}^+$, we may assume that $\{u, v\}$ is not a subset of either $|\mathbf{P}^+|$ or $|\mathbf{Q}|$. The remaining cases are also straightforward.

**Claim 3:** Assume $u \in |\mathbf{Q}|$ and $v \in |\mathbf{P}^+|$.

1. For $k = 1, 2$ and any $u' \in |\mathbf{Q}|$, if $u' \preceq_k u$ and $u \prec\!\prec_k v$ then $u' \prec\!\prec_k v$.
2. For $k = 1, 2$ and any $v' \in |\mathbf{P}^+|$, if $u \preceq v' \preceq_{k-1} v$ and $u \prec\!\prec_k v$ then $u \prec\!\prec_k v'$.
3. For $k = 1, 2$, if $u \prec\!\prec_k v$ then $v \preceq_{k-1} p$ and $u \preceq_k p$ where $p$ is the least element of $|\mathbf{P}|$ such that $v \preceq p$.
4. Assume $u \neq v$.

$$\begin{aligned} u \preceq_1 v &\iff \exists a \in |\mathbf{P}^+| \text{ such that } u \prec\!\prec_1 a \preceq_1 v \\ &\iff u \prec\!\prec_1 v \end{aligned}$$

5. Assume $u \neq v$.

$$u \preceq_2 v \iff \exists a \in |\mathbf{P}^+| \text{ such that } u \lessdot_2 a \preceq_2 v$$

6. Assume $u \neq v$. For $k = 0, 1, 2$, if $p$ is minimal in $|\mathbf{P}|$ such that $v \preceq p$ then

$$u \preceq_k v \implies u \preceq_k p$$

7. Assume $u \neq v$. Assume $p$ is minimal in $|\mathbf{P}|$ such that $v \preceq p$. For $k = 1, 2$,

$$\begin{aligned} v \preceq_k u &\iff \exists a \in |\mathbf{P}| \text{ such that } v \preceq_k a \preceq_k u \\ &\iff v \preceq_k p \preceq_k u \end{aligned}$$

8. Assume $u \neq v$. For $k = 0, 1, 2$, if $p$ is minimal in $|\mathbf{P}|$ such that $u \preceq p$ then

$$v \preceq_k u \implies v \preceq_k p$$

(Part 1) Straightforward.

(Part 2) Straightforward.

(Part 3) Assume $k \in \{1, 2\}$, $a \in |\mathbf{P}|$, $v \preceq_{k-1} a$ and $u \preceq_k a$. Let $p$ be the least element of $\mathbf{P}$ such that $v \preceq p$. Since $\mathbf{P}^+$ is a continuous extension of $\mathbf{P}$, $p \preceq_{k-1} a$ implying that $v \preceq_{k-1} p$ and, since $u \preceq_k a$, $u \preceq_k p$.

We will need the following subclaim in parts 4, 5 and 6.

**Subclaim:** Under our assumptions that $u \in \mathbf{Q}$ and $v \in \mathbf{P}^+$, if $k \in \{1, 2\}$ and there exists $a \in |\mathbf{P}^+|$ such that $u \lessdot_k a \preceq_k v$ then $u \preceq_k p$ where $p$ is the least element of $|\mathbf{P}|$ such that $v \preceq p$.

To prove the subclaim, assume $k \in \{1, 2\}$ and $a \in |\mathbf{P}^+|$ and $u \lessdot_k a \preceq_k v$. By part 3, $u \preceq_k p'$ and $a \preceq_{k-1} p'$ where $p'$ is the least element of $\mathbf{P}$ such that $a \preceq p'$. If $v \preceq p'$ then we are done, so suppose $p' \prec v$. Since $\mathbf{P}^+$ is a continuous extension of $\mathbf{P}$, $p' \preceq_k p$ where $p$ is the least element of $|\mathbf{P}|$ such that $v \preceq p$. Since $u, p', p \in |\mathbf{Q}|$, $u \preceq_k p$.

(Part 4) Assume $u \neq v$. We first notice that the last two conditions are equivalent. Clearly, the third condition implies the second. By the Subclaim, the second condition implies the third.

Since $u \neq v$, all three conditions are false if $u$ is not indecomposable. So, we may assume $u$ is indecomposable.

First assume that $u \in |\mathbf{P}^+|$. By part 1 of Claim 1, $u \in |\mathbf{P}|$. Since $\mathbf{P}^+$ is a continuous extension of $\mathbf{P}$, $u \preceq_1 v$ implies $u \lll_1 v$. Since $u, v \in \mathbf{P}^+$, $u \lll_1 v$ implies $u \preceq_1 v$.

Now assume that $u \notin |\mathbf{P}^+|$. We will consider the cases $v \in |\mathbf{Q}|$ and $v \notin |\mathbf{Q}|$.

Assume $v \in |\mathbf{Q}|$.

We first show $u \preceq_1 v$ implies $u \lll_1 v$. Assume $u \preceq_1 v$. By part 3 of Claim 1, $p \preceq v$ where $p$ is the least element of $|\mathbf{P}|$ such that $u \preceq p$. Since $\mathbf{Q}$ is a continuous extension of $\mathbf{P}$, $p \preceq_1 p'$ where $p'$ is the least element of $|\mathbf{P}|$ such that $v \preceq p'$. Since $u, p, v \in |\mathbf{Q}|$, $u \preceq_1 p$. Since $u, p, p' \in |\mathbf{Q}|$, this implies that $u \preceq_1 p'$. Therefore, $u \lll_1 v$.

Now assume $u \lll_1 v$. There exists $p \in |\mathbf{P}|$ such that $v \preceq p$ and $u \preceq_1 p$. Since $u, v, p \in |\mathbf{Q}|$, $u \preceq_1 v$.

Finally, assume $v \notin |\mathbf{Q}|$.

In this case, the first two conditions are equivalent by the definition of $\mathbf{Q}^+$.

(Part 5) Assume $u \neq v$. Both conditions are false if either $u$ or $v$ is not indecomposable. So, we may assume $u$ and $v$ are indecomposable.

If $u \in |\mathbf{P}^+|$ then $u \in |\mathbf{P}|$ by part 1 of Claim 1 implying $u \lll_2 u \preceq_2 v$. If $v \in |\mathbf{Q}|$ then $v \in |\mathbf{P}|$ implying $u \lll_2 v$ so that $u \lll_2 v \preceq_2 v$. So, we may assume that $u \notin |\mathbf{P}^+|$ and $v \notin |\mathbf{Q}|$. The equivalence of the two conditions now follows from the definition of $\mathbf{Q}^+$.

(Part 6) Assume $u \neq v$. Also suppose $k \in \{0, 1, 2\}$, and $p$ is minimal in $|\mathbf{P}|$ such that $v \preceq p$ and $u \preceq_k v$. For $k = 0$, $u \preceq_0 p$ is immediate. For $k = 1$, $u \preceq_1 p$ follows from part 4. Suppose $k = 2$. By part 5, there exists $a \in |\mathbf{P}^+|$ such that $u \lll_2 a \preceq_2 v$. By the Subclaim, $u \preceq_2 p$.

(Part 7) Assume $u \neq v$. Also suppose $k \in \{1, 2\}$. We begin by showing the last two conditions are equivalent. The third condition trivially implies the second. To show that the second condition implies the third, assume $a \in |\mathbf{P}|$ and $v \preceq_k a \preceq_k u$. Since $\mathbf{P}^+$ is a continuous extension of $\mathbf{P}$, $p \preceq_k a$. Since $v, p, a \in |\mathbf{P}^+|$, $v \preceq_k p$. Since $p, a, u \in |\mathbf{Q}|$, $p \preceq_k u$.

If $v$ is not indecomposable, then each of the three conditions is false. So we may assume $v$ is indecomposable. If $v \in |\mathbf{Q}|$ then by part 3 of Claim 1, $v \in |\mathbf{P}|$ implying $v = p$ which in turn implies that $v \preceq_k u$ is equivalent to $v \preceq_k p \preceq_k u$. So, we may also assume $v \notin |\mathbf{Q}|$. If $u \notin |\mathbf{P}^+|$ then the equivalence of the first two conditions follows from the definition of $\mathbf{Q}^+$. So, we may assume that $u \in |\mathbf{P}^+|$. Since $v, p, u \in |\mathbf{P}^+|$, $v \preceq_k p \preceq_k u$ implies $v \preceq_k u$. It remains to show that $v \preceq_k u$ implies $v \preceq_k p \preceq_k u$.

Assume $v \preceq_k u$. Since $\mathbf{P}^+$ is a continuous extension of $\mathbf{P}$, $p \preceq_k p'$ where $p'$ is the least element of $|\mathbf{P}|$ such that $u \preceq p'$.

First suppose that $k = 1$. Since $v, p, u, p' \in |\mathbf{P}^+|$, $v \preceq_1 p$ and $p \preceq_1 u$.

Now suppose that $k = 2$. $u$ is indecomposable and an element of $|\mathbf{P}^+| \cap |\mathbf{Q}|$. By part 1 of Claim 1, $u \in |\mathbf{P}|$. Hence, $u = p'$ implying $p \preceq_2 u$. Since $v, p, u \in |\mathbf{P}^+|$, $v \preceq_2 p$.

(Part 8) Assume $u \neq v$. Also suppose $k \in \{0, 1, 2\}$ and $p$ is the least element of $|\mathbf{P}|$ such that $u \preceq p$ and assume that $v \preceq_k u$. We will show that $v \preceq_k p$. This is trivial if $k = 0$. So, we may assume $k \neq 0$. Let $p'$ be the least element of $|\mathbf{P}|$ such that $v \preceq p'$. By part 7, $v \preceq_k p' \preceq_k u$. Since $\mathbf{Q}$ is a continuous extension of $\mathbf{P}$, $p' \preceq_k p$. Since $v, p', p \in |\mathbf{P}^+|$, $v \preceq_k p$.

**Claim 4:** $\mathbf{Q}^+$ is a continuous extension of $\mathbf{Q}$.

Assume $k \in \{1, 2\}$, $x \preceq_k y$, $q \in |\mathbf{Q}|$ and $(-\infty, q)^{\mathbf{Q}} \prec x \preceq q \prec y$. Let $p$ be the least element of $|\mathbf{P}|$ such that $y \preceq p$ and let $q'$ be the least element of $|\mathbf{Q}|$ such that $y \preceq q'$. Notice that $q' \preceq p$. We will show that $q \preceq_k q'$. We will argue by cases depending on whether each of $x$ and $y$ are in $|\mathbf{Q}|$ or not.

Assume $x, y \in |\mathbf{Q}|$. In this case, $q = x$ and $q' = y$ so that $q \preceq_k q'$.

Assume $x \in |\mathbf{Q}|$ and $y \notin |\mathbf{Q}|$. In this case, $q = x$. By part 6 of Claim 3, $q \preceq_k p$. For $k = 1$, this implies $q \preceq_k q'$ since $q, q', p \in |\mathbf{Q}|$. If $k = 2$ then $y$ is an indecomposable element of $|\mathbf{P}^+|$ implying, by part 2 of Claim 1, $q' = p$ so that $q \preceq_k q'$.

Assume $x \notin |\mathbf{Q}|$ and $y \in |\mathbf{Q}|$. In this case, $y = q'$. By part 2 of Claim 1, $q \in |\mathbf{P}|$. By part 7 of Claim 3, $x \preceq_k q \preceq_k y$. In particular, $q \preceq_k y = q'$.

Assume $x \notin |\mathbf{Q}|$ and $y \notin |\mathbf{Q}|$. By part 2 of Claim 1, $q \in |\mathbf{P}|$. Since $x, y \in |\mathbf{P}^+|$ and $\mathbf{P}^+$ is a continuous extension of $\mathbf{P}$, $q \preceq_k p$. For $k = 1$, this implies that $q \preceq_k q'$ since $q, q', p \in |\mathbf{Q}|$ and $q \preceq q' \preceq p$. For $k = 2$, $y$ is indecomposable so that part 2 of Claim 1 implies $q' = p$ and, hence, $q \preceq_k q'$.

To verify conditions 4-7 of Definition 5.6, first notice that $\preceq_1^{\mathbf{Q}^+}$ is contained in $\preceq^{\mathbf{Q}^+}$ and $\preceq_2^{\mathbf{Q}^+}$ is contained in $\preceq_1^{\mathbf{Q}^+}$.

$\preceq_1^{\mathbf{Q}^+}$ and $\preceq_2^{\mathbf{Q}^+}$ are reflexive by definition and they are easily seen to be antisymmetric. Hence, to verify conditions 4 and 5, which state that $\preceq_1$ and $\preceq_2$ are partial orderings, it is enough to show they are both transitive. First notice that for $k = 1, 2$, all $u, u' \in |\mathbf{Q}|$ and all $v, v' \in |\mathbf{P}^+|$

$$u' \preceq_k u \preceq_k v \preceq_k v' \implies u' \preceq_k v'$$

and

$$v' \preceq_k v \preceq_k u \preceq_k u' \implies v' \preceq_k u'$$

The first implication follows from parts 1, 4 and 5 of Claim 3. The second follows from part 7 of Claim 3. To verify conditions 4 and 5, assume $k \in \{1, 2\}$ and $u \preceq_k v \preceq_k w$. We will show $u \preceq_k w$ by considering the cases depending on whether each of $u$, $v$ and $w$ are in $|\mathbf{P}^+|$ or $|\mathbf{Q}|$ (of course, these cases are not exclusive). Since $u \preceq_k w$ follows trivially if either $u = v$ or $v = w$, we may assume that $u$, $v$ and $w$ are distinct.

