Ranked Partial Structures

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Abstract. The theory of ranked partial structures allows a reinterpretation of several of the standard results of model theory and first-order logic and is intended to provide a proof-theoretic method which allows for the intuitions of model theory. A version of the downward Löwenheim-Skolem theorem is central to our development. In this paper we will present the basic theory of ranked partial structures and their logic including an appropriate version of the completeness theorem.

Proof theory and model theory for universal theories are more transparent than for general theories. Using Skolem's method [9] (also see [10]) one can transform any first-order theory into a universal theory. More precisely, the theory based on the Skolemization of a set of axioms for the original theory has the property that its consequences in the original language comprise exactly the original theory. And more, the models of the original theory are exactly the reducts to the original language of the models of the universal theory based on Skolemizations. However, Skolemized theories have the disadvantage, aesthetic to say the least, of having an array of new function symbols. In this paper we present a model theory of ranked partial structures which is an attempt to capture some of the properties of universal theories without expanding the language. Ranked partial structures are stratified structures which are supposed to correspond to applying Skolem functions stage by stage. Our approach can be traced back to the methods of Kirby and Paris [3] for producing cuts in models of arithmetic by using a kind of indiscernibility. We have also been influenced by Silver's proof [8] of the Paris-Harrington Theorem [6] and Mycielski's finitization technique [5].

We will extend this work in several directions in future papers. We will investigate the determination of formulas (a version of cut-elimination for ranked partial structures). We will apply the theory of ranked partial structures to arithmetic where our treatment of cut-elimination will be extended to provide new insight into the work of Ketonen and Solovay [2] and Mills [4]. Also, we

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intend to investigate an approach to proof theory and ordinal analysis which replaces sequent calculus with methods involving ranked partial structures.

1 Preliminaries

We will use the word *language* to refer to any first-order language.

While the theory of ranked partial structures can be presented in relation to first-order logic, there is a more natural connection with the *logic of partial structures* (for a more detailed description see [1] where this logic is called the logic of partial terms). By the logic of partial structures, which we will denote by \( PS \), we mean the logic of those structures which satisfy the same requirements as first-order structures except that the interpretation of a function symbol need only be a partial operation on the universe of the structure. Since we will view constant symbols as 0-ary function symbols, this means that constants are allowed to go undefined. The formulas in the logic of partial structures for a language \( \mathcal{L} \) are generated exactly as the formulas in first-order logic except there are additional atomic formulas of the form \( t \downarrow \), read "\( t \) converges", where \( t \) is a term. We will use \( t \uparrow \), read "\( t \) diverges", as an abbreviation for \( \neg t \downarrow \). These formulas are given the obvious semantic interpretations: a term will converge in a structure under a given assignment provided it can be computed "from bottom up" i.e. when trying to compute from bottom up, one always remains in the domains of the relevant functions. We then define the semantics for all formulas as usual with the addition that for an atomic formula \( \Phi(t_1, \ldots, t_n) \) to be satisfied we require that all the terms \( t_1, \ldots, t_n \) converge.

Henceforth, the word *formula* will refer to a formula with respect to the logic of partial structures. When we speak of formulas of first-order logic we will use the phrase "first-order formula". By the *complexity* of a formula we mean the height of the tree constructing the formula from atomic formulas. The *height* of a term is the height of the tree constructing the term e.g. variables and constant symbols have height 0.

Our syntax will be based on the logical connectives for conjunction, disjunction, and negation along with both universal and existential quantifiers. And we will have a fixed list of variables \( v_0, v_1, \ldots, v_m, \ldots (n \in \omega) \). A *partial assignment* in \( A \) is any function which maps a set of variables into \( A \). A partial assignment in \( A \) whose domain is the set of all variables will be called an *assignment* in \( A \). If \( \sigma \) and \( \tau \) are functions we define \( \sigma \tau \) to be the function \( \rho \) whose domain is the union of the domains of \( \sigma \) and \( \tau \) where

\[
\rho(x) = \begin{cases} 
\sigma(x) & \text{if } x \in \text{dom}(\sigma) \setminus \text{dom}(\tau) \\
\tau(x) & \text{if } x \in \text{dom}(\tau)
\end{cases}
\]

When \( x_1, \ldots, x_n \) are distinct variables \( (x_1, \ldots, x_n|a_1, \ldots, a_n) \) denotes the function \( \sigma \) with domain \( \{x_1, \ldots, x_n\} \) defined by \( \sigma(x_i) = a_i \) for \( i = 1, \ldots, n \). When \( t_1, \ldots, t_n \) are terms of a language \( \mathcal{L} \) and \( \phi \) is a formula of \( \mathcal{L} \), we will use
\( \phi(x_1, \ldots, x_n|t_1, \ldots, t_n) \) to denote the formula obtained by simultaneously replacing all free occurrences of \( x_i \) by \( t_i \) for \( i = 1, \ldots, n \). Similarly, if \( t \) is a term of \( \mathcal{L} \), \( t(x_1, \ldots, x_n|t_1, \ldots, t_n) \) is the term obtained by simultaneously replacing all occurrences of \( x_i \) by \( t_i \) for \( i = 1, \ldots, n \).

We write \( \mathcal{M} \models \phi[s] \) when \( \mathcal{M} \) is a partial structure which satisfies a formula \( \phi \) under an assignment \( s \) in \( |\mathcal{M}| \) (\( |\mathcal{M}| \) is the universe of \( \mathcal{M} \)). When \( t \) is a term and \( \mathcal{M} \models t[s] \) we write \( t^t[\mathcal{M}, s] \) for the value of \( t \) in \( \mathcal{M} \) under the assignment \( s \). When \( \Phi \) is a set of first-order formulas and \( \phi \) is a first-order formula, we use standard notation and write \( \Phi \models \phi \) when \( \Phi \) implies \( \phi \) in first-order logic. We will write \( \Phi \models_{PS} \phi \) when \( \Phi \) is a set of formulas and \( \phi \) is a formula such that \( \Phi \) implies \( \phi \) in the logic of partial structures.

**Definition 1.1** Assume \( \mathcal{L} \) is a language. \( \Sigma_n \) and \( \Pi_n \) are sets of formulas of \( \mathcal{L} \) defined simultaneously by induction on \( n \in \omega \). \( \Sigma_0 \) and \( \Pi_0 \) are both the collection of quantifier free formulas. Given \( \Sigma_n \) and \( \Pi_n \), \( \Sigma_{n+1} \) and \( \Pi_{n+1} \) are generated by the following clauses.

- \( \phi \in \Sigma_n \cup \Pi_n \Rightarrow \phi \in \Sigma_{n+1} \).
- \( \phi \in \Pi_{n+1} \Rightarrow \neg \phi \in \Sigma_{n+1} \).
- \( \phi, \psi \in \Sigma_{n+1} \Rightarrow \phi \land \psi, \phi \lor \psi \in \Sigma_{n+1} \).
- For each variable \( x \), \( \phi \in \Sigma_{n+1} \Rightarrow \exists x \phi \in \Sigma_{n+1} \).
- \( \phi \in \Sigma_n \cup \Pi_n \Rightarrow \phi \in \Pi_{n+1} \).
- \( \phi \in \Pi_{n+1} \Rightarrow \neg \phi \in \Pi_{n+1} \).
- \( \phi, \psi \in \Pi_{n+1} \Rightarrow \phi \land \psi, \phi \lor \psi \in \Pi_{n+1} \).
- For each variable \( x \), \( \phi \in \Pi_{n+1} \Rightarrow \forall x \phi \in \Pi_{n+1} \).

In the logic of partial structures, every formula is equivalent to a formula in which negation is only applied to atomic formulas.

**Definition 1.2** Assume \( \mathcal{L} \) is a language. A formula of \( \mathcal{L} \) is in **negation normal form** if the only applications of negation within the formula are to atomic formulas. The **negation normal form**, \( \phi^{nnf} \), of a formula \( \phi \) of \( \mathcal{L} \) is defined by induction on the complexity of \( \phi \) so that

- If \( \phi \) is atomic then \( \phi^{nnf} = \phi \).
- \( (\alpha \land \beta)^{nnf} = \alpha^{nnf} \land \beta^{nnf} \).
- \( (\alpha \lor \beta)^{nnf} = \alpha^{nnf} \lor \beta^{nnf} \).
- \( (\forall x \psi)^{nnf} = \forall x \psi^{nnf} \).
• $(\exists x \psi)^{\text{nf}} = \exists x \psi^{\text{nf}}$.
• If $\phi$ is the negation of an atomic formula then $\phi^{\text{nf}} = \phi$.
• $((\neg (\alpha \land \beta))^{\text{nf}} = (\neg \alpha)^{\text{nf}} \lor (\neg \beta)^{\text{nf}}$.
• $(\neg (\alpha \lor \beta))^{\text{nf}} = (\neg \alpha)^{\text{nf}} \land (\neg \beta)^{\text{nf}}$.
• $(\neg \forall x \phi)^{\text{nf}} = \exists x (\neg \phi)^{\text{nf}}$.
• $(\neg \exists x \phi)^{\text{nf}} = \forall x (\neg \phi)^{\text{nf}}$.
• $(\neg \neg \psi)^{\text{nf}} = \psi^{\text{nf}}$.

$\phi$ and $\phi^{\text{nf}}$ are equivalent in the logic of partial structures. This equivalence will also hold in the logic of ranked partial structures.

Notice that $\phi$ is in $\Sigma_n$ (respectively, $\Pi_n$) iff its negation normal form is in $\Sigma_n$ (respectively, $\Pi_n$).

In general, formulas in negation normal form are easier to work with than arbitrary formulas.

**Definition 1.3** A formula $\phi$ of a language $\mathcal{L}$ is in *variable normal form* if no variable which occurs free in $\phi$ also occurs quantified in $\phi$ and no variable is quantified at two separate locations in $\phi$.

Of course, every formula has an alphabetic variant which is in variable normal form (an alphabetic variant of a formula is a formula obtained by renaming bound variables so as to respect the quantifications). So, every formula is equivalent in the logic of partial structures to a formula in negation normal form which is also in variable normal form. We want to consider formulas in negation normal form and variable normal form because discussing Skolem forms is more convenient for such formulas.

The semantics for ranked partial structures is motivated in part by the method of “Skolemizing” formulas. There are several ways of doing this and we will need to be precise as to which method we have in mind.

Let $\mathcal{L}$ be a first-order language. A Skolemization of a formula $\phi$ of $\mathcal{L}$ is a formula in a language extending $\mathcal{L}$ which is obtained, roughly, by replacing each existential assertion in $\phi$ by a new function symbols which is supposed to pick out a witness to the existential assertion if it is true. To illustrate several ways of doing this, consider the formula $\forall x \exists y \exists z \mathbf{P}(w, x, y, z)$. The following four formulas may all be considered Skolemizations of $\phi$.

1. $\forall x \mathbf{P}(w, x, f(x), g(x))$
2. $\forall x \mathbf{P}(w, x, f(w, x), g(w, x))$
3. $\forall x \mathbf{P}(w, x, f(x), g(x, f(x)))$
4. \( \forall x P(w, x, f(w, x), g(w, x, f(w, x))) \)

Examples 2 and 4 differ from 1 and 3 by allowing the choice of \( y \) and \( z \) to depend on \( w \). This dependence will be important for our semantics. Example 4 differs from example 2 by allowing the choice of \( z \) to depend on the choice of \( y \). Of course, this dependence is not real as the choice of \( y \) can be recovered from \( w \) and \( x \) in example 4. For this reason, example 2 may appear to be more natural. However, the semantics for ranked partial structures is much smoother if we base it on example 4 rather than example 2.

This discussion of Skolemizations has to be modified slightly in the context of partial structures. We would like a Skolemization to have the property that a sentence is satisfied in a partial structure if and only if there is an expansion of the structure which satisfies a Skolemization of the formula. Consider the formula \( \exists x \neg P(x) \). The Skolemizations discussed above will be of the form \( \neg P(c) \) where \( c \) is a constant symbol. \( \neg P(c) \) will be true in any expansion in which \( c \) is undefined, even though \( \exists x \neg P(x) \) may be false in the original partial structure. This situation can be remedied by adding a conjunct which says that the Skolem term converges.

Some annoying technicalities are avoided by restricting the following definition to formulas which are in both variable normal form and negation normal form.

**Definition 1.4** Assume \( L \) is a language. For \( \phi \) a formula of \( L \) which is in variable normal form and negation normal form define the set, possibly empty, of formulas of \( L \) which are **Skolem forms** of \( \phi \) by induction on the complexity of \( \phi \) so that the following conditions hold.

1. If \( \phi \) is atomic or the negation of an atomic formula then the only formula of \( L \) which is a Skolem form of \( \phi \) is \( \phi \) itself.