First consider the case when $u \in |\mathbf{P}^+|$, $v \in |\mathbf{Q}|$ and $w \in |\mathbf{P}^+|$. Let $p$ be the least element of $|\mathbf{P}|$ such that $u \preceq p$. By part 7 of Claim 3, $u \preceq_k p \preceq_k v$. By the first of the implications above, $p \preceq_k w$. Since $u, p, w \in |\mathbf{P}^+|$, $u \preceq_k w$.

Now consider the case when $u \in |\mathbf{Q}|$, $v \in |\mathbf{P}^+|$ and $w \in |\mathbf{Q}|$. Let $p$ be the least element of $|\mathbf{P}|$ such that $v \preceq p$. By part 7 of Claim 3, $v \preceq_k p \preceq_k w$. By the first of the implications above, $u \preceq_k p$. Since $u, p, w \in |\mathbf{Q}|$, $u \preceq_k w$.

The remaining cases follow immediately from the two implications above.

Conditions 6 and 7 say that for all $u, v, w \in |\mathbf{Q}^+|$

$$u \preceq v \preceq w \text{ and } u \preceq_1 w \implies u \preceq_1 v$$

and

$$u \preceq v \preceq_1 w \text{ and } u \preceq_2 w \implies u \preceq_2 v$$

Assume $k \in \{1, 2\}$, $u, v, w \in |\mathbf{Q}^+|$, $u \preceq v \preceq_{k-1} w$ and $u \preceq_k w$. Let $p$, $p'$ and $p''$ be the least elements of $|\mathbf{P}|$ such that $u \preceq p$, $v \preceq p'$ and $w \preceq p''$ respectively. We will show that $u \preceq_k v$. Since $u \preceq_k v$ follows trivially if either $u = v$ or $v = w$, we may assume that $u$, $v$ and $w$ are distinct.

Notice that $u$ is indecomposable.

We will consider several cases depending on whether each of $u$ and $v$ are in $|\mathbf{P}^+|$ or $|\mathbf{Q}|$. If either $u, v, w \in |\mathbf{Q}|$ or $u, v, w \in |\mathbf{P}^+|$ then $u \preceq_k v$ follows from the fact that $\mathbf{Q}$ and $\mathbf{P}^+$ are substructures of $\mathbf{Q}^+$ which are patterns. So, we may assume that at least one of $u$, $v$ or $w$ is not in $|\mathbf{Q}|$ and at least one is not in $|\mathbf{P}^+|$. We are left with the following 6 cases depending on whether each of $u$, $v$ and $w$ are in $|\mathbf{Q}|$ or $|\mathbf{P}|$.

**Case 1:** Assume $u \in |\mathbf{P}^+|$, $v \in |\mathbf{P}^+|$ and $w \in |\mathbf{Q}|$.

By part 8 of Claim 3, $u \preceq_k p''$ and $v \preceq_{k-1} p''$. Since $u, v, p'' \in |\mathbf{P}^+|$, $u \preceq_k v$.

**Case 2:** Assume $u \in |\mathbf{P}^+|$, $v \in |\mathbf{Q}|$ and $w \in |\mathbf{P}^+|$.

By part 2 of Claim 1, $p \preceq v$. By Lemma 7.2, $u \preceq_k p \preceq_k p''$. By part 6 of Claim 3, $v \preceq_{k-1} p''$. Since $p, v, p'' \in |\mathbf{Q}|$, $p \preceq_k v$. By part 7 of Claim 3, $u \preceq_k v$.

**Case 3:** Assume $u \in |\mathbf{P}^+|$, $v \in |\mathbf{Q}|$ and $w \in |\mathbf{Q}|$.

By part 2 of Claim 1, $p \preceq v$. By part 7 of Claim 3, $u \preceq_k p \preceq_k w$. Since $v \preceq_{k-1} w$ and $p, v, w \in |\mathbf{Q}|$, $p \preceq_k v$. By part 7 of Claim 3, $u \preceq_k v$.

**Case 4:** Assume $u \in |\mathbf{Q}|$, $v \in |\mathbf{P}^+|$ and $w \in |\mathbf{P}^+|$.

By parts 4 and 5 of Claim 3, there exists $a \in |\mathbf{P}^+|$ such that $u \lll_k a \preceq_k w$.

First suppose $a \preceq v$. Since $a, v, w \in |\mathbf{P}^+|$, $a \preceq_k v$. By parts 4 and 5 of Claim 3, $u \preceq_k v$.

Now suppose $v \prec a$. Since $v, a, w \in |\mathbf{P}^+|$, $v \preceq_{k-1} a$. By part 2 of Claim 3, $u \lll_k v$ implying $u \preceq_k v$.

**Case 5:** Assume $u \in |\mathbf{Q}|$, $v \in |\mathbf{P}^+|$ and $w \in |\mathbf{Q}|$.

We first show there is $x \in |\mathbf{P}|$ such that $u \preceq x \preceq w$. Let $a$ be the largest indecomposable in $\mathbf{Q}^+$ such that $a \preceq v$. Notice $u \preceq a$. By part 1 of Lemma 4.7, $a \in |\mathbf{P}^+|$. Let $x$ be the least element of $|\mathbf{Q}|$ such that $a \preceq x$. Notice $x \preceq w$. By part 2 of Claim 1, $x \in |\mathbf{P}|$.

Since $\mathbf{Q}$ is a continuous extension of $\mathbf{P}$, Lemma 7.2 and the existence of $x \in |\mathbf{P}|$ with $u \preceq x \preceq w$ implies that $u \preceq_k p''$. By part 8 of Claim 3, $v \preceq p''$. Therefore, $u \lll_k v$ implying $u \preceq_k v$.

**Case 6:** Assume $u \in |\mathbf{Q}|$, $v \in |\mathbf{Q}|$ and $w \in |\mathbf{P}^+|$.

By part 6 of Claim 3, $u \preceq_k p''$ and $v \preceq_{k-1} p''$. Since $u, v, p'' \in |\mathbf{Q}|$, $u \preceq_k v$.

We have established part 1 of the lemma .

For part 2, assume that $\mathbf{Q}^*$ is a minimal lifting of $\mathbf{P}|\mathbf{P}^+$ to $\mathbf{Q}$. Since $\mathbf{Q}^+$ is a minimal lifting of $\mathbf{P}|\mathbf{P}^+$ to $\mathbf{Q}$, Lemma 12.5 implies that $\mathbf{Q}^*$ is isomorphic to $\mathbf{Q}^+$ over $\mathbf{Q}$. Since $\mathbf{Q}^+$ is a continuous extension of $\mathbf{Q}$, $\mathbf{Q}^*$ is a continuous extension of $\mathbf{Q}$.

For part 3, assume that $\mathbf{Q}^*$ is a lifting of $\mathbf{P}|\mathbf{P}^+$ to $\mathbf{Q}$ and $\mathbf{P}^+$ is an extension of $\mathbf{P}$ at $a$. Lemma 4.15 implies that that $\mathbf{Q}^*$ is an extension of $\mathbf{Q}$ at $a$.

Part 4 is straightforward. **QED**

# 13 Generating the Valid Rules

**Definition 13.1** Assume $\mathcal{D}$ is a family of continuous rules. $\mathcal{D}$ is *closed under compositions* if whenever $\mathbf{P}_1|\mathbf{P}_2$ and $\mathbf{P}_2|\mathbf{P}_3$ are in $\mathcal{D}$ then so is $\mathbf{P}_1|\mathbf{P}_3$.

**Lemma 13.2** *The family of cofinally valid continuous rules is closed under compositions.*

**Proof.** By part 2 of Lemma 6.4 and part 1 of Lemma 7.8. **QED**

**Definition 13.3** Assume $\mathcal{D}$ is a family of continuous rules. $\mathcal{D}$ is *closed under restrictions* if $\mathbf{P}_1|\mathbf{P}_2$ is in $\mathcal{D}$ whenever $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ are patterns such that

$\mathbf{P}_1$ is a closed substructure of $\mathbf{P}_2$, $\mathbf{P}_2$ is a closed substructure of $\mathbf{P}_3$ and $\mathbf{P}_1|\mathbf{P}_3$ is in $\mathcal{D}$.

**Lemma 13.4** *The family of cofinally valid continuous rules is closed under restrictions.*

**Proof.** Assume $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ are patterns such that $\mathbf{P}_1$ is a closed substructure of $\mathbf{P}_2$, $\mathbf{P}_2$ is a closed substructure of $\mathbf{P}_3$ and $\mathbf{P}_1|\mathbf{P}_3$ is cofinally valid. By part 2 of Lemma 7.6, $\mathbf{P}_1|\mathbf{P}_2$ is a continuous rule. It follows that $\mathbf{P}_1|\mathbf{P}_2$ is cofinally valid. **QED**

**Definition 13.5** Assume $\mathcal{D}$ is a family of continuous rules. $\mathcal{D}$ is *closed under upward 2-reflection* if $\mathbf{P}|\mathbf{P}^*$ is in $\mathcal{D}$ whenever there is a rule $\mathbf{P}|\mathbf{P}^+$ in $\mathcal{D}$ such that $\mathbf{P}|\mathbf{P}^*$ is obtained by 2-reflecting $\mathbf{P}|\mathbf{P}^+$ upward from $a$ to $b$ for some $a$ and $b$.

**Lemma 13.6** *The family of cofinally valid continuous rules is closed under upward 2-reflection.*

**Proof.** By Lemma 10.3. **QED**

**Definition 13.7** Assume $\mathcal{D}$ is a family of continuous rules. $\mathcal{D}$ is *closed under liftings* if whenever $\mathbf{P}|\mathbf{P}^+$ is in $\mathcal{D}$ and $h$ is a continuous embedding of $\mathbf{P}$ in $\mathbf{Q}$ then $\mathbf{Q}|\mathbf{Q}^+$ is in $\mathcal{D}$ for some $\mathbf{Q}^+$ which is a lifting of $\mathbf{P}|\mathbf{P}^+$ to $\mathbf{Q}$ with respect to $h$.

**Lemma 13.8** *The family of cofinally valid continuous rules is closed under liftings. More precisely, if $\mathbf{P}|\mathbf{P}^+$ is a cofinally valid continuous rule and $h$ is a continuous embedding of $\mathbf{P}$ in $\mathbf{Q}$ then $\mathbf{Q}|\mathbf{Q}^+$ is a cofinally valid continuous rule whenever $\mathbf{Q}^+$ is a minimal lifting of $\mathbf{P}|\mathbf{P}^+$ to $\mathbf{Q}$ with respect to $h$.*

**Proof.** Assume $\mathbf{P}|\mathbf{P}^+$ is a confinally valid continuous rule and $h$ is a continuous embedding of $\mathbf{P}$ in $\mathbf{Q}$. Let $\mathbf{Q}^+$ be a minimal lifting of $\mathbf{P}|\mathbf{P}^+$ to $\mathbf{Q}$ with respect to $h$ and let $h^+$ be the lifting map. By identifying $\mathbf{P}$ and $\mathbf{P}^+$ with their images under $h^+$, we may assume that $h^+$ is the identity map on $|\mathbf{P}^+|$.

Assume $f$ is a covering of $\mathbf{Q}$ into $\mathcal{R}_2$ and suppose $\varphi$ is a regressive function on the nonzero indecomposables in the range of $f$. We will show that there is a covering of $\mathbf{Q}^+$ in $\mathcal{R}_2$ which extends $f$ above $\varphi$. Without loss of generality, we may assume that for any indecomposable $x$ in $\mathbf{Q}$, $f[(-\infty, x)^{\mathbf{Q}}] \leq \varphi(f(x))$.

Since $\mathbf{P}|\mathbf{P}^+$ is cofinally valid, there is a covering $g$ of $\mathbf{P}^+$ in $\mathcal{R}_2$ which extends the restriction of $f$ to $\mathbf{P}$ above the restriction of $\varphi$ to the indecomposables in $f[|\mathbf{P}|]$.

The restriction of $f \cup g$ to the indecomposables of $\mathbf{Q}^+$ is order preserving. This map extends to a unique embedding of the arithmetic part of $\mathbf{Q}^+$ into $\mathcal{R}_0$ which must extend both $f$ and $g$. Hence, $f \cup g$ is an embedding of the arithmetic part of $\mathbf{Q}^+$ in $\mathcal{R}_0$.

We claim $f \cup g$ is a covering of $\mathbf{Q}^+$ into $\mathcal{R}_2$. To see this, let $\mathbf{Q}^*$ be the structure with the same arithmetic part as $\mathbf{Q}^+$ such that $f \cup g$ is an embedding of $\mathbf{Q}^*$ into $\mathcal{R}_2$. It suffices to show that $\mathbf{Q}^*$ is a covering of $\mathbf{Q}^+$. Since $f$ is a covering of $\mathbf{Q}$ and $g$ is a covering of $\mathbf{P}^+$, the intersection of $\mathbf{Q}^+$ and $\mathbf{Q}^*$ is a lifting of $\mathbf{P}|\mathbf{P}^+$ to $\mathbf{Q}$. Since $\mathbf{Q}^+$ is a minimal lifting, this implies that $\mathbf{Q}^*$ is a covering of $\mathbf{Q}^+$.

To see that $f \cup g$ extends $f$ above $\varphi$, assume $x \in |\mathbf{Q}^+| \setminus |\mathbf{Q}|$ is indecomposable in $\mathbf{Q}^+$. Any such $x$ must be an indecomposable of $\mathbf{P}^+$ in $|\mathbf{P}^+| \setminus |\mathbf{P}|$. Let $a$ be the least indecomposable of $\mathbf{P}$ such that $x \prec^{\mathbf{P}^+} a$. It follows that $a$ is the least indecomposable $y$ of $\mathbf{Q}$ such that $x \prec^{\mathbf{Q}^+} y$. Therefore, $\varphi((f \cup g)(a)) = \varphi(g(a)) < g(x) = (f \cup g)(x)$. **QED**

**Definition 13.9** Assume $k \in \{1, 2\}$. A continuous rule $\mathbf{P}|\mathbf{P}^+$ is a *downward $k$-reflection rule* if $\mathbf{P}^+$ is obtained from $\mathbf{P}$ by $k$-reflecting $X$ downward from $b$ to $a$ for some $X$, $a$ and $b$.