2. If \( \phi \) is \( \alpha \land \beta \) then a formula of \( L \) is a Skolem form of \( \phi \) if it has the form \( \alpha^* \land \beta^* \) where \( \alpha^* \) is a Skolem form of \( \alpha \), \( \beta^* \) is a Skolem form of \( \beta \), no function symbol which occurs in \( \alpha^* \) but not \( \alpha \) also occurs in \( \beta^* \), and no function symbol which occurs in \( \beta^* \) but not \( \beta \) also occurs in \( \alpha^* \).

3. If \( \phi \) is \( \alpha \lor \beta \) then a formula of \( L \) is a Skolem form of \( \phi \) if it has the form \( \alpha^* \lor \beta^* \) where \( \alpha^* \) is a Skolem form of \( \alpha \), \( \beta^* \) is a Skolem form of \( \beta \), no function symbol which occurs in \( \alpha^* \) but not \( \alpha \) also occurs in \( \beta^* \), and no function symbol which occurs in \( \beta^* \) but not \( \beta \) also occurs in \( \alpha^* \).

4. If \( \phi \) is \( \forall x \alpha \) then a formula of \( L \) is a Skolem form of \( \phi \) if it has the form \( \forall x \alpha^* \) where \( \alpha^* \) is a Skolem form of \( \alpha \).

5. If \( \phi \) is \( \exists x \alpha \) then a formula of \( L \) is a Skolem form of \( \phi \) if it is of the form

\[
\mathcal{f}(y_1, \ldots, y_n) \land \alpha^*(x|f(y_1, \ldots, y_n))
\]
where $\alpha^*$ is a Skolem form of $\alpha$, $f$ is a function symbol which does not occur in $\alpha^*$, and $y_1, \ldots, y_m$ lists without repetitions all the free variables of $\phi$.

Notice that when we assume that $\phi^*$ is a Skolem form of $\phi$ we are also implicitly assuming that $\phi$ is in negation normal form and variable normal form. If $\phi$ is in negation normal form and variable normal form then a Skolem form of $\phi$ is also. Moreover, a Skolem form of $\phi$ is $\Pi_1$ implying, since it is in negation normal form, that it contains no existential quantifiers.

Roughly, a Skolem form $\phi^*$ for $\phi$ is obtained by dropping all existential quantifiers and replacing each variable which was existentially quantified by an appropriate Skolem term. When the Skolem term for a variable $z$ is of the form $f(t_1, \ldots, t_n)$ we will refer to $f$ as the skolem function symbol corresponding to $z$ in $\phi^*$.

The definition of Skolem form must be modified slightly for first-order logic: the Skolem form will not have subformulas concerning the convergence of the Skolem terms. So, if a variable $z$ is existentially quantified but in a trivial way (that is, with no free occurrences of $z$ in the scope of the quantifier) then there is no way of recovering a Skolem term or Skolem function symbol for $z$ from $\phi^*$.

**Definition 1.5** Assume $\phi$ is a formula of $\mathcal{L}$ and $\mathcal{L}^*$ is a language extending $\mathcal{L}$. A Skolem form $\phi^*$ of $\phi$ is a Skolem form of $\phi$ over $\mathcal{L}$ if every function symbol in $\phi^*$ which does not occur in $\phi$ is not in $\mathcal{L}$.

2 A Semantics for Ranked Partial Structures

In this section we present a semantics for ranked partial structures and examine its relationship to the semantics of partial structures.

While ranked partial structures were developed with first-order logic in mind, we are naturally lead to present them in the wider context of the logic of partial structures.

**Definition 2.1** Assume $\mathcal{L}$ is a first-order language. $\mathcal{A}$ is a ranked partial structure for $\mathcal{L}$ if $\mathcal{A}$ is of the form $(M, X_i \ (i \in I), \prec)$ where $M$ is a partial structure for $\mathcal{L}$, each $X_i$ is a nonempty subset of the universe of $M$, $\prec$ is a strict linear ordering of $I$, $X_i \subseteq X_j$ whenever $i \prec j$, and the condition (*) below holds.

(*) If $i \in I$, $a_1, \ldots, a_n \in \bigcup_{j \prec i} X_j$, $f$ is an $n$-place function symbol of $\mathcal{L}$, and $(a_1, \ldots, a_n)$ is in the domain of $f^M$ then $f^M(a_1, \ldots, a_n) \in X_i$.

Define $|\mathcal{A}|$, the universe of $\mathcal{A}$, to be $\bigcup_{i \in I} X_i$ and let $\mathcal{A}|\mathcal{L}$ be the substructure of $M$ whose universe is $|\mathcal{A}|$. The bounded part of $\mathcal{A}$ is the union of those $X_i$ where $i$ is not the maximal element of $(I, \prec)$ (if it exists). $M$ will be referred to as the underlying partial structure of $\mathcal{A}$. $(I, \prec)$ is the rank ordering of $\mathcal{A}$. If
the rank ordering of $\mathcal{A}$ is a well-ordering, the ordinal with the same type as the rank ordering will be called the \textit{height} of $\mathcal{A}$. The indexed family $X_i (i \in I)$ is the \textit{rank hierarchy} of $\mathcal{A}$ and the $X_i$ are the \textit{ranks} of $\mathcal{A}$.

We allow the universe of $\mathcal{A}$ to be different than that of $\mathcal{M}$ only for notational convenience. The elements of the universe of $\mathcal{M}$ which are not in the universe of $\mathcal{A}$ will play no role in the semantics for $\mathcal{A}$.

Suppose $\mathcal{A} = (\mathcal{M}, X_i (i \in I), \prec)$ is a ranked partial structure for $\mathcal{L}$. We will often write $R_{X_i}$ for $X_i$ or even write $R_i$ when there will be no confusion. Similarly, we will define $R_{X_i}^A$ to be $\bigcup_{j \prec i} X_j$ and write $R_{A,j}$ when $A$ is understood.

When $\prec$ is clear from the context, we will sometimes write $(\mathcal{M}, X_i (i \in I))$ for $(\mathcal{M}, X_i (i \in I), \prec)$.

The most natural example of a ranked partial structure is obtained by taking a first-order structure and appending part of the hierarchy generated by applying the interpretations of the function symbols stage by stage. Another class of important examples comes from applying a collection of Skolem functions stage by stage. So, for us the most natural situation is when $(I, \prec)$ is a well ordering. In fact, we will be most concerned with ranked partial structures where $I$ is of the form $\{0, 1, \ldots, n\}$ for some natural number $n$ (ordered as usual). In this case, we will sometimes write $(\mathcal{M}, X_0, \ldots, X_n)$ for $(\mathcal{M}, X_i (i \leq n))$.