**Definition 13.10** The family of *Generating Rules* is the smallest family $\mathcal{G}$ of continuous rules such that

1. $\mathcal{G}$ contains all rules $\mathbf{P}|\mathbf{P}^+$ where $\mathbf{P}^+$ is a 1-correct arithmetic extension of $\mathbf{P}$.

2. $\mathcal{G}$ contains all downward $k$-reflection rules for $k = 1, 2$.

3. $\mathcal{G}$ is closed under upward 2-reflection.

4. If $\mathbf{P} = \mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_n$ are patterns such that $\mathbf{P}_i|\mathbf{P}_{i+1}$ is in $\mathcal{G}$ for $i < n$ and $\mathbf{P}^+$ is a closed substructure of $\mathbf{P}_n$ which is an extension of $\mathbf{P}$ at $a$ for some $a$ then $\mathbf{P}|\mathbf{P}^+$ is in $\mathcal{G}$.

5. If $\mathbf{P}|\mathbf{P}^+$ is in $\mathcal{G}$ and $h$ is a continuous embedding of $\mathbf{P}$ in $\mathbf{Q}$ then $\mathbf{Q}|\mathbf{Q}^+$ is in $\mathcal{G}$ whenever $\mathbf{Q}^+$ is a minimal lifting of $\mathbf{P}|\mathbf{P}^+$ to $\mathbf{Q}$ with respect to $h$.

Notice any continuous rule of the form $\mathbf{P}|\mathbf{P}$ is a generating rule by part 1 of the definition.

**Lemma 13.11** *Assume* $\mathbf{P}|\mathbf{Q}$ *is a generating rule.*

1. *If* $\mathbf{P}^*$ *and* $\mathbf{Q}^*$ *are patterns such that* $\mathbf{P}^*$ *is a substructure of* $\mathbf{Q}^*$ *and there is an isomorhism of* $\mathbf{P}$ *and* $\mathbf{P}^*$ *which extends to an isomorphism of* $\mathbf{Q}$ *and* $\mathbf{Q}^*$ *then* $\mathbf{P}^*|\mathbf{Q}^*$ *is a generating rule.*

2. $\mathbf{P}$ *is a closed substructure of* $\mathbf{Q}$*,* $\mathbf{Q}$ *is a continuous extension of* $\mathbf{P}$ *and* $\mathbf{P}|\mathbf{Q}$ *is cofinally valid.*

**Proof.** Both parts can be proved by induction on the generation of the generating rules. Part 2 uses Lemmas 8.4, 9.3, 9.6, 10.3, 13.2, 13.4 and 13.8. **QED**

# 14 Generating the Core

In this section, we show how to generate any initial segment of the core as the union of an infinite chain of patterns.

**Definition 14.1** A pattern $\mathbf{P}$ is *covered* if there is a covering of $\mathbf{P}$ in $\mathcal{R}_2$.

**Definition 14.2** Assume $\mathbf{P}$ and $\mathbf{P}^+$ are patterns. $\mathbf{P}$ *exactly generates* $\mathbf{P}^+$ if there is a finite sequence of patterns $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_n$ such that $\mathbf{P}_0 = \mathbf{P}$, $\mathbf{P}_i|\mathbf{P}_{i+1}$ is a generating rule for $i < n$ and $\mathbf{P}_n = \mathbf{P}^+$. $\mathbf{P}$ *generates* $\mathbf{P}^+$ if $\mathbf{P}$ is a closed substructure of $\mathbf{P}^+$ and $\mathbf{P}$ exactly generates some pattern $\mathbf{Q}$ such that $\mathbf{P}^+$ is a closed substructure of $\mathbf{Q}$.

**Lemma 14.3** *Assume* $\mathbf{P}$ *and* $\mathbf{Q}$ *are patterns. If* $\mathbf{P}$ *generates* $\mathbf{Q}$ *then* $\mathbf{P}$ *is a closed substructure of* $\mathbf{Q}$ *and* $\mathbf{Q}$ *is a continuous extension of* $\mathbf{P}$*.*

**Proof.** By part 2 of Lemma 13.11, part 2 of Lemma 2.4 and Lemma 7.6. **QED**

**Lemma 14.4** *Assume* $\mathbf{P}^*$ *and* $\mathbf{Q}^*$ *are patterns such that* $\mathbf{P}^*$ *is a substructure of* $\mathbf{Q}^*$ *and there is an isomorhism of* $\mathbf{P}$ *and* $\mathbf{P}^*$ *which extends to an isomorphism of* $\mathbf{Q}$ *and* $\mathbf{Q}^*$*. If* $\mathbf{P}$ *generates* $\mathbf{Q}$ *then* $\mathbf{P}^*$ *generates* $\mathbf{Q}^*$*.*

**Proof.** By part 1 of Lemma 13.11. **QED**

**Lemma 14.5** *Assume* $\mathbf{P}$ *generates* $\mathbf{P}^+$*. Any covering of* $\mathbf{P}$ *in* $\mathcal{R}_2$ *extends to a covering of* $\mathbf{P}^+$ *in* $\mathcal{R}_2$*.*

**Proof.** By part 2 of Lemma 13.11. **QED**

Notice that if $\mathbf{P}^+$ is an extension of $\mathbf{P}$ at $a$ then $\mathbf{P}$ generates $\mathbf{P}^+$ iff $\mathbf{P}|\mathbf{P}^+$ is a generating rule. Hence, in this case, $\mathbf{P}$ generates $\mathbf{P}^+$ iff $\mathbf{P}$ exactly generates $\mathbf{P}^+$. We will establish this equivalence in general elsewhere.

**Lemma 14.6** *Assume* $\mathbf{P}$ *is a covered pattern. If* $\mathbf{P}$ *generates* $\mathbf{P}^+$ *and* $\mathbf{Q}$ *is a closed substructure of* $\mathbf{P}^+$ *which is a covering of* $\mathbf{P}$ *then* $|\mathbf{P}| \preceq_{pw}^{\mathbf{P}^+} |\mathbf{Q}|$*.*

**Proof.** Argue by contradiction and assume that $\mathbf{P}$ generates $\mathbf{P}^+$, $\mathbf{Q}$ is a closed substructure of $\mathbf{P}^+$ which is a covering of $\mathbf{P}$ and $\mathbf{P} \not\preceq_{pw}^{\mathbf{P}^+} \mathbf{Q}$. Choose $i$ such that the $i^{th}$ element of $\mathbf{Q}$ is less than the $i^{th}$ element of $\mathbf{P}$. Consider the nonempty collection of all ordinals which occur as the $i^{th}$ element of some substructure of $\mathcal{R}_2$ which is a cover of $\mathbf{P}$. By the previous lemma, this collection of ordinals is without a minimal element – contradiction. **QED**

**Definition 14.7** Assume $\mathbf{P}_n$ $(n \in \omega)$ is an increasing sequence of patterns such that $\mathbf{P}_n$ is a closed substructure of $\mathbf{P}_{n+1}$ for each $n \in \omega$. The sequence $\mathbf{P}_n$ $(n \in \omega)$ is *fair* if for each $n \in \omega$ and generating rule of the form $\mathbf{P}_n|\mathbf{P}^+$ there is $k \geq n$ such that $\mathbf{P}_{k+1}$ is a lifting of $\mathbf{P}_n|\mathbf{P}^+$ to $\mathbf{P}_k$.

**Lemma 14.8** *Assume* $\mathbf{P}_n$ $(n \in \omega)$ *is an increasing sequence of patterns such that* $\mathbf{P}_n|\mathbf{P}_{n+1}$ *is a generating rule for each* $n \in \omega$*. Let* $\mathbf{P}_\infty$ *be the union of* $\mathbf{P}_n$ $(n \in \omega)$*.*

1. $\mathbf{P}_\infty$ *is a model of reflection.*
2. $\mathbf{P}$ *is a closed substructure of* $\mathbf{P}_\infty$ *and* $\mathbf{P}_\infty$ *is a continuous extension of* $\mathbf{P}_n$ *for* $n \in \omega$*.*
3. *Any covering of* $\mathbf{P}_0$ *in* $\mathcal{R}_2$ *extends to a covering of* $\mathbf{P}_\infty$ *in* $\mathcal{R}_2$*.*
4. *If* $\mathbf{P}_0$ *is covered then* $(|\mathbf{P}_\infty|, \preceq^{\mathbf{P}_\infty})$ *is order isomorphic to an ordinal.*
5. *If* $\mathbf{P}_n$ $(n \in \omega)$ *is fair then the interpretation of each function symbol in* $\mathbf{P}_\infty$ *is total.*

6. *If* $\mathbf{P}_0$ *is covered and* $\mathbf{Q}$ *is a closed substructure of* $\mathbf{P}_\infty$ *which is a covering of* $\mathbf{P}_n$ *then* $|\mathbf{P}_n| \preceq_{pw}^{\mathbf{P}_\infty} |\mathbf{Q}|$.

7. *If* $\mathbf{P}_n$ $(n \in \omega)$ *is fair and* $k \in \{1, 2\}$ *then*

$$a \preceq_k^{\mathbf{P}_\infty} b \quad \Longrightarrow \quad a \preceq_k^{\infty} b$$

*for all* $a, b \in |\mathbf{P}_\infty|$.

8. *If* $\mathbf{P}_0$ *is covered,* $\mathbf{P}_n$ $(n \in \omega)$ *is fair and* $k \in \{1, 2\}$ *then*

$$a \preceq_k^{\infty} b \quad \Longrightarrow \quad a \preceq_k^{\mathbf{P}_\infty} b$$

*for all* $a, b \in |\mathbf{P}_\infty|$.

**Proof.** Part 1 is clear since each $\mathbf{P}_n$, being a pattern, is a model of reflection.

Part 2 follows from the fact that $\mathbf{P}_n$ is a closed substructure of $\mathbf{P}_{n+1}$ and $\mathbf{P}_{n+1}$ is a continuous extension of $\mathbf{P}_n$ for $n \in \omega$.

To prove part 3, use part 2 of Lemma 13.11 to find a nested sequence of functions $h_n$ $(n \in \omega)$ such that $h_0 = h$ and $h_n$ is a covering of $\mathbf{P}_n$ into $\mathcal{R}_2$ for $n \in \omega$. The union of the $h_n$ is the desired covering.

Part 4 follows from part 3.

For part 5, assume $\mathsf{f}$ is an $m$-ary function symbol of $\mathcal{L}_0$ and $a_1, \ldots, a_m \in |\mathbf{P}_\infty|$. There is $k \in \omega$ such that $a_1, \ldots, a_m \in |\mathbf{P}_k|$. By Lemma 8.5, there is a 1-correct arithmetic extension $\mathbf{P}^+$ of $\mathbf{P}_k$ such that $\mathsf{f}^{\mathbf{P}^+}(a_1, \ldots, a_m)$ is defined. Since $\mathbf{P}_n$ $(n \in \omega)$ is fair, there is $i \geq k$ such that $\mathbf{P}_{i+1}$ is a lifting of $\mathbf{P}_k|\mathbf{P}^+$ to $\mathbf{P}_i$ with respect to the inclusion of $|\mathbf{P}_k|$ in $|\mathbf{P}_i|$. This implies that $\mathsf{f}^{\mathbf{P}_{i+1}}(a_1, \ldots, a_m)$ is defined.

Part 6 follows from Lemmas 14.5 and 14.6.

To establish part 7, assume $\mathbf{P}_n$ $(n \in \omega)$ is fair, $a, b \in |\mathbf{P}_\infty|$ and $a \preceq_k^{\mathbf{P}_\infty} b$ where $k \in \{1, 2\}$. We will show that $a \preceq_k^{\infty} b$.

**Case 1:** Assume $k = 1$.

Suppose $X$ and $Y$ are finite subsets of $|\mathbf{P}_\infty|$ such that $X \subseteq (-\infty, a)^{\mathbf{P}_\infty}$, $Y \subseteq [a, b)^{\mathbf{P}_\infty}$ and $X \cup Y$ is a closed subset of $\mathbf{P}_\infty$. Fix $m$ such that $a, b \in |\mathbf{P}_m|$ and $X, Y \subseteq |\mathbf{P}_m|$. Let $\mathbf{P}^+$ be obtained from $\mathbf{P}_m$ by 1-reflecting $Y$ downward from $b$ to $a$ and choose $k > m$ such that $\mathbf{P}_{k+1}$ is a lifting of $\mathbf{P}_m|\mathbf{P}^+$ to $\mathbf{P}_k$.

There is a subset $\tilde{Y}$ of $|\mathbf{P}_{k+1}|$ such that $X \prec^{\mathbf{P}_{k+1}} \tilde{Y} \prec^{\mathbf{P}_{k+1}} a$, $X \cup \tilde{Y}$ is a closed subset of $\mathbf{P}_{k+1}$ and $X \cup \tilde{Y}$ is a covering of $X \cup Y$.

**Case 2:** Assume $k = 2$.

Suppose $X$ and $Y$ are finite subsets of $|\mathbf{P}_\infty|$ such that $X \subseteq (-\infty, a)^{\mathbf{P}_\infty}$, $Y \subseteq [a, b)^{\mathbf{P}_\infty}$ and $X \cup Y$ is a closed subset of $\mathbf{P}_\infty$. Fix $m$ such that $a, b \in |\mathbf{P}_m|$ and $X, Y \subseteq |\mathbf{P}_m|$. Let $\mathbf{P}^+$ be obtained from $\mathbf{P}_m$ by 2-reflecting $Y$ downward from $b$ to $a$ and choose $k > m$ such that $\mathbf{P}_{k+1}$ a lifting of $\mathbf{P}_m|\mathbf{P}^+$ to $\mathbf{P}_k$ with respect to the inclusion of $|\mathbf{P}_m|$ in $|\mathbf{P}_k|$. There is a subset $\tilde{Y}$ of $|\mathbf{P}_{k+1}|$ such that $X \prec^{\mathbf{P}_{k+1}} \tilde{Y} \prec^{\mathbf{P}_{k+1}} a$, $X \cup \tilde{Y}$ is a closed subset of $\mathbf{P}_{k+1}$, $X \cup \tilde{Y}$ is a covering of $X \cup Y$ and if $y \in Y$ has the property that $y \preceq_1^{\mathbf{P}_k} b$ then $\tilde{y} \preceq_1^{\mathbf{P}_{k+1}} a$ where $\tilde{y}$ corresponds to $y$ under the covering of $X \cup Y$ onto $X \cup \tilde{Y}$.

Now suppose $X \subseteq (-\infty, a)^{\mathbf{P}_\infty}$, $\mathbf{Q}$ is a pattern and there are cofinally many subsets $Y$ of $(-\infty, a)^{\mathbf{P}_\infty}$ such that $X \cup Y$ is a closed subset of $\mathbf{P}_\infty$ and $X \cup Y$ is a covering of $\mathbf{Q}$. Notice that the least element of any such $Y$ must be closed under the interpretations of the function symbols in $X \cup Y$ implying that it is indecomposable. To show that there are cofinally many subsets $Y$ of $(-\infty, b)^{\mathbf{P}_\infty}$ such that $X \cup Y$ is a closed subset of $\mathbf{P}_\infty$ and $X \cup Y$ is a covering of $\mathbf{Q}$, assume that $c \prec^{\mathbf{P}_\infty} b$. Choose $m$ such that $X \subseteq |\mathbf{P}_m|$ and $a, b, c \in |\mathbf{P}_m|$. Now choose $i > m$ such that there is a subset $Y^+$ of $|\mathbf{P}_i|$ such that

- $(-\infty, a)^{\mathbf{P}_m} \prec^{\mathbf{P}_i} Y^+ \prec^{\mathbf{P}_i} a$
- $X \cup Y^+$ is a closed subset of $\mathbf{P}_i$.
- $X \cup Y^+$ is a covering of $\mathbf{Q}$.

Notice that $X \prec^{\mathbf{P}_i} Y^+$ since $X \subseteq (-\infty, a)^{\mathbf{P}_m}$. Let $\mathbf{P}^+$ be the substructure of $\mathbf{P}_i$ with universe $|\mathbf{P}_m| \cup Y^+$. Notice that $\mathbf{P}_m|\mathbf{P}^+$ is a generating rule.

Let $\mathbf{P}_m|\mathbf{P}^*$ be obtained by 2-reflecting $\mathbf{P}_m|\mathbf{P}^+$ upward from $a$ to $b$. Notice that there is a subset $Y^*$ of $|\mathbf{P}^*|$ such that

- $(-\infty, b)^{\mathbf{P}_m} \prec^{\mathbf{P}^*} Y^* \prec^{\mathbf{P}^*} b$
- $X \cup Y^*$ is a closed subset of $\mathbf{P}^*$.
- $X \cup Y^*$ is a covering of $\mathbf{Q}$.

Since $c \in (-\infty, b)^{\mathbf{P}_m}$, $c \prec^{\mathbf{P}^*} Y^*$.

Choose $k > m$ such that $\mathbf{P}_{k+1}$ is a lifting of $\mathbf{P}_m|\mathbf{P}^*$ to $\mathbf{P}_k$. Letting $Y$ be the image of $Y^*$ under the lifting map for $(\mathbf{P}^*, \mathbf{P}_m, h, \mathbf{P}_k, \mathbf{P}_{k+1})$ where $h$ is the inclusion map for $|\mathbf{P}_m|$ as a subset of $|\mathbf{P}_k|$ we have

- $(-\infty, b)^{\mathbf{P}_m} \prec^{\mathbf{P}_{k+1}} Y \prec^{\mathbf{P}_{k+1}} b$
- $X \cup Y$ is a closed subset of $\mathbf{P}_{k+1}$.
- $X \cup Y$ is a covering of $\mathbf{Q}$.
- $c \prec^{\mathbf{P}_{k+1}} Y$

By the first point, $X \prec^{\mathbf{P}_{k+1}} Y$.

For part 8, assume $\mathbf{P}_0$ is covered, $\mathbf{P}_n$ $(n \in \omega)$ is fair and $k \in \{1, 2\}$. We must show that $a \preceq_1^\infty b$ implies $a \preceq_1^{\mathbf{P}_\infty} b$ for all $a, b \in |\mathbf{P}_\infty|$.

**Case 1:** Assume $k = 1$.

Argue by contradiction and assume that there are $a, b \in |\mathbf{P}_\infty|$ such that

$$a \preceq_1^\infty b \quad \text{and} \quad a \not\preceq_1^{\mathbf{P}_\infty} b$$

Since $a \prec_1^\infty b$, $a$ is indecomposable in $\mathbf{P}_\infty$. Choose $m$ such that $a, b \in |\mathbf{P}_m|$. We may assume that $b$ is minimal in $|\mathbf{P}_m|$ such that there is an $x \in |\mathbf{P}_m|$ with $x \preceq_1^\infty b$ and $x \not\preceq_1^{\mathbf{P}_m} b$ and that, given this choice of $b$, $a$ is minimal in $|\mathbf{P}_m|$ such that $a \preceq_1^\infty b$ and $a \not\preceq_1^{\mathbf{P}_\infty} b$.

**Claim 1 for Case 1:** If $a \preceq^{\mathbf{P}_m} x \prec^{\mathbf{P}_m} b$ then $a \preceq_1^{\mathbf{P}_m} x$ and $x \not\preceq_1^{\mathbf{P}_m} b$.

Suppose $a \preceq^{\mathbf{P}_m} x \prec^{\mathbf{P}_m} b$. Since $a \preceq_1^\infty b$, $a \preceq_1^\infty x$. By the choice of $b$, this implies that $a \preceq_1^{\mathbf{P}_m} x$. Since $a \not\preceq_1^{\mathbf{P}_m} b$, this implies that $x \not\preceq_1^{\mathbf{P}_m} b$.

Let $X = (-\infty, a)^{\mathbf{P}_m}$ and $Y = [a, b)^{\mathbf{P}_m}$. There exists $\tilde{Y}$ such that $X \prec^{\mathbf{P}_\infty} \tilde{Y} \prec^{\mathbf{P}_\infty} a$, $X \cup \tilde{Y}$ is closed in $\mathbf{P}_\infty$ and $X \cup \tilde{Y}$ is a covering of $X \cup Y$. Let $h$ be a covering of $X \cup Y$ onto $X \cup \tilde{Y}$. Define a function $f$ from the indecomposables of $\mathbf{P}_m$ into the indecomposables of $\mathbf{P}_\infty$ so that

$$f(x) = h(x) \text{ for } x \in X \cup Y \ (= (-\infty, b)^{\mathbf{P}_m})$$

and

$$f(x) = x \text{ for } x \in [b, \infty)^{\mathbf{P}_m}$$

By Lemma 4.5, $f$ extends to a closed embedding $f^+$ of the arithmetic part of $\mathbf{P}_m$ into the arithmetic part of $\mathbf{P}_\infty$. By Lemma 4.4, $f^+$ extends $h$ and $f^+(u) \preceq^{\mathbf{P}_\infty} u$ for all $u \in |\mathbf{P}_m|$.

**Claim 2 for Case 1:** $f^+$ is a covering of $\mathbf{P}_m$.

Assume $x, y \in |\mathbf{P}_m|$ and $x \prec_i^{\mathbf{P}_m} y$ where $i \in \{1, 2\}$. Notice that $x$ must be indecomposable in $\mathbf{P}_m$. If $x, y \in (-\infty, b)^{\mathbf{P}_m}$ then $f^+(x) \preceq_i^{\mathbf{P}_\infty} f^+(y)$ since $f^+$ extends $h$. So we may assume that $y \in [b, \infty)^{\mathbf{P}_m}$. By the claim, $x \notin [a, b)^{\mathbf{P}_m}$ which implies that $f^+(x) = x$.

Suppose $i = 1$. Since $f^+(y) \preceq^{\mathbf{P}_\infty} y$ and $f^+(x) = x$, $f^+(x) \preceq_1^{\mathbf{P}_\infty} f^+(y)$.

Now suppose $i = 2$. This implies that $y$ is also indecomposable in $\mathbf{P}_m$. Since $y \in [b, \infty)^{\mathbf{P}_m}$, $f^+(y) = y$. Hence, $f^+(x) \preceq_i^{\mathbf{P}_m} f^+(y)$.

Claim 2 contradicts part 6 since $f^+(a) = h(a) \prec^{\mathbf{P}_\infty} a$.

**Case 2:** Assume $k = 2$.

Argue by contradiction and assume that there are $a, b \in |\mathbf{P}_\infty|$ such that

$$a \preceq_2^\infty b \quad \text{and} \quad a \not\preceq_2^{\mathbf{P}_\infty} b$$

Since $a \prec_2^\infty b$, $a$ and $b$ are indecomposable in $\mathbf{P}_\infty$. Choose $m$ such that $a, b \in |\mathbf{P}_m|$. As in Case 1, we may assume that $b$ is minimal such that there is an $x \in |\mathbf{P}_m|$ with $x \preceq_2^\infty b$ and $x \not\preceq_2^{\mathbf{P}_m} b$ and that, given this choice of $b$, $a$ is minimal such that $a \preceq_2^\infty b$ and $a \not\preceq_2^{\mathbf{P}_m} b$.

**Claim for Case 2:** If $a \preceq^{\mathbf{P}_m} x \prec_1^{\mathbf{P}_m} b$ then $a \preceq_2^{\mathbf{P}_m} x$ and $x \not\preceq_2^{\mathbf{P}_m} b$.

Assume $a \preceq^{\mathbf{P}_\infty} x \prec_1^{\mathbf{P}_\infty} b$. By part 7, $x \preceq_1^\infty b$ and $a \preceq_2^\infty b$. This implies that $a \preceq_2^\infty x$. By the minimality of $b$, $a \preceq_2^{\mathbf{P}_m} x$. Since $a \not\preceq_2^{\mathbf{P}_m} b$, this implies that $x \not\preceq_2^{\mathbf{P}_m} b$

**Case 2.1:** Assume that for any $d \in [b, \infty)^{\mathbf{P}_m}$, if $a \preceq_1^{\mathbf{P}_m} d$ then $b \preceq_1^{\mathbf{P}_m} d$.

Let $X = (-\infty, a)^{\mathbf{P}_m}$ and $Y = [a, b)^{\mathbf{P}_m}$. Since $a \preceq_2^\infty b$, there exists $\tilde{Y}$ such that $X \prec^{\mathbf{P}_\infty} \tilde{Y} \prec^{\mathbf{P}_\infty} a$, $X \cup \tilde{Y}$ is closed in $\mathbf{P}_\infty$, $X \cup \tilde{Y}$ is a covering of $X \cup Y$ and, letting $h$ be the covering of $X \cup Y$ onto $X \cup \tilde{Y}$, $h(y) \preceq_1^{\mathbf{P}_\infty} a$ whenever $y \in Y$ has the property that $y \preceq_1^{\mathbf{P}_\infty} b$. Define an increasing function $f$, as in Case 1 above, from the indecomposables of $\mathbf{P}_m$ into the indecomposables of $\mathbf{P}_\infty$ so that

$$f(x) = h(x) \text{ for } x \in X \cup Y \ (= (-\infty, b)^{\mathbf{P}_m})$$

and

$$f(x) = x \text{ for } x \in [b, \infty)^{\mathbf{P}_m}$$

By Lemma 4.5, $f$ extends to a closed embedding $f^+$ of the arithmetic part of $\mathbf{P}_m$ into the arithmetic part of $\mathbf{P}_\infty$. By Lemma 8.2, $f^+$ extends $h$ and $f^+(u) \preceq^{\mathbf{P}_\infty} u$ for all $u \in |\mathbf{P}_m|$.

We claim that $f^+$ is a covering of $\mathbf{P}_m$. This will contradict part 6 since $f^+(a) = h(a) \prec^{\mathbf{P}_\infty} a$.

Assume $x, y \in |\mathbf{P}_m|$ and $x \prec_i^{\mathbf{P}_m} y$ where $i \in \{1, 2\}$. Notice that $x$ must be indecomposable in $\mathbf{P}_m$. If $x, y \in (-\infty, b)^{\mathbf{P}_m}$ then $f^+(x) \preceq_i^{\mathbf{P}_\infty} f^+(y)$ since $f^+$ extends $h$. So we may assume that $y \in [b, \infty)^{\mathbf{P}_m}$.

First suppose that $x \in [a, b)^{\mathbf{P}_m}$. This assumption implies that $i = 1$ by the claim for this case. We must have $x \preceq_1^{\mathbf{P}_m} b$ implying that $f^+(x) \preceq_1^{\mathbf{P}_\infty} a$ since $f^+$ extends $h$. Since $a \preceq_1^{\mathbf{P}_\infty} x$, $f^+(x) \preceq_1^{\mathbf{P}_\infty} x$ implying $f^+(x) \preceq_1^{\mathbf{P}_\infty} y$. Since $f^+(y) \preceq^{\mathbf{P}_\infty} y$, $f^+(x) \preceq_1^{\mathbf{P}_\infty} f^+(y)$.