\textbf{Definition 2.2} Assume $\mathcal{F}$ is a collection of partial operations on a set $A$ i.e. each element of $\mathcal{F}$ is a partial function from $A^n$ into $A$ for some $n \in \omega$ ($n = 0$ is allowed). If $(I, \prec)$ is a linear ordering and $X_i (i \in I)$ is a family of subsets of $A$ with $X_i \subseteq X_j$ for $i \prec j$ then $X_i (i \in I)$ is \textit{closed under} $\mathcal{F}$ with respect to $(I, \prec)$ if $f(a_1, \ldots, a_n) \in X_j$ whenever $f(a_1, \ldots, a_n)$ is defined and $a_1, \ldots, a_n \in \bigcup_{i \prec j} X_i$.

A ranked partial structure $\mathcal{A} = (\mathcal{M}, X_i (i \in I), \prec)$ is \textit{closed under} $\mathcal{F}$ if the universe of $\mathcal{A}$ is a subset of $A$ and $X_i (i \in I)$ is closed under $\mathcal{F}$ with respect to $(I, \prec)$.

When the ordering on $I$ is understood, we often omit reference to $\prec$ and say that $X_i (i \in I)$ is closed under $\mathcal{F}$. In fact, the definition is independent of $\prec$ since $\prec$ can be recovered sufficiently by the fact that the $X_i$ are linearly ordered by $\subseteq$.

Notice that the condition $(*)$ of definition 2.1 says that $R_i (i \in I)$ is closed under the interpretations of the function symbols of $\mathcal{L}$ in the underlying structure.

\textbf{Definition 2.3} Assume $\mathcal{A}$ is a ranked partial structure for $\mathcal{L}$ and $\mathcal{N}$ is a partial structure for $\mathcal{L}$. $\mathcal{A}$ is an \textit{approximation} of $\mathcal{N}$ if $\mathcal{A} | \mathcal{L}$ is a partial substructure of $\mathcal{N}$ and $\mathcal{A}$ is closed under the set of interpretations of the function symbols of $\mathcal{L}$ in $\mathcal{N}$.

Before defining the full semantics for ranked partial structures we will consider formulas of the form $t \uparrow$ and $t_i$. Recall that $t \uparrow$ is a abbreviation for $\neg t_i$. 

While the next definition will appear somewhat technical, the lemma given afterwards gives an equivalence which is more intuitive.

**Definition 2.4** Assume $\mathcal{A}$ is a ranked partial structure for $\mathcal{L}$. For $t$ a term of $\mathcal{L}$ and $s$ an assignment in $[\mathcal{A}]$ we say that $t$ **converges** in $\mathcal{A}$ under $s$ provided that whenever $f(t_1, \ldots, t_m)$ is a subterm of $t$ with the property that each of $t_1^{(\mathcal{A|L}, s)}, \ldots, t_m^{(\mathcal{A|L}, s)}$ is defined and all are in the bounded part of $\mathcal{A}$ then $(t_1^{(\mathcal{A|L}, s)}, \ldots, t_m^{(\mathcal{A|L}, s)})$ is in the domain of $f^{\mathcal{A|L}}$. We say that $t$ **diverges** in $\mathcal{A}$ under $s$ if $t^{(\mathcal{A|L}, s)}$ is not defined.

Notice that a term $t$ may converge in a ranked partial structure $\mathcal{A}$ under some $s$ even though $t^{(\mathcal{A|L}, s)}$ is not defined. In this case, $t$ both converges and diverges in $\mathcal{A}$ under $s$. For example, suppose $\mathcal{A}$ is a ranked partial structure whose rank ordering has a largest element $i$ and suppose $a \in R_i \setminus R_{<i}$. If $f$ is a unary function symbol such that $f^{\mathcal{A|L}}(a)$ is undefined and $s$ is an assignment with $s(x) = a$ then $f(x)^{\mathcal{A|L}, s}$ is undefined while $\mathcal{A} \vDash f(x)^{\mathcal{A|L}, s}$.

**Lemma 2.5** Assume $\mathcal{A}$ is a ranked partial structure for $\mathcal{L}$. If $t$ is a term and $s$ is an assignment in $[\mathcal{A}]$ then

1. $t$ converges in $\mathcal{A}$ under $s$ iff there is a partial structure $\mathcal{N}$ such that $\mathcal{A}$ is an approximation of $\mathcal{N}$ and $\mathcal{N} \vDash t^{\mathcal{N}, s}$.

2. $t$ diverges in $\mathcal{A}$ under $s$ iff there is a partial structure $\mathcal{N}$ such that $\mathcal{A}$ is an approximation of $\mathcal{N}$ and $\mathcal{N} \models t^{\mathcal{N}, s}$.

**Proof.** (1) For the forward direction, let $\mathcal{N}$ be a one element extension of $\mathcal{A|L}$ by adding a new element $\infty$. Suppose $f$ is an $n$-ary function symbol of $\mathcal{L}$. We define $f^\mathcal{N}$ to be the extension of $f^{\mathcal{A|L}}$ to the tuples $(a_1, \ldots, a_n)$ of elements of $[\mathcal{N}]$ which are not in the domain of $f^{\mathcal{A|L}}$ and where not all of $a_1, \ldots, a_n$ are in the bounded part of $\mathcal{A}$ so that $f^\mathcal{N}(a_1, \ldots, a_n) = \infty$. In particular, $f^\mathcal{N}(a_1, \ldots, a_n) = \infty$ whenever some $a_i$ is $\infty$. A straightforward induction on the height of terms $t$ shows that if $\mathcal{A} \vDash t^{\mathcal{N}, s}$ then $\mathcal{N} \vDash t^{\mathcal{N}, s}$.

The reverse direction is straightforward.

(2) Clear since $\mathcal{A}$ approximates $\mathcal{A|L}$ and $\mathcal{A|L}$ is a partial substructure of $\mathcal{N}$ whenever $\mathcal{A}$ approximates $\mathcal{N}$. $\square$

The equivalences given in the lemma provide the motivation for the preceding definition. We did not choose them for our definition in order to avoid quantifying over all approximated structures.

We will also use equivalences of the form given in the lemma to motivate our definition of when $\phi$ is satisfied in a ranked partial structure when $\phi$ is the negation of an atomic formula or an atomic formula which isn’t of the form $t_1 = t_2$. While we could use the same strategy for the semantics of $t_1 = t_2$, simplicity and uniformity can be gained by taking a slightly different approach.
Moreover, one can always choose to work with equality as an undistinguished binary predicate which satisfies the usual axioms for a congruence.