Now suppose $x \notin [a, b)^{\mathbf{P}_m}$. Since $x$ is indecomposable, $f^+(x) = x$. If $i = 1$ then $f^+(x) \preceq_1^{\mathbf{P}_\infty} f^+(y)$ since $f^+(y) \preceq^{\mathbf{P}_\infty} y$. So, we may assume $i = 2$. This assumption implies $y$ is indecomposable. Since $y \in [b, \infty)^{\mathbf{P}_m}$, $f^+(y) = y$ implying that $f^+(x) \preceq_2^{\mathbf{P}_\infty} f^+(y)$.

**Case 2.2:** Assume there is $d \in [b, \infty)^{\mathbf{P}_m}$ such that $a \preceq_1^{\mathbf{P}_m} d$ and $b \not\preceq_1^{\mathbf{P}_m} d$.

Let $c$ be the largest element of $\mathbf{P}_m$ such that $b \preceq_1^{\mathbf{P}_m} c$. Notice that $c \prec^{\mathbf{P}_m} d$. Let $X = (-\infty, a)^{\mathbf{P}_m}$ and $Y = [a, c]^{\mathbf{P}_m}$. Since $a \preceq_1^\infty d$ by part 7, there are cofinally many $Y^+$ below $a$ such that

- $X \preceq^{\mathbf{P}_\infty} Y^+$
- $X \cup Y^+$ is a closed subset of $\mathbf{P}_\infty$.
- $X \cup Y^+$ is a covering of $X \cup Y$

Since $a \preceq_2^\infty b$, there are cofinally many $Y^*$ below $b$ such that

- $X \preceq^{\mathbf{P}_\infty} Y^*$
- $X \cup Y^*$ is a closed subset of $\mathbf{P}_\infty$.
- $X \cup Y^*$ is a covering of $X \cup Y$

Choose such a $Y^*$ such that

$$(-\infty, b)^{\mathbf{P}_m} \prec^{\mathbf{P}_\infty} Y^*$$

Let $h$ be the covering of $X \cup Y$ onto $X \cup Y^*$. Define $f$ from the indecomposables of $\mathbf{P}_m$ into the indecomposables of $\mathbf{P}_\infty$ so that

$$f(x) = h(x) \text{ if } x \in [b,c]^{\mathbf{P}_m}$$

and

$$f(x) = x \text{ if } x \notin [b,c]^{\mathbf{P}_m}$$

By Lemma 4.5, there is a closed embedding $f^+$ of the arithmetic part of $\mathbf{P}_m$ into the arithmetic part of $\mathbf{P}_\infty$ which extends $f$. By Lemma 4.4, $f^+(x) \preceq^{\mathbf{P}_\infty} h(x)$ for all $x \in (-\infty, c]^{\mathbf{P}_m}$ and $f^+(x) \preceq^{\mathbf{P}_\infty} x$ for all $x \in |\mathbf{P}_m|$.

We will show that $f^+$ is a covering of $\mathbf{P}_m$ in $\mathbf{P}_\infty$. This will contradict part 6 since $f^+(b) \prec^{\mathbf{P}_\infty} b$.

Suppose $x, y \in |\mathbf{P}_m|$ and $x \prec_i^{\mathbf{P}_m} y$. We will show that $f^+(x) \prec_i^{\mathbf{P}_\infty} f^+(y)$.

**Case 2.2.1:** Assume $i = 1$.

Notice that $x$ is indecomposable in $\mathbf{P}_m$.

If $f^+(x) = x$ then $f^+(x) \preceq_1^{\mathbf{P}_\infty} f^+(y)$ since $f^+(y) \preceq^{\mathbf{P}_\infty} y$. So, we may assume that $f^+(x) \neq x$. This implies that $x \in [b,c]^{\mathbf{P}_m}$ and, hence, that $f^+(x) = f(x) = h(x)$. By the choice of $c$ and since $x \preceq_1^{\mathbf{P}_m} y$, we must have $y \in [b,c]^{\mathbf{P}_m}$. Since $h$ is a covering, $f^+(x) = h(x) \preceq_1^{\mathbf{P}_\infty} h(y)$. Since $y \in [b,c]^{\mathbf{P}_m}$, $f^+(y) \preceq^{\mathbf{P}_\infty} h(y)$ which implies $f^+(x) \preceq_1^{\mathbf{P}_\infty} f^+(y)$.

**Case 2.2.2:** Assume $i = 2$.

In this case, both $x$ and $y$ are indecomposable in $\mathbf{P}_m$. By choice of $c$, if $x \in [b,c]^{\mathbf{P}_m}$ then $y \in [b,c]^{\mathbf{P}_m}$. By the Claim for Case 2, if $x \in [a,b)^{\mathbf{P}_m}$ then $y \notin [b,c]^{\mathbf{P}_m}$. The following three cases remain.

**Case 2.2.2.1:** Assume $x, y \notin [b,c]^{\mathbf{P}_m}$.

In this case, $f^+(x) = x$ and $f^+(y) = y$ implying $f^+(x) \preceq_2^{\mathbf{P}_\infty} f^+(y)$.

**Case 2.2.2.2:** Assume $x, y \in [b,c]^{\mathbf{P}_m}$.

In this case, $f^+(x) = h(x)$ and $f^+(y) = h(y)$. Since $h$ is a covering, $f^+(x) \preceq_2^{\mathbf{P}_\infty} f^+(y)$.

**Case 2.2.2.3:** Assume $x \in (-\infty, a)^{\mathbf{P}_m}$ and $y \in [b,c]^{\mathbf{P}_m}$.

In this case, $f^+(x) = x = h(x)$ and $f^+(y) = h(y)$. Since $h$ is a covering, $f^+(x) \preceq_2^{\mathbf{P}_\infty} f^+(y)$. **QED**

**Lemma 14.9** *Assume $\mathbf{P}_n$ $(n \in \omega)$ is a fair sequence of patterns such that $\mathbf{P}_0$ is covered and $\mathbf{P}_n|\mathbf{P}_{n+1}$ is a generating rule for each $n \in \omega$. There is an isomorphism $h$ of the union $\mathbf{P}_\infty$ of $\mathbf{P}_n$ $(n \in \omega)$ with an initial substructure $\mathbf{P}^*_\infty$ of $\mathcal{R}_2$ such that, letting $\mathbf{P}^*_n$ be the substructure of $\mathcal{R}_2$ with universe $h[|\mathbf{P}_n|]$,*

1. *The universe of $\mathbf{P}^*_\infty$ is the least indecomposable ordinal $\lambda$ such that $h(x) < \lambda$ for $x \in |\mathbf{P}_0|$.*

2. *For $n \in \omega$, if $\mathbf{Q}$ is a closed substructure of $\mathcal{R}_2$ which is a covering of $\mathbf{P}_n$ then then $|\mathbf{P}^*_n| \leq_{pw} |\mathbf{Q}|$.*

3. *For $n \in \omega$, $\mathbf{P}^*_n$ is an isominimal substructure of $\mathcal{R}_2$.*

4. *For $k = 1, 2$, $\mathcal{R}_2$ is a $k$-correct extension of $\mathbf{P}^*_\infty$.*

**Proof.** By part 4 of the previous lemma, there is an order preserving map of the indecomposables of $\mathbf{P}_\infty$ onto an initial segment of the indecomposables of $\mathcal{R}_2$. Let $\lambda$ be the first indecomposable not in the range of this map. By Lemma 4.5, this map extends to an embedding $h$ of the arithmetic part of $\mathbf{P}_\infty$ into $\mathcal{R}_0$. Since $\mathbf{P}_\infty$ is a continuous extension of $\mathbf{P}_0$ by part two of the previous lemma, the largest indecomposable of $\mathbf{P}_0$ is the largest indecomposable of $\mathbf{P}_\infty$. Therefore, $\lambda$ is the least indecomposable such that $h(x) < \lambda$ for $x \in |\mathbf{P}_0|$. By part 5 of the previous lemma, the range of this embedding is $\lambda$. Therefore, we may assume that the arithmetic part of $\mathbf{P}_\infty$ is the substructure of $\mathcal{R}_0$ with universe $\lambda$.

A straightforward induction shows that the restriction of $\mathbf{P}_\infty$ to $\alpha$ is the same as the restriction of $\mathcal{R}_2$ to $\alpha$ for $\alpha \leq \lambda$. To verify the step from $\alpha$ to $\alpha+1$, use parts 7 and 8 of the previous lemma to see that the interpretations of $\preceq_1$ agree and to see that the interpretations of $\preceq_2$ agree. Thus, $\mathbf{P}_\infty$ is (isomorphic to) the initial substructure of $\mathcal{R}_2$ with universe $\lambda$.

Under our simplifying assumption that $\mathbf{P}_\infty$ has universe $\lambda$, $h$ is the identity on $\lambda$, $\mathbf{P}^*_n = \mathbf{P}_n$ and $\mathbf{P}^*_\infty = \mathbf{P}_\infty$.

For part 2, suppose $\mathbf{Q}$ is a closed substructure of $\mathcal{R}_2$ such that there is a covering of $\mathbf{P}_n$ onto $\mathbf{Q}$. By part 3 of the previous lemma, there is a covering $h$ of $\mathbf{P}_\infty$ into $\mathcal{R}_2$ which maps $\mathbf{P}_n$ onto $\mathbf{Q}$. Since $h$ is order preserving, $|\mathbf{P}_n| \leq_{pw} |\mathbf{Q}|$.

For part 3, fix $n \in \omega$. By part 2 of Lemma 13.11, $\mathbf{P}_m$ is a closed substructure of $\mathbf{P}_{m+1}$ for $m \in \omega$. By part 2 of Lemma 2.4, $\mathbf{P}_n$ is a closed substructure of $\mathbf{P}_m$ whenever $n \leq m$. This implies that $\mathbf{P}_n$ is a closed substructure of $\mathbf{P}_\infty$ for all $n \in \omega$. This, combined with part 2, implies that $\mathbf{P}_n$ is an isominimal substructure of $\mathbf{P}_\infty$.

For part 4, assume $k \in \{1, 2\}$. Notice that since $\mathbf{P}_\infty$ has universe $\lambda$, to see that $\mathcal{R}_2$ is a $k$-correct extension of $\mathbf{P}_\infty$, it is enough to show that whenever $\alpha < \lambda$ and $\alpha \leq_1 \beta$ then $\beta < \lambda$. Arguing by contradiction, suppose $\alpha < \lambda \leq \beta$

and $\alpha \leq_1 \beta$. $\alpha \in \mathbf{P}_n$ for some $n \in \omega$. Since $|\mathbf{P}_n| < \lambda \leq \beta$, the fact that $\alpha \leq_1 \beta$ implies there is a covering of $\mathbf{P}_n$ below $\alpha$, contradicting part 2. **QED**

**Theorem 14.10** *If $\mathbf{P}$ is a covered pattern then there is a substructure $\mathbf{P}^*$ of $\mathcal{R}_2$ such that*

1. *$\mathbf{P}^*$ is isomorphic to $\mathbf{P}$.*
2. *$|\mathbf{P}^*| \leq_{pw} |\mathbf{Q}|$ whenever $\mathbf{Q}$ is a closed substructure of $\mathcal{R}_2$ which is a covering of $\mathbf{P}^*$.*
3. *$\mathbf{P}^*$ is an isominimal substructure of $\mathcal{R}_2$.*
4. *For $k = 1, 2$, $\mathcal{R}_2$ is a $k$-correct extension of $\mathbf{P}^*$.*
5. *If $\mathbf{Q}$ is a closed substructure of $\mathcal{R}_2$ such that $\mathbf{P}^*$ is a substructure of $\mathbf{Q}$ and the largest indecomposable of $\mathbf{Q}$, if there is one, is the same as the largest indecomposable of $\mathbf{P}^*$ then $\mathbf{P}^*$ generates $\mathbf{Q}$.*

**Proof.** Let $\mathbf{P}_n$ $(n \in \omega)$ be a fair sequence of patterns with $\mathbf{P} = \mathbf{P}_0$ such that $\mathbf{P}_n | \mathbf{P}_{n+1}$ is a generating rule for $n \in \omega$. Let $\mathbf{P}_\infty$ be the union of $\mathbf{P}_n$ $(n \in \omega)$. Fix an isomorphism $h$ of $\mathbf{P}_\infty$ with an initial segment $\mathbf{P}^*_\infty$ of $\mathcal{R}_2$ as in the previous lemma. Let $\mathbf{P}^*_n$ be the the image of $\mathbf{P}_n$ under $h$ for $n \in \omega$ and set $\mathbf{P}^* = \mathbf{P}^*_0$. By part 1 of Lemma 13.11, $\mathbf{P}^*_n$ $(n \in \omega)$ is a fair sequence such that $\mathbf{P}^*_n | \mathbf{P}^*_{n+1}$ is a generating rule for $n \in \omega$.

Parts 2 and 3 follow from parts 2 and 3 of the previous lemma.

For part 4, assume $k \in \{1, 2\}$. Notice that $\mathcal{R}_2$ is a $k$-correct extension of $\mathbf{P}^*_\infty$ by part 4 of the previous lemma. Hence, showing that $\mathbf{P}^*_\infty$ is a continuous extension of $\mathbf{P}^*$ is sufficient. By part 2 Lemma 13.11, $\mathbf{P}^*_{n+1}$ is a continuous extension of $\mathbf{P}^*_n$ for $n \in \omega$. By part 1 of Lemma 7.6, $\mathbf{P}^*_n$ is a continuous extension of $\mathbf{P}^*$ for $n \in \omega$. This easily implies that $\mathbf{P}^*_\infty$ is a continuous extension of $\mathbf{P}^*$.