**Definition 2.6** Suppose $\mathcal{L}^+$ is a language which extends another language $\mathcal{L}$, $\mathcal{A}$ is a ranked partial structure for $\mathcal{L}$, and $\mathcal{A}^+$ is a ranked partial structure for $\mathcal{L}^+$. $\mathcal{A}^+$ is an expansion of $\mathcal{A}$ if $\mathcal{A}^+$ and $\mathcal{A}$ have the same rank ordering, say $(I, \prec)$, $R_i^{\mathcal{A}^+} = R_i^\mathcal{A}$ for all $i \in I$, and $\mathcal{A}^+\mid \mathcal{L}^+$ is an expansion of $\mathcal{A}\mid \mathcal{L}$.

We will now define the semantics for ranked partial structures. In addition to the motivation for defining the semantics of atomic formulas and negations of atomic formulas discussed above, we will be guided by the following:

1. Conjunction, disjunction and universal quantification are to be interpreted as in first-order logic.
2. The negation of a formula is to be viewed as the dual of the original formula.
3. A formula $\phi$ in negation normal form should be satisfied in a ranked partial structure $\mathcal{A}$ under an assignment $s$ iff there is an expansion of $\mathcal{A}$ which satisfies a Skolem form of (an alphabetic variant of) $\phi$ under $s$.

Condition 2 implies that each formula will be equivalent to its negation normal form. Given the semantics for atomic formulas and negations of atomic formulas, condition 1 will determine the semantics of formulas in negation normal form which contain no existential quantifiers. Since a Skolem form of a formula in negation normal form contains no existential quantifiers, condition 3 now completely determines a semantics of ranked partial structures.

Rather than adhering strictly to principle 3, we will choose another semantics which provides advantages which include greater elegance. We will compare our choice with alternative semantics, including that based on principle 3, in the final section. The clauses in our definition will be standard except for the existential quantifier where we will require that if the interpretations of the free variables of a formula of the form $\exists x \phi$ are in the bounded part then a witness for $x$ must exist soon afterward in the rank hierarchy.

**Definition 2.7** Assume $\mathcal{A} = (\mathcal{M}, X_i (i \in I), \prec)$ is a ranked partial structure for $\mathcal{L}$. We define when $\mathcal{A}$ satisfies a formula $\phi$ under an assignment $s$, written $\mathcal{A} \models \phi[s]$, by induction on the complexity of the formula $\phi$ simultaneously for all assignments $s$ in $|\mathcal{A}|$ so that the following clauses hold:

1. $\mathcal{A} \models t_i[s]$ iff $t_i$ converges in $\mathcal{A}$ under $s$.
2. $\mathcal{A} \models \mathbf{P}(t_1, \ldots, t_n)[s]$ iff $t_i$ converges in $\mathcal{A}$ under $s$ for $i = 1, \ldots, n$ and if all of $t_1^{(\mathcal{A}\mid \mathcal{L}, s)}, \ldots, t_n^{(\mathcal{A}\mid \mathcal{L}, s)}$ are defined then $(t_1^{(\mathcal{A}\mid \mathcal{L}, s)}, \ldots, t_n^{(\mathcal{A}\mid \mathcal{L}, s)}) \in \mathbf{P}\mathcal{M}$.
3. $\mathcal{A} \models \phi \land \psi[s]$ iff $\mathcal{A} \models \phi[s]$ and $\mathcal{A} \models \psi[s]$. 

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4. $\mathcal{A} \models \phi \lor \psi[s]$ if $\mathcal{A} \models \phi[s]$ or $\mathcal{A} \models \psi[s]$.

5. $\mathcal{A} \models \forall x \phi[s]$ if $\mathcal{A} \models \phi[s(x[a])]$ for all $a \in |\mathcal{A}|$.

6. $\mathcal{A} \models \exists x \phi[s]$ if whenever $i \in I$ has the property that $s$ maps the free variables of $\exists x \phi$ into $R_{<i}$ then there is $a \in R_i$ such that $\mathcal{A} \models \phi[s(x[a])]$.

7. $\mathcal{A} \models \neg t[s]$ if $t$ diverges in $\mathcal{A}$ under $s$.

8. $\mathcal{A} \models \neg \mathbf{P}(t_1, \ldots, t_n)[s]$ if either $t_i$ diverges in $\mathcal{A}$ under $s$ for some $i$ or else each of $t_1^{\mathcal{A}[\mathcal{L}, s]}, \ldots, t_n^{\mathcal{A}[\mathcal{L}, s]}$ is defined and $(t_1^{\mathcal{A}[\mathcal{L}, s]}, \ldots, t_n^{\mathcal{A}[\mathcal{L}, s]}) \notin \mathbf{P}^M$.

9. $\mathcal{A} \models \neg (\phi \land \psi)[s]$ if $\mathcal{A} \models \neg \phi[s]$ or $\mathcal{A} \models \neg \psi[s]$.

10. $\mathcal{A} \models \neg (\phi \lor \psi)[s]$ if $\mathcal{A} \models \neg \phi[s]$ and $\mathcal{A} \models \neg \psi[s]$.

11. $\mathcal{A} \models \neg \forall x \phi[s]$ if whenever $i \in J$ has the property that $s$ maps the free variables of $\neg \forall x \phi$ into $R_{<i}$ then there is $a \in R_i$ such that $\mathcal{A} \models \neg \phi[s(x[a])]$.

12. $\mathcal{A} \models \neg \exists x \phi[s]$ if $\mathcal{A} \models \neg \phi[s(x[a])]$ for all $a \in |\mathcal{A}|$.

13. $\mathcal{A} \models \neg \phi[s]$ if $\mathcal{A} \models \phi[s]$.

We consider $=$ to be a special predicate symbol whose interpretation in a partial structure $\mathcal{M}$ is the relation of equality restricted to $|\mathcal{M}|$. Therefore, clauses 2 and 8 determine when formulas of the form $t_1 = t_2$ or $t_1 \neq t_2$ are satisfied.

An important fact to keep in mind is that $\mathcal{A} \models \exists x \phi[s]$ holds trivially if there is no $i$ as in clause 6 of definition 2.7 i.e. if $s(y)$ fails to be in the bounded part of $\mathcal{A}$ for some variable $y$ which occurs free in $\exists x \phi$ (in this case, the rank ordering must have a largest element). We will use this fact frequently when constructing examples below.

As with first-order semantics, whether or not $\mathcal{A} \models \phi[s]$ does not depend on the values of $s$ on variables which do not occur free in $\phi$.

In section 6 we will see that the logic of ranked partial structures and the logic of partial structures are connected in a natural way:

A set of sentences is satisfiable in a partial structure if it is satisfiable in a ranked partial structure of height $n$ for each positive natural number $n$.

**Lemma 2.8** Assume $\mathcal{A}$ is a ranked partial structure for $\mathcal{L}$. If $\phi$ is a formula of $\mathcal{L}$ then

$$\mathcal{A} \models \phi[s] \quad \text{iff} \quad \mathcal{A} \models \phi^{nmf}[s]$$

whenever $s$ is an assignment in $|\mathcal{A}|$. 

10
Proof. Induction on the complexity of $\phi$ (notice that $\phi$ and $\phi^{m}$ have the same free variables).

This lemma can be used to simplify many inductive proofs by allowing us to restrict to formulas in negation normal form.

**Definition 2.9** Assume $\mathcal{A}$ is a ranked partial structure for $\mathcal{L}$, $\phi$ is a formula of $\mathcal{L}$, and $\sigma$ is a partial assignment in $|\mathcal{A}|$ whose domain includes each variable which occurs free in $\phi$. Define $\mathcal{A} \models \phi[\sigma]$ if $\mathcal{A} \models \phi[s]$ for all assignments $s$ in $|\mathcal{A}|$ which extend $\sigma$. Define $\mathcal{A} \models \phi$ if $\mathcal{A} \models \phi[\emptyset]$.

When $\sigma$ is $(x_{1}, \ldots, x_{n}|a_{1}, \ldots, a_{n})$ we will drop the parentheses and write $\mathcal{A} \models \phi[x_{1}, \ldots, x_{n}|a_{1}, \ldots, a_{n}]$ for $\mathcal{A} \models \phi[\sigma]$.

**Remark 2.10** We can now verify that our semantics is in accord with our earlier discussion concerning atomic formulas. Assume $\mathcal{A}$ is a ranked partial structure for $\mathcal{L}$, $\phi$ is the negation of an atomic formula or an atomic formula which is not of the form $t_{1} = t_{2}$ and $s$ is an assignment in $|\mathcal{A}|$. We will argue that $\mathcal{A} \models \phi[s]$ iff there is a partial structure $\mathcal{N}$ such that $\mathcal{A}$ is an approximation of $\mathcal{N}$ and $\mathcal{N} \models \phi[s]$.

The reverse direction is straightforward using lemma 2.5 (and holds true also for formulas of the form $t_{1} = t_{2}$).

For the forward direction, first suppose $\phi$ is an atomic formula and $\mathcal{A} \models \phi[s]$. Let $\mathcal{N}$ be a one element extension of $\mathcal{A}|\mathcal{L}$ obtained by adding a new element $\Sigma$ and extending the interpretations of the function symbols as in the proof of part 1 of lemma 2.5 so that we will have $\mathcal{N} \models t_{1}[s]$ whenever $\mathcal{A} \models t_{1}[s]$. One can easily extend the predicate in $\phi$ to $\mathcal{N}$ so that $\mathcal{N} \models \phi[s]$.

When $\phi$ is the negation of an atomic formula one can verify the forward direction by letting $\mathcal{N}$ be $\mathcal{A}|\mathcal{L}$.

To see that the equivalence does not extend to the case $t_{1} = t_{2}$ in general, suppose $\mathcal{A}$ is a ranked partial structure, $f$ is a function symbol of $\mathcal{L}$, and $a, b \in |\mathcal{A}|$ where $a$ is not in the bounded part of $\mathcal{A}$ and $f^{\mathcal{A}|\mathcal{L}}$ is undefined at $a$. Consider the formula $f(x) = y$. $\mathcal{A} \models f(x) = y[x, y|a, b]$ but there is no partial structure $\mathcal{N}$ of which $\mathcal{A}$ is an approximation with $\mathcal{N} \models f(x) = y[x, y|a, b]$.

**Definition 2.11** Assume $\mathcal{A} = (\mathcal{M}, X_{i} (i \in I), \prec)$ is a ranked partial structure. For $J$ a nonempty subset of $I$, the localization of $\mathcal{A}$ to $J$ is the ranked partial structure $(\mathcal{M}, X_{i} (i \in J), \prec)$. A ranked partial structure is a localization of $\mathcal{A}$ if it is the localization of $\mathcal{A}$ to $J$ for some $J \subseteq I$. A ranked partial structure $\mathcal{B}$ is an extension of $\mathcal{A}$ if $\mathcal{A}$ is a localization of $\mathcal{B}$. $\mathcal{B}$ is an end extension of $\mathcal{A}$ if $\mathcal{B}$ is an extension of $\mathcal{A}$ and the rank ordering of $\mathcal{B}$ is an end extension of the rank ordering of $\mathcal{A}$.

The following lemma appears natural, but its proof is sensitive to the particular way we have defined the satisfaction predicate for ranked partial structures.
Lemma 2.12 (localization lemma) Assume \( A \) is a ranked partial structure for \( \mathcal{L} \) and \( B \) is a localization of \( A \). If \( \phi \) is a formula and \( s \) is an assignment in \( |B| \) such that \( A \models \phi[s] \) then \( B \models \phi[s] \).

Proof. Straightforward induction on the complexity of \( \phi \) using lemma 2.5 for the cases when \( \phi \) has one of the forms \( t \downarrow \) or \( t \uparrow \). \( \square \)

The previous lemma is important for our subsequent development. Our definition of the semantics of negation was complicated by the desire to have a formula equivalent to its negation normal form so that the lemma would hold. If we had defined the semantics of negation by

\[
A \models \neg \phi[s] \Leftrightarrow A \not\models \phi[s]
\]

the lemma would have failed. For example, let \( A = (\mathcal{M}, \{a\}, \{a,b\}) \) be a ranked partial structure for a language including a unary predicate symbol \( P \) such that the interpretation of \( P \) is \( \{a\} \). We see that \( (\mathcal{M}, \{a\}, \{a,b\}) \not\models \forall x P(x) \) while \( (\mathcal{M}, \{a\}) \models \forall x P(x) \). So, if negation had been defined as above we would have \( (\mathcal{M}, \{a\}, \{a,b\}) \models \neg \forall x P(x) \) while \( (\mathcal{M}, \{a\}) \not\models \neg \forall x P(x) \).