For part 5, let $\lambda$ be the universe of $\mathbf{P}^*_\infty$. By part 1 of the previous lemma, $\lambda$ is an indecomposable ordinal. Since the largest indecomposable in $\mathbf{Q}$ is in $\mathbf{P}^*$, $|\mathbf{Q}| \subseteq \lambda$. Therefore, $|\mathbf{Q}| \subseteq |\mathbf{P}^*_n|$ for some $n$ implying that $\mathbf{P}^*$ generates $\mathbf{Q}$. **QED**

We return for a moment to an earlier discussion regarding the definition of isominimality. Suppose that $\mathbf{P}$ is an isominimal substructure of $\mathcal{R}_2$. Can part 2 of the lemma be generalized to an arbitrary substructure $\mathbf{Q}$? In other words, does it follow that $|\mathbf{P}| \leq_{pw} |\mathbf{Q}|$ for any substructure $\mathbf{Q}$, not necessarily closed, of $\mathcal{R}_2$ which is a covering of $\mathbf{P}$? The answer is *yes* if the following simple conditions are met.

- For any ordinal $\alpha$, if the substructure of $\mathcal{R}_0$ with universe $\{\alpha\}$ is isomorphic to a substructure whose universe consists of a single indecomposable then there is an indecomposable $\kappa$ with $\kappa \leq \alpha$.
- For any ordinals $\alpha$ and $\beta$ with $\alpha < \beta$, if the substructure of $\mathcal{R}_0$ with universe $\{\alpha, \beta\}$ is isomorphic to a substructure consisting of two indecomposables then there is an indecomposable $\kappa$ with $\alpha < \kappa \leq \beta$.

The conditions imply that for any finite substructure $\mathbf{Q}$ of $\mathcal{R}_2$ which is a pattern there is a closed substructure $\mathbf{Q}'$ which is a covering of $\mathbf{Q}$ and pointwise below $\mathbf{Q}$ (map each indemposable $\alpha$ of $\mathbf{Q}$ to the largest indecomposable $\kappa$ of $\mathcal{R}_2$ with $\kappa \leq \alpha$. The structures we are interested in satisfy these conditions e.g. both $\mathcal{R}_0 = (ORD, 0, \leq, +)$ and $\mathcal{R}_0 = (ORD, 0, \leq, +, \varphi)$ (for the latter structure, every subset which is isomorphic to a closed subset is closed).

**Theorem 14.11** *Assume* $\mathbf{P}$ *is a covered pattern and* $\mathbf{P}^*$ *is the isominimal substructure of* $\mathcal{R}_2$ *which is isomorphic to* $\mathbf{P}$*. Also assume* $\mathbf{P}$ *is a closed substructure of a pattern* $\mathbf{Q}$*. The following are equivalent.*

1. $\mathbf{P}$ *generates* $\mathbf{Q}$*.*
2. $\mathbf{P}|\mathbf{Q}$ *is a cofinally valid continuous rule.*
3. $\mathbf{P}|\mathbf{Q}$ *is a valid continuous rule.*
4. *(a)* $\mathbf{Q}$ *is covered,*
   *(b) if* $\mathbf{Q}$ *has an indecomposable then the largest indecomposable of* $\mathbf{Q}$ *is in* $|\mathbf{P}|$*, and*
   *(c) if* $\mathbf{Q}^*$ *is the isominimal substructure of* $\mathcal{R}_2$ *which is isomorphic to* $\mathbf{Q}$ *then the isomorphism of* $\mathbf{Q}$ *and* $\mathbf{Q}^*$ *extends the isomorphism of* $\mathbf{P}$ *and* $\mathbf{P}^*$*.*

**Proof.** $(1 \Rightarrow 2)$ By part 2 of Lemma 13.11, Lemma 13.2 and Lemma 13.4.

$(2 \Rightarrow 3)$ Immediate.

$(3 \Rightarrow 4)$ Assume $\mathbf{P}|\mathbf{Q}$ is a valid continuous rule. Since $\mathbf{P}$ is covered, this implies that $\mathbf{Q}$ is covered. Since $\mathbf{Q}$ is a continuous extension of $\mathbf{P}$, the largest indecomposable in $\mathbf{Q}$, if there is one, is in $|\mathbf{P}|$. Let $\mathbf{P}^*$ and $\mathbf{Q}^*$ be the isominimal substructures of $\mathcal{R}_2$ which are isomorphic to $\mathbf{P}$ and $\mathbf{Q}$ respectively and let $f$ be the isomorphism of $\mathbf{Q}$ and $\mathbf{Q}^*$. Since $\mathbf{P}|\mathbf{Q}$ is valid, there is a covering $h$ of $\mathbf{Q}$ in $\mathcal{R}_2$ which extends the isomorphism of $\mathbf{P}$ and $\mathbf{P}^*$. Let $\mathbf{Q}'$ be the substructure of $\mathcal{R}_2$ whose universe is the range of $h$. By Part 2 of the previous theorem, $|\mathbf{Q}^*| \leq_{pw} |\mathbf{Q}'|$. Let $\mathbf{P}'$ be the substructure of $\mathbf{Q}^*$ whose universe is $f[|\mathbf{P}|]$. Since $|\mathbf{Q}^*| \leq_{pw} |\mathbf{Q}'|$, we must have $|\mathbf{P}'| \leq_{pw} |\mathbf{P}^*|$. On the other hand, by part 2 of the previous theorem again, $|\mathbf{P}^*| \leq_{pw} |\mathbf{P}'|$. Therefore, $\mathbf{P}' = \mathbf{P}^*$ implying the isomorphism of $\mathbf{Q}$ and $\mathbf{Q}^*$ extends the isomorphism of $\mathbf{P}$ and $\mathbf{P}^*$.

($4 \Rightarrow 1$) By Part 5 of the previous theorem, $\mathbf{P}^*$ generates $\mathbf{Q}^*$ implying that $\mathbf{P}$ generates $\mathbf{Q}$ by Lemma 14.4. **QED**

**Corollary 14.12** *Assume* $\mathbf{P}$*,* $\mathbf{Q}$ *and* $\mathbf{R}$ *are covered patterns. Also suppose* $\mathbf{P}$ *is a closed substructure of* $\mathbf{Q}$ *and* $\mathbf{Q}$ *is a closed substructure of* $\mathbf{R}$*.*

1. *If* $\mathbf{P}$ *generates* $\mathbf{R}$ *then* $\mathbf{P}$ *generates* $\mathbf{Q}$*.*

2. *If* $\mathbf{P}$ *generates* $\mathbf{Q}$ *and* $\mathbf{Q}$ *generates* $\mathbf{R}$ *then* $\mathbf{P}$ *generates* $\mathbf{R}$*.*

**Proof.** Immediate from the equivalence of parts 1 and 3 of the theorem. **QED**

The corollary can be proved without the assumptions that $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$ are covered, but we will not need that here. Part 1 follows directly from the definition and part 2 follows from the fact, mentioned earlier, that for patterns $\mathbf{P}$ and $\mathbf{Q}$, $\mathbf{P}$ exactly generates $\mathbf{Q}$ iff $\mathbf{P}$ generates $\mathbf{Q}$.

**Corollary 14.13** *Assume* $\mathbf{P}^+$ *is a pattern and* $\mathbf{P}$ *is a closed substructure of* $\mathbf{P}^+$ *such that either* $\mathbf{P}^+$ *is an arithmetic extension of* $\mathbf{P}$ *or* $\mathbf{P}^+$ *is an extension of* $\mathbf{P}$ *at $a$ for some $a$.* $\mathbf{P}|\mathbf{P}^+$ *is valid iff* $\mathbf{P}|\mathbf{P}^+$ *is a generating rule.*

**Proof.** ($\Longrightarrow$) Assume $\mathbf{P}|\mathbf{P}^+$ is valid. By the theorem, $\mathbf{P}$ generates $\mathbf{P}^+$. By Lemma 13.11, $\mathbf{P}^+$ is a continuous extension of $\mathbf{P}$.

First suppose $\mathbf{P}^+$ is an arithmetic extension of $\mathbf{P}$. Since $\mathbf{P}^+$ is a 1-correct extension of $\mathbf{P}$, $\mathbf{P}^+|\mathbf{P}$ is a generating rule.

If $\mathbf{P}^+$ is an extension of $\mathbf{P}$ at $a$ then the definition of the generating rules immediately implies that $\mathbf{P}^+|\mathbf{P}$ is a generating rule.

($\Longleftarrow$) Immediate from the theorem. **QED**

**Theorem 14.14** *If there exists a $\kappa$ such that $\kappa \leq_1 \infty$ then the least such $\kappa$ is the core of $\mathcal{R}_2$. Otherwise, the core of $\mathcal{R}_2$ is $ORD$.*

*Proof.* Clearly, if $\kappa \leq_1 \infty$ then the core of $\mathcal{R}_2$ is contained in $\kappa$.

Next, notice that the core of $\mathcal{R}_2$ is an initial segment of $ORD$. To see this, suppose $\mathbf{P}$ is an isominimal substructure of $\mathcal{R}_2$. Fix a fair sequence $\mathbf{P}_n$ $(n \in \infty)$ such that $\mathbf{P}_0 = \mathbf{P}$ and $\mathbf{P}_{n+1}$ is exactly generated from $\mathbf{P}_n$ for $n \in \omega$. Let $\mathbf{P}_\infty$ be the union of $\mathbf{P}_n$ $(n \in \omega)$. By Lemma 14.9, there is an isomorphism $h$ of $\mathbf{P}_\infty$ with an initial segment $\mathbf{P}^*_\infty$ of $\mathcal{R}_2$ which fixes $\mathbf{P}$. Lemma 14.9 also states that the image of each $\mathbf{P}_n$ under $h$ is isominimal in $\mathcal{R}_2$. Therefore, $\mathbf{P}^*_\infty$ is contained in the core. Since both $\mathbf{P}$ and its image under $h$ are isominimal, they must be equal implying that $[0, max(\mathbf{P})]$ is contained in the core.

Suppose there is an ordinal which is not in the core and let $\kappa$ be the least such ordinal. We will show that $\kappa \leq_1 \infty$.

Suppose $X$ and $Y$ are finite sets of ordinals such that $X < \kappa \leq Y$. We will show that there is $\tilde{Y}$ such that $X < \tilde{Y} < \kappa$ and $X \cup \tilde{Y} \cong X \cup Y$. Without loss of generality, we may assume that $X \cup Y$ is a closed substructure of $\mathcal{R}_2$. Notice that any finite union of isominimal substructures of $\mathcal{R}_2$ is isominimal. Since $X$ is contained in a finite union of isominimal patterns, we may assume that $X$ is isominimal. Now let $\tilde{X} \cup \tilde{Y}$ be the isominimal copy of $X \cup Y$ where $\tilde{X}$ corresponds to $X$ and $\tilde{Y}$ corresponds to $Y$ under the isomorphism. We must have $\tilde{X} \leq_{pw} X$. Since $X$ is isominimal, $\tilde{X} = X$. Since $\tilde{X} \cup \tilde{Y}$ is contained in the core, $\tilde{Y} < \kappa$. **QED**

# 15 Computability and the Core

Assume $\mathcal{L}_2$ is finite for this section.

We now consider the complexity of the core of $\mathcal{R}_2$. The fundamental question is

Is the core a computable structure?

More precisely, we ask if the substructure of $\mathcal{R}_2$ whose universe is the core of $\mathcal{R}_2$ is isomorphic to a computable structure. In our base theory of $\mathsf{KP}\omega$, the answer can be "no" for the simple reason that the core may be $\omega_1^{ck}$ or the class of all ordinals.

We first show that every proper initial segment of the core is isomorphic to a computable structure. We later show that the core is isomorphic to a computable structure under certain assumptions beyond $\mathsf{KP}\omega$.

**Lemma 15.1** *Assume the set of arithmetic structures in* $\mathsf{H}(\omega)$ *is computable. The family of generating rules is computably enumerable.*

**Proof.** Notice that since the family of arithmetic structures in $\mathsf{H}(\omega)$ is computable, the family of patterns in $\mathsf{H}(\omega)$ is computable.

The conclusion of the lemma now follows from the fact that the following families are evidently computable.

- The family of continuous arithmetic rules.
- For $k = 1, 2$, the family of downward $k$-reflection rules.
- The family of tuples $(\mathbf{P}, \mathbf{P}^+, \mathbf{P}^*)$ where $\mathbf{P}|\mathbf{P}^*$ is obtained by 2-reflecting $\mathbf{P}|\mathbf{P}^+$ upward from $a$ to $b$ for some $a$ and $b$.
- The family of tuples $(\mathbf{P}, \mathbf{P}^+, \mathbf{Q})$ where $\mathbf{P}|\mathbf{P}^+$ is a continuous rule, $\mathbf{P}$ is a closed substructure of $\mathbf{Q}$ and $\mathbf{Q}$ is a closed substructure of $\mathbf{P}^+$.
- The family of tuples $(\mathbf{P}^+, \mathbf{P}, h, \mathbf{Q}, \mathbf{Q}^+)$ where $\mathbf{Q}^+$ is a minimal lifting of $\mathbf{P}|\mathbf{P}^+$ to $\mathbf{Q}$ with respect to $h$.

**QED**

**Theorem 15.2** *Assume the collection of arithmetic structures in* $\mathsf{H}(\omega)$ *is computable. Every proper initial segment of the core is isomorphic to a computable structure.*

**Proof.** Assume $\alpha$ is in the core of $\mathcal{R}_2$. Let $\mathbf{P}^*$ be an isominimal substructure of $\mathcal{R}_2$ such that $\alpha \in |\mathbf{P}^*|$. Let $\mathbf{P}$ be a pattern in $\mathsf{H}(\omega)$ which is isomorphic to $\mathbf{P}^*$. By the previous lemma, there is a computably enumerable fair sequence $\mathbf{P}_n$ $(n \in \omega)$ of patterns in $\mathsf{H}(\omega)$ such that $\mathbf{P}_0 = \mathbf{P}$. Let $\mathbf{P}_\infty$ be the union of $\mathbf{P}_n$ $(n \in \omega)$. Notice that $\mathbf{P}_\infty$ is computably enumerable and, therefore, isomorphic to a computable structure. By Lemma 14.8, there is an isomorphism $h$ of $\mathbf{P}_\infty$ with an initial segment of $\mathcal{R}_2$ whose universe is the least indecomposable ordinal $\lambda$ such that $h(x) < \lambda$ for all $x \in |\mathbf{P}|$. Moreover, the image of each $\mathbf{P}_n$ under $h$ is an isominimal substructure of $\mathcal{R}_2$. Therefore, $h$ extends the isomorphism of $\mathbf{P}$ and $\mathbf{P}^*$ implying $\alpha < \lambda$. **QED**

**Definition 15.3** A *pointed pattern* is a pair $(\mathbf{P}, a)$ where $\mathbf{P}$ is a pattern which is an element of $\mathsf{H}(\omega)$, the family of hereditarily finite sets, and $a \in |\mathbf{P}|$.

For $(\mathbf{P},a)$ a pointed pattern where $\mathbf{P}$ is covered, define $\iota(\mathbf{P},a)$ to be the image of $a$ under the isomorphism of $\mathbf{P}$ with the isominimal substructure of $\mathcal{R}_2$ which is isomorphic to it. $\mathcal{C}_2$ is the pseudostructure for the language $\mathcal{L}_2$ whose universe is the domain of $\iota$ such that $\iota$ is a homomorphism of $\mathcal{C}_2$ into $\mathcal{R}_2$:

1. For any $n$-ary relation symbol $\mathsf{R}$

$$\mathsf{R}^{\mathcal{C}_2}((\mathbf{P}_1,a_1),\ldots,(\mathbf{P}_n,a_n)) \quad \text{iff} \quad \mathsf{R}^{\mathcal{R}_2}(\iota(\mathbf{P}_1,a_1),\ldots,\iota(\mathbf{P}_n,a_n))$$

   for all pointed patterns $(\mathbf{P}_1,a_1),\ldots,(\mathbf{P}_n,a_n) \in \mathsf{H}(\omega)$ such that $\mathbf{P}_i$ is covered for $i=1,\ldots,n$.

2. For any $n$-ary function symbol $\mathsf{f}$

$$\mathsf{f}^{\mathcal{C}_2}((\mathbf{P}_1,a_1),\ldots,(\mathbf{P}_n,a_n),(\mathbf{Q},b))$$

   iff

$$\mathsf{f}^{\mathcal{R}_2}(\iota(\mathbf{P}_1,a_1),\ldots,\iota(\mathbf{P}_n,a_n)) = \iota(\mathbf{Q},b)$$

   for all pointed patterns $(\mathbf{P}_1,a_1),\ldots,(\mathbf{P}_n,a_n),(\mathbf{Q},b) \in \mathsf{H}(\omega)$ such that $\mathbf{P}_i$ is covered for $i=1,\ldots,n$ and $\mathbf{Q}$ is covered.

Notice that condition 1 of the definition is

$(\mathbf{P}_1,a_1) =^{\mathcal{C}_2} (\mathbf{P}_2,a_2)$ iff $\iota(\mathbf{P}_1,a_1) = \iota(\mathbf{P}_2,a_2)$

when $\mathsf{R}$ is the equality symbol.

$\iota$ may be too large to be a set, however $\mathsf{KP}\omega$ proves that $\iota$ is a $\Sigma$ definable class. So, even though the domain of $\iota$, which is the universe of $\mathcal{C}_2$, may not be a set, it is a $\Sigma$ definable class along with the interpretations of the other function and relation symbols in $\mathcal{C}_2$.

**Lemma 15.4** $=^{\mathcal{C}_2}$ *is a congruence on* $\mathcal{C}_2$.

**Proof.** Straightforward. **QED**

**Definition 15.5** Assume $\mathbf{Q}$ is a pattern and $\mathbf{P}$ is a closed substructure of $\mathbf{Q}$. $\mathbf{P}$ is *exact in* $\mathbf{Q}$ if $\mathbf{P}$ generates $(-\infty, max(\mathbf{P})]^{\mathbf{Q}}$ and $(-\infty, max(\mathbf{P})]^{\mathbf{Q}}$ is 1-correct in $\mathbf{Q}$. If $f$ is an embedding of $\mathbf{P}$ into a pattern $\mathbf{R}$ then $f$ is said to be *exact* with respect to $\mathbf{R}$ if the range of $f$ is an exact subpattern of $\mathbf{R}$.

**Lemma 15.6**

1. *If* $\mathbf{P}$ *generates* $\mathbf{Q}$ *then* $\mathbf{P}$ *is exact in* $\mathbf{Q}$.

2. *Assume* $\mathbf{P}$, $\mathbf{Q}$ *and* $\mathbf{R}$ *are patterns such that* $\mathbf{P}$ *is a closed substructure of* $\mathbf{Q}$ *and* $\mathbf{Q}$ *is a closed substructure of* $\mathbf{R}$. *If* $\mathbf{P}$ *is exact in* $\mathbf{R}$ *then* $\mathbf{P}$ *is exact in* $\mathbf{Q}$.

3. *If* $\mathbf{P}$ *is exact in* $\mathbf{Q}$ *then* $\mathbf{P}$ *is a closed substructure of* $\mathbf{Q}$.

*Proof.* Part 1 follows from Lemma 14.3.

Parts 2 is straightforward.

For part 3, notice that Lemma 14.3 implies that $\mathbf{P}$ is a closed substructure of $(-\infty, max(\mathbf{P})]^{\mathbf{Q}}$. Since $(-\infty, max(\mathbf{P})]^{\mathbf{Q}}$ is a closed substructure of $\mathbf{Q}$, this implies that $\mathbf{P}$ is a closed substructure of $\mathbf{Q}$. **QED**

Using the fact that the relation of one pattern generating another is transitive (mentioned earlier but not proved), exactness can be seen to be transitive. We will not require this fact.

**Lemma 15.7** *Assume* $\mathbf{P}$ *is an isominimal substructure of* $\mathcal{R}_2$. *If* $\beta \in \mathbf{P}$ *has the property that* $(-\infty, \beta]^{\mathbf{P}}$ *is 1-correct in* $\mathbf{P}$ *then* $(-\infty, \beta]^{\mathbf{P}}$ *is an isominimal substructure of* $\mathcal{R}_2$.

*Proof.* Let $\mathbf{Q}$ be the isominimal substructure of $\mathcal{R}_2$ which is isomorphic to $(-\infty, \beta]^{\mathbf{P}}$. Let $h$ be an isomorphism of $(-\infty, \beta]^{\mathbf{P}}$ and $\mathbf{Q}$. Notice that $h(\xi) \leq \xi$ for all $\xi \in (-\infty, \beta]^{\mathbf{P}}$ since $|\mathbf{Q}| \leq_{pw} (-\infty, \beta]^{\mathbf{P}}$.

Define a function $f$ mapping the indecomposables of $\mathbf{P}$ into the indecomposables of $\mathcal{R}_2$ so that

$$f(\alpha) = \alpha \text{ for } \alpha \in (\beta, \infty)^{\mathbf{P}}$$

and

$$f(\alpha) = h(\alpha) \text{ for } \alpha \in (-\infty, \beta]^{\mathbf{P}}$$

By Lemma 4.5, there is a closed embedding $f^+$ of the arithmetic part of $\mathbf{P}$ into the arithmetic part of $\mathcal{R}_2$ which extends $f$. By Lemma 4.4, $f^+$ extends $h$ and $f^+(\xi) \leq \xi$ for all $\xi \in |\mathbf{P}|$.

We claim that $f^+$ is a covering of $\mathbf{P}$ in $\mathcal{R}_2$. Since $f^+(\xi) \leq \xi$ for all $\xi \in |\mathbf{P}|$ and $\mathbf{P}$ is isominimal, this will imply that $f^+(\xi) = \xi$ for all $\xi \in (-\infty, \beta]^{\mathbf{P}}$ and, consequently, $h$ is the identity.

To show $f^+$ is a covering, suppose $k \in \{1,2\}$, $\xi, \eta \in |\mathbf{P}|$ and $\xi \leq_k \eta$. We will show $f^+(\xi) \leq_k f^+(\eta)$. Since the case $\xi = \eta$ is trivial, we may assume $\xi < \eta$. Notice that we cannot have $\xi \in (-\infty, \beta]^{\mathbf{P}}$ and $\eta \in (\beta, \infty)^{\mathbf{P}}$. If $\xi, \eta \in (-\infty, \beta]^{\mathbf{P}}$ then $f^+(\xi) = h(\xi)$ and $f^+(\eta) = h(\eta)$ implying $f^+(\xi) \leq_k f^+(\eta)$. So, we may assume that $\xi, \eta \in (\beta, \infty)^{\mathbf{P}}$. Notice that $\xi$ is indecomposable.

First suppose $k = 1$. Since $\xi = f^+(\xi) \leq f^+(\eta) \leq \eta$ and $\xi \leq_1 \eta$, $f^+(\xi) \leq_1 f^+(\eta)$.

Now suppose $k = 2$. In this case, both $\xi$ and $\eta$ must be indecomposable. Therefore, $f^+(\xi) = \xi$ and $f^+(\eta) = \eta$ implying $f^+(\xi) \leq_2 f^+(\eta)$. **QED**

**Definition 15.8** Assume $\mathbf{P}_1, \ldots, \mathbf{P}_n$ are patterns. An *amalgamation* of $\mathbf{P}_1, \ldots, \mathbf{P}_n$ is a pattern $\mathbf{Q}$ along with embeddings $f_i$ of $\mathbf{P}_i$ into $\mathbf{Q}$ for $i = 1, \ldots, n$ such that, letting $\mathbf{P}_i^*$ be the substructure of $\mathbf{Q}$ with universe $f_i[|\mathbf{P}_i|]$,

1. $|\mathbf{Q}| = |\mathbf{P}_1^*| \cup \cdots \cup |\mathbf{P}_n^*|$ and

2. $\mathbf{P}_i^*$ is exact in $\mathbf{Q}$ for $i = 1, \ldots, n$.

We will often say that $\mathbf{Q}, \mathbf{P}_1^*, \ldots, \mathbf{P}_n^*$ is an amalgamation of $\mathbf{P}_1, \ldots, \mathbf{P}_n$ when $\mathbf{P}_i^*$ is a substructure of $\mathbf{Q}$ which is isomorphic to $\mathbf{P}_i$ for $i = 1, \ldots, n$ and $\mathbf{Q}, f_1, \ldots, f_n$ is an amalgamation of $\mathbf{P}_1, \ldots, \mathbf{P}_n$ where $f_i$ is an isomorphism of $\mathbf{P}_i$ with $\mathbf{P}_i^*$ for $i = 1, \ldots, n$. We will also call $\mathbf{Q}$ an amalgamation of $\mathbf{P}_1, \ldots, \mathbf{P}_n$ if there exist $f_1, \ldots, f_n$ such that $\mathbf{Q}, f_1, \ldots, f_n$ is an amalgamation of $\mathbf{P}_1, \ldots, \mathbf{P}_n$.

There are two simple facts worth mentioning here. The first is that if $\mathbf{Q}, \mathbf{P}_1^*, \ldots, \mathbf{P}_n^*$ is an amalgamation of $\mathbf{P}_1, \ldots, \mathbf{P}_n$ and $\tilde{\mathbf{P}}_i$ is isomorphic to $\mathbf{P}_i$ for $i = 1, \ldots, n$ then $\mathbf{Q}, \mathbf{P}_1^*, \ldots, \mathbf{P}_n^*$ is an amalgamation of $\tilde{\mathbf{P}}_1, \ldots, \tilde{\mathbf{P}}_n$. This will allow us to make simplifying assumptions e.g. that $\mathbf{P}_i^* = \mathbf{P}_i$ for $i = 1, \ldots, n$. The second fact is that $\mathbf{P}_1 \cup \cdots \cup \mathbf{P}_n, \mathbf{P}_1, \ldots, \mathbf{P}_n$ is an amalgamation of $\mathbf{P}_1, \ldots, \mathbf{P}_n$ whenever each $\mathbf{P}_i$ is exact in some fixed pattern $\mathbf{Q}$.

**Theorem 15.9** *Assume* $\mathbf{P}_1, \ldots, \mathbf{P}_n$ *are covered patterns.*

1. *There is an amalgamation of* $\mathbf{P}_1, \ldots, \mathbf{P}_n$. *In particular, if* $\mathbf{P}_i^*$ *is the iso-minimal substructure of* $\mathcal{R}_2$ *which is isomorphic to* $\mathbf{P}_i$ *for* $i = 1, \ldots, n$ *then* $(\mathbf{P}_1^* \cup \cdots \cup \mathbf{P}_n^*), \mathbf{P}_1^*, \ldots, \mathbf{P}_n^*$ *is an amalgamation of* $\mathbf{P}_1, \ldots, \mathbf{P}_n$.

2. *The amalgamations of* $\mathbf{P}_1, \ldots, \mathbf{P}_n$ *are unique up to isomorphism in the sense that if* $\mathbf{Q}_1, \mathbf{P}_1^1, \ldots, \mathbf{P}_n^1$ *and* $\mathbf{Q}_2, \mathbf{P}_1^2, \ldots, \mathbf{P}_n^2$ *are both amalgamations of* $\mathbf{P}_1, \ldots, \mathbf{P}_n$ *then there is an isomorphism* $f$ *of* $\mathbf{Q}_1$ *and* $\mathbf{Q}_2$ *which maps* $\mathbf{P}_i^1$ *onto* $\mathbf{P}_i^2$ *for* $i = 1, \ldots, n$.