Notice that for the structure \( A \) above we also have \( A \not\models \exists x \neg P(x) \) or, equivalently, \( A \not\models \neg \forall x P(x) \). Letting \( \phi \) be \( \forall x P(x) \), we see that \( A \) does not determine whether \( \phi \) is true or false i.e. \( A \not\models \phi \) and \( A \not\models \neg \phi \). The interplay between the semantics of a formula and its negation will be discussed further in the following section.

The semantic consequences of substituting equals for equals is somewhat complicated in ranked partial structures. The difficulty has to do with the existential quantifier. Even though \( A \models t_1 = t_2 [s] \), the conditions which determine whether \( A \models \exists y \phi(x,t_1)[s] \) may differ with those for \( A \models \exists y \phi(x,t_2)[s] \) in regard to where the witness for \( y \) must appear in the rank hierarchy (this will depend on which variables occur in \( t_1 \) and \( t_2 \)). With this in mind, one should expect that substituting equals for equals should rarely be legitimate.

For example, assume \( A = (\mathcal{M}, \{a\}) \) is a ranked partial structure for a language containing a binary predicate symbol \( P \) and a constant symbol \( c \) so that \( (a,a) \) is not in the interpretation of \( P \) and \( c \) converges in \( A \) with value \( a \). Trivially, \( A \models \exists y P(x,y)[x|a] \) (there is no \( i \) as in the defining clause for the existential quantifier in definition 2.7). Also, \( A \models x = c[x|a] \). On the other hand \( A \not\models \exists y P(c,y) \) since otherwise there would have to be a witness for \( y \) in \( R_0 = \{a\} \) which there is not.

Lemma 2.13 (substitution lemma) Assume \( A \) is a ranked partial structure for \( \mathcal{L} \) and \( t_1 \) and \( t_2 \) are terms of \( \mathcal{L} \). If \( \phi \) is a formula of \( \mathcal{L} \), \( s \) is an assignment in \( |A| \), \( x \) is a variable such that both \( t_1 \) and \( t_2 \) are substitutable for \( x \) in \( \phi \),

\[
(a) \ A \models t_1 \uparrow [s] \implies A \models t_2 \uparrow [s],
\]

and
(b) \( \mathcal{A} \models t_1 \downarrow [s] \) implies \( \mathcal{A} \models t_1 = t_2 \downarrow [s] \) (in particular, \( \mathcal{A} \models t_2 \downarrow [s] \))

then

1. if \( \phi \) is \( \Pi_1 \) then

\[
\mathcal{A} \models \phi(x|t_1) \downarrow [s] \Rightarrow \mathcal{A} \models \phi(x|t_2) \downarrow [s],
\]

2. letting \( X_k \) be the set of variables occurring in \( t_k \) for \( k = 1, 2 \), if \( s[X_2] \subseteq R_i \)

implies \( s[X_1] \subseteq R_i \) for all \( i \) (in particular, if every variable occurring in \( t_1 \) occurs in \( t_2 \)) then

\[
\mathcal{A} \models \phi(x|t_1) \downarrow [s] \Rightarrow \mathcal{A} \models \phi(x|t_2) \downarrow [s],
\]

and

3. if \( \mathcal{B} \) is the localization of \( \mathcal{A} \) to \( J \) where \( J \) consists of all \( i \) in the rank

ordering such that either \( s \) does not map the variables of \( t_2 \) into \( R_{-i} \) or \( s \)

does map the variables of \( t_1 \) into \( R_{-i} \) then

\[
\mathcal{A} \models \phi(x|t_1) \downarrow [s] \Rightarrow \mathcal{B} \models \phi(x|t_2) \downarrow [s].
\]

Proof. Notice that \( (\phi(x|t))^{\text{nf}} = \phi^{\text{nf}}(x|t) \) for any term \( t \) and variable \( x \).

Therefore, lemma 2.8 allows us to assume that \( \phi \) is in negation normal form.

Claim: Suppose (a) and (b) hold. If \( t \) is a term and \( x \) is a variable then, setting \( \bar{t}_i = t(x|t_i) \) for \( i = 1, 2 \),

- \( \mathcal{A} \models \bar{t}_1 \uparrow [s] \) implies \( \mathcal{A} \models \bar{t}_2 \uparrow [s] \), and
- \( \mathcal{A} \models \bar{t}_1 \downarrow [s] \) implies \( \mathcal{A} \models \bar{t}_1 = \bar{t}_2 \downarrow [s] \)

The proof of the claim is a straightforward induction on the height of \( t \).

We can now establish part 1 of the conclusion by induction on the complexity of \( \phi \) simultaneously for all \( s \) where \( \phi \) is a \( \Pi_1 \) formula in negation normal form.

When \( \phi \) is atomic or the negation of an atomic formula the claim implies the desired conclusion. This handles the case where \( \phi \) is of the form \( \neg \psi \) since \( \phi \) is in negation normal form. The remaining cases are straightforward.

As part 2 is a special case of part 3, we turn our attention to part 3.

We establish part 3 by induction on the complexity of \( \phi \) simultaneously for all \( s \). Suppose \( \phi, s, t_1, \) and \( t_2 \) are as in the hypothesis of the lemma. The cases when \( \phi \) is an atomic formula or the negation of an atomic formula are taken care of by part 1 and the localization lemma. The cases where \( \phi \) is a disjunction, conjunction, or of the form \( \forall y \psi \) are straightforward.

Assume \( \phi \) has the form \( \exists x \theta \) and \( \mathcal{A} \models \phi(x|t_1) \downarrow [s] \). In order to establish that \( \mathcal{B} \models \phi(x|t_2) \downarrow [s] \), suppose that \( j \in J \) and \( s \) maps the free variables of \( \phi(x|t_2) \)

into \( R_{-j} \). By the choice of \( J \), \( s \) maps the free variables of \( \phi(x|t_1) \) into \( R_{-j} \).
Therefore, there is an \( a \in R_j \) such that \( A \models \theta(x[t_1]) [s(z[a])] \). By the induction hypothesis, \( B \models \theta(x[t_2]) [s(z[a])] \). This implies that \( B \models \phi(x[t_2]) [s] \). \( \square \)

An important consequence of part 2 of the substitution lemma is that alphabetic variants are equivalent in ranked partial structures i.e. if \( \hat{\phi} \) is an alphabetic variant of \( \phi \) then \( A \models \phi [s] \) iff \( A \models \hat{\phi} [s] \) for all \( A \) and \( s \). We leave the straightforward proof by induction to the reader.