*Proof.* For part 1, notice that $\mathbf{P}_1^* \cup \cdots \cup \mathbf{P}_n^*$ is closed in $\mathcal{R}_2$ and, consequently, is a pattern. By 4 and 5 of Theorem 14.10, each $\mathbf{P}_i^*$ is exact in $\mathbf{P}_1^* \cup \cdots \cup \mathbf{P}_n^*$.

For part 2, we will show that any amalgamation of $\mathbf{P}_1, \ldots, \mathbf{P}_n$ is isomorphic to $\mathbf{P}_1^* \cup \cdots \cup \mathbf{P}_n^*, \mathbf{P}_1^*, \ldots, \mathbf{P}_n^*$. So, assume $\mathbf{Q}, \mathbf{P}_1{}', \ldots, \mathbf{P}_n{}'$ is an amalgamation of $\mathbf{P}_1, \ldots, \mathbf{P}_n$. For some $k$, $max(\mathbf{P}_k') = max(\mathbf{Q})$. Since $\mathbf{P}_k'$ is exact in $\mathbf{Q}$, $\mathbf{P}_k'$ generates $\mathbf{Q}$. By Lemma 14.5, this implies that $\mathbf{Q}$ is covered. By Theorem 14.10, we may assume $\mathbf{Q}$ is an isominimal substructure of $\mathcal{R}_2$. We will show that $\mathbf{P}_i' = \mathbf{P}_i^*$ for $i = 1, \ldots, n$. Fix $i$ with $1 \le i \le n$. By Lemma 15.7, $(-\infty, max(\mathbf{P}_i')]^{\mathbf{Q}}$ is an isominimal substructure of $\mathcal{R}_2$. Since $\mathbf{P}_i'$ generates $(-\infty, max(\mathbf{P}_i')]^{\mathbf{Q}}$, Theorem 14.11 implies that $\mathbf{P}_i' = \mathbf{P}_i^*$. **QED**

**Lemma 15.10** *1. Assume* $\mathsf{R}$ *is an* $n$*-ary relation symbol and* $(\mathbf{P}_1, a_1), \ldots, (\mathbf{P}_n, a_n) \in \mathsf{H}(\omega)$ *are pointed patterns such that* $\mathbf{P}_i$ *is covered for* $i = 1, \ldots, n$.

$$\mathsf{R}^{\mathcal{C}_2}((\mathbf{P}_1, a_1), \ldots, (\mathbf{P}_n, a_n)) \quad \textit{iff} \quad \mathsf{R}^{\mathbf{Q}}(f_1(a_1), \ldots, f_n(a_n))$$

*whenever* $\mathbf{Q}, f_1, \ldots, f_n$ *is an amalgamation of* $\mathbf{P}_1, \ldots, \mathbf{P}_n$.

*2. Assume* $\mathsf{f}$ *is an* $n$*-ary function symbol and* $(\mathbf{P}_1, a_1), \ldots, (\mathbf{P}_n, a_n), (\mathbf{Q}, b) \in \mathsf{H}(\omega)$ *are pointed patterns such that* $\mathbf{P}_i$ *is covered for* $i = 1, \ldots, n$ *and* $\mathbf{Q}$ *is covered.*

$$\mathsf{f}^{\mathcal{C}_2}((\mathbf{P}_1, a_1), \ldots, (\mathbf{P}_n, a_n), (\mathbf{Q}, b))$$

*iff*

$$\mathsf{f}^{\mathbf{R}}(f_1(a_1), \ldots, f_n(a_n)) = g(b)$$

*whenever* $\mathbf{R}, f_1, \ldots, f_n, g$ *is an amalgamation of* $\mathbf{P}_1, \ldots, \mathbf{P}_n, \mathbf{Q}$.

**Proof.** For part 1, let $\mathbf{P}_i^*$ be the isominimal substructure of $\mathcal{R}_2$ which is isomorphic to $\mathbf{P}_i$ and let $h_i$ be an isomorphism of $\mathbf{P}_i$ and $\mathbf{P}_i^*$ for $i = 1, \ldots, n$. Also let $\mathbf{P}^*$ be $\mathbf{P}_1^* \cup \cdots \cup \mathbf{P}_n^*$. Suppose that $\mathbf{Q}, \mathbf{P}_1', \ldots, \mathbf{P}_n'$ is an amalgamation of $\mathbf{P}_1, \ldots, \mathbf{P}_n$. By the previous theorem, there is an isomorphism $g$ of $\mathbf{P}^*$ and $\mathbf{Q}$ such that $g(h_i(x)) = f_i(x)$ for $i = 1, \ldots, n$ and $x \in |\mathbf{P}_i|$.

$$\begin{array}{lll}
\mathsf{R}^{\mathcal{C}_2}((\mathbf{P}_1, a_1), \ldots, (\mathbf{P}_n, a_n)) & \text{iff} & \mathsf{R}^{\mathcal{R}_2}(\iota(\mathbf{P}_1, a_1), \ldots, \iota(\mathbf{P}_n, a_n)) \\
& \text{iff} & \mathsf{R}^{\mathcal{R}_2}(h_1(a_1), \ldots, h_n(a_n)) \\
& \text{iff} & \mathsf{R}^{\mathbf{P}^*}(h_1(a_1), \ldots, h_n(a_n)) \\
& \text{iff} & \mathsf{R}^{\mathbf{Q}}(g(h_1(a_1)), \ldots, g(h_n(a_n))) \\
& \text{iff} & \mathsf{R}^{\mathbf{Q}}(f_1(a_n), \ldots, f_n(a_n))
\end{array}$$

The proof of part 2 is similar. **QED**

**Lemma 15.11** *Assuming* $\mathsf{ZF}$*, every pattern is covered.*

*Proof.* By the reflection principal, there are arbitrarily large ordinals $\kappa$ such that $\mathcal{R}_2|\kappa \leq_{\Sigma_1} \mathcal{R}_2$. Clearly, $\kappa{\leq_1}\infty$ for any such $\kappa$. Write $\lambda{\leq_2}\infty$ to indicate that $\lambda{\leq_2}\kappa$ whenever $\lambda \leq \kappa$ and $\kappa{\leq_1}\infty$. By the reflection principal, there are arbitrarily large ordinals $\lambda$ with $\mathcal{R}_2|\lambda{\leq_2}\mathcal{R}_2$. An easy argument shows that $\lambda{\leq_2}\infty$ for such $\lambda$. It follows that $\lambda_1{\leq_2}\lambda_2$ whenever $\lambda_1 \leq \lambda_2$ and $\lambda_i{\leq_2}\infty$ for $i = 1, 2$.

If $\mathbf{P}$ is a pattern let $h$ be an embedding of the arithmetic part of $\mathbf{P}$ into $\mathcal{R}_0$ which maps the indecomposables of $\mathbf{P}$ to those $\lambda$ with $\lambda{\leq_2}\infty$. Such an embedding is a covering of $\mathbf{P}$ in $\mathcal{R}_2$. **QED**

**Lemma 15.12** *Assume the set of arithmetic structures in* $\mathsf{H}(\omega)$ *is computable.*

1. *Assume every pattern is covered. The family of generating rules is computable.*

2. *The collection of pairs* $(\mathbf{P}, \mathbf{P}^+) \in \mathsf{H}(\omega)$ *such that* $\mathbf{P}$ *generates* $\mathbf{P}^+$ *is computably enumerable.*

3. *Assume every pattern is covered. The collection of pairs* $(\mathbf{P}, \mathbf{P}^+) \in \mathsf{H}(\omega)$ *such that* $\mathbf{P}$ *generates* $\mathbf{P}^+$ *is computable.*

4. *The collection of tuples*

$$(\mathbf{P}_1, \ldots \mathbf{P}_n, \mathbf{Q}, \mathbf{P}_1^*, \ldots \mathbf{P}_n^*) \in \mathsf{H}(\omega)$$

   *such that* $\mathbf{Q}, \mathbf{P}_1^*, \ldots, \mathbf{P}_n^*$ *is an amalgamation of* $\mathbf{P}_1, \ldots, \mathbf{P}_n$ *is computably enumerable.*

5. *Assume every pattern is covered. The collection of tuples*

$$(\mathbf{P}_1, \ldots \mathbf{P}_n, \mathbf{Q}, \mathbf{P}_1^*, \ldots \mathbf{P}_n^*) \in \mathsf{H}(\omega)$$

   *such that* $\mathbf{Q}, \mathbf{P}_1^*, \ldots, \mathbf{P}_n^*$ *is an amalgamation of* $\mathbf{P}_1, \ldots, \mathbf{P}_n$ *is computable.*

**Proof.** Notice that since the family of arithmetic structures in $\mathsf{H}(\omega)$ is computable, the family of patterns in $\mathsf{H}(\omega)$ is computable.

Parts 2 and 4 follow from Lemma 15.1.

For part 3, assume every pattern is covered. Notice that $\mathbf{P}$ generates $\mathbf{Q}$ iff $\mathbf{Q}, \mathbf{P}, \mathbf{Q}$ is an amalgamation of $\mathbf{P}, \mathbf{Q}$ by Part 1 of Lemma 15.6. By part 1 of Theorem 15.9, there is an amalgamation of $\mathbf{P}, \mathbf{Q}$. This easily implies there is an amalgamation of $\mathbf{P}, \mathbf{Q}$ which has the form $\mathbf{R}, \mathbf{P}', \mathbf{Q}$. Part 2 of Theorem 15.9 implies that $\mathbf{Q}, \mathbf{P}, \mathbf{Q}$ is an amalgamation of $\mathbf{P}, \mathbf{Q}$ iff $\mathbf{P}' = \mathbf{P}$. Therefore,

$$\mathbf{P} \text{ does not generate } \mathbf{Q} \quad \text{iff} \quad \begin{array}{l} \text{there is an amalgamation } \mathbf{R}, \mathbf{P}', \mathbf{Q} \text{ of} \\ \mathbf{P}, \mathbf{Q} \text{ in } \mathsf{H}(\omega) \text{ such that } \mathbf{P}' \neq \mathbf{P} \end{array}$$

This shows that the complement of the set of pairs $(\mathbf{P}, \mathbf{P}^+)$ where $\mathbf{P}$ generates $\mathbf{P}^+$ is computably enumerable. This and part 2 give the desired result.

By Theorem 14.11 and Corollary 14.13, part 1 follows from part 3.

Part 5 follows from part 3. **QED**

**Theorem 15.13** *Assume that the collection of arithmetic structures in $\mathsf{H}(\omega)$ is computable. If $X \subseteq \mathsf{H}(\omega)$ is a computable collection of pointed patterns $(\mathbf{P}, a)$ where $\mathbf{P}$ is covered then the restriction of $\mathcal{C}_2$ to $X$ is computable. In particular, if every pattern is covered then $\mathcal{C}_2$ is computable.*

*Proof.* Since the set of arithmetic structures in $\mathsf{H}(\omega)$ is computable, the set of patterns in $\mathsf{H}(\omega)$ is computable. The theorem now follows from part 1 of Theorem 15.9, Lemma 15.10 and part 4 of the previous lemma. **QED**

**Corollary 15.14** *Assume the collection of arithmetic structures in $\mathsf{H}(\omega)$ is computable. If every pattern is covered then the restriction of $\mathcal{R}_2$ to the core is isomorphic to a computable structure and, hence, the core is a computable ordinal.*

*Proof.* By the theorem, $\mathcal{C}_2$ is a computable structure. Let $X$ be a computable subset of the set of hereditarily finite sets such that $X$ contains one element from each equivalence class of $=^{\mathcal{C}_2}$ e.g. let $X$ consist of the first element of each equivalence class under some computable enumeration of the hereditarily finite sets. The restriction of $\mathcal{C}_2$ to $X$ is clearly isomorphic to the substructure of $\mathcal{R}_2$ whose universe is the core. This also implies that the core is a computable ordinal. **QED**

**Corollary 15.15** *Assume* $\mathsf{ZF}$*. If the collection of arithmetic structures in* $\mathsf{H}(\omega)$ *is computable then the substructure of* $\mathcal{R}_2$ *whose universe is the core is isomorphic to a computable structure and, hence, the core is a computable ordinal.*

*Proof.* By Lemma 15.11, every pattern is covered. **QED**

# 16 Conclusion

There is a natural extension of $\mathcal{R}_2$ that yields patterns of resemblance of all finite orders. First extend the language $\mathcal{L}_0$ to $\mathcal{L}_\omega$ by adding binary relation symbols $\preceq_n$ for $(n \in \omega^+)$. The structure $\mathcal{R}_0$ can be extended to a structure $\mathcal{R}_\omega = (\mathcal{R}_0, \leq_n \ (n \in \omega^+))$ for $\mathcal{L}_\omega$ by inductively defining the interpretations $\leq_n$ of $\preceq_n$ for $n \in \omega^+$ so that

$$\alpha \leq_n \beta \ \text{ iff } \ \mathcal{R}_2|\alpha \preceq_{\Sigma_n} \mathcal{R}_2|\beta$$

for all ordinals $\alpha$ and $\beta$. However, under this definition, the core will not generally be an initial segment of the ordinals (e.g. if one assumes $\mathsf{ZF}$). For this reason, one defines a "collapsed" version $\preceq_n^\infty$ of $\preceq_{\Sigma_n}$ in analogy with the definitions for $n = 1, 2$. Under this definition, much of this paper remains correct with slight modifications. The main gap which remains is part 8 of Lemma 14.8.

Even without the results corresponding to those in this paper, the generating rules in the general case can be used to generate large notation systems. However, we view the equivalence of the resulting notation systems with the core as a fundamental issue.

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