**Remark 2.14** Individual constant symbols and variables play distinct roles in our semantics. However, they can almost be used interchangeably in some situations.

Assume \( A \) is a ranked partial structure for \( \mathcal{L} \), \( \phi \) is a formula of \( \mathcal{L} \), and \( c \) is a constant symbol such that \( c^A \mathcal{L} \) is defined with value \( c \). By part 2 of the previous lemma

\[
A \models \phi(x[c])[s] \Rightarrow A \models \phi(s(x[c]))
\]

whenever \( s \) is an assignment in \( |A| \). The example before the previous lemma shows that the converse doesn’t hold in general. However, part 3 of the previous lemma implies the converse holds if the rank ordering doesn’t have a minimal element. Moreover, if the rank ordering does have a minimal element then by part 3 of the lemma

\[
A \models \phi [s(x[c])] \Rightarrow B \models \phi(x[c]) [s]
\]

where \( B \) is the localization of \( A \) to the set of nonminimal elements of the rank ordering.

As with substitution, universal instantiation can be problematic in ranked partial structures (as the example before the previous lemma shows): one should not expect \( A \models \forall x \phi [s] \) to imply \( A \models \phi(x[t]) [s] \) in general. The next lemma describes some exceptions.

**Lemma 2.15** Assume \( A \) is a ranked partial structure for \( \mathcal{L} \), \( s \) is an assignment in \( |A| \), \( \forall x \phi \) is a formula of \( \mathcal{L} \), and \( t \) is a term which is substitutable for \( x \) in \( \phi \).

1. If \( A \models t[t] [s], A \models t[t] [s] \), and \( t \) contains an occurrence of a variable then

\[
A \models \forall x \phi[s] \Rightarrow A \models \phi(x[t]) [s]
\]

2. If \( A \models t[t] [s], A \models t[t] [s] \), and \( t \) is a closed term then

\[
A \models \forall x \phi[s] \Rightarrow B \models \phi(x[t]) [s]
\]

when \( B \) is the localization of \( A \) to the set of nonminimal elements of the rank ordering.
3. If \( \phi \) is \( \Pi_1 \) and \( \mathcal{A} \models t_i[s] \) then 
\[
\mathcal{A} \models \forall x \phi[s] \Rightarrow \mathcal{A} \models \phi(x|t)[s]
\]

Proof. We may assume that \( x \) does not occur in \( t \) (otherwise, replace \( \forall x \phi \) by an alphabetic variant \( \forall \bar{x} \phi(x|\bar{x}) \) where \( \bar{x} \) does not occur in \( t \).

Assume the hypotheses of part 1. Also suppose \( \mathcal{A} \models \forall x \phi[s] \). Let \( y \) be a variable which occurs in \( t \) and set \( a = s(y) \). We will use part 2 of the substitution lemma with \( t_1 = x \) and \( t_2 = t \) and \( s \) replaced by \( s(x|a) \).

To see that hypotheses (a) and (b) of the substitution lemma hold, notice that \( \mathcal{A} \not\models x^+ [s(x|a)] \) and \( \mathcal{A} \models x = t [s(x|a)] \) (both are immediate from definition 2.7 using for the latter the fact that \( \mathcal{A} \models t^+ [s] \) which implies \( \mathcal{A} \models t^+ [s(x|a)] \)).

Since \( \mathcal{A} \models \forall \phi[s] \), \( \mathcal{A} \models \phi(s(x|a)) \). By part 2 of the substitution lemma, \( \mathcal{A} \models \phi(x|t) [s(x|a)] \). Since \( x \) does not occur in \( t \), it does not occur free in \( \phi(x|t) \). Therefore, \( \mathcal{A} \models \phi(x|t)[s] \).

Assume the hypotheses of part 2. \( t \) must contain some constant symbol \( c \). Since \( \mathcal{A} \models \downarrow t[s] \), \( c^\mathcal{A}c \subseteq \mathcal{L} \) is defined and equals some \( a \in [\mathcal{A}] \). Notice that \( a \in \bigcap_{i \in I} R_i \). Let \( z \) be some variable which does not occur in \( \phi \) or \( t \). We may assume that \( s(z) = a \). Part 3 of the substitution lemma applies taking \( t_1 \) to be \( z \) and \( t_2 \) to be \( t \). The conclusion of part 1 follows since \( J \) will be the set of nonminimal elements of the rank ordering.

Assume the hypothesis of part 3. Let \( z \) be a variable which does not occur in \( \phi \) or \( t \). Since \( \mathcal{A} \models t_i[s] \), we may assume that \( \mathcal{A} \models t = z[s] \) (otherwise replace \( s \) by \( s(z|a) \) for an appropriate \( a \) ). Apply part 1 of the substitution lemma taking \( t_1 \) to be \( z \) and \( t_2 \) to be \( t \). \( \square \)

We will now show that we have come close to satisfying principle 3 (which says that a formula is satisfied in a structure iff there is an expansion which satisfies some Skolem form). That principle 3 does not hold outright can be seen by considering the formula \( \forall x \exists y (P(x) \land \neg \neg P(x)) \). \( \exists y (P(x) \land \neg \neg P(x)) \) is trivially satisfied by any assignment in any ranked partial structure of height 1 (since there is no \( i \) as in the defining clause for the existential quantifier in definition 2.7). Therefore, \( \forall x \exists y (P(x) \land \neg \neg P(x)) \) is satisfied in every ranked partial structure of height 1. However, its Skolem form, \( \forall x (f(x)_i \land P(x) \land \neg \neg P(x)) \), is not satisfied in any ranked partial structure.

We now extend to ranked partial structures the standard way of adding Skolem functions to a partial structure.

**Definition 2.16** Assume \( \mathcal{A} \) is a ranked partial structure for \( \mathcal{L} \), \( \phi \) is a formula of \( \mathcal{L} \), and \( \phi^* \) a formula of \( \mathcal{L} \) which is a Skolem form of \( \phi \). If \( s \) is an assignment in \( [\mathcal{A}] \) we define whether \( \phi^* \) witnesses \( \phi \) in \( \mathcal{A} \) at \( s \) by induction on the complexity of \( \phi \) so that the following clauses hold.
1. If $\phi$ is atomic or the negation of an atomic formula then $\phi^*$ witnesses $\phi$ in $A$ at $s$ iff $A \models \phi[s]$.

2. Suppose $\phi$ is $\alpha \land \beta$. $\phi^*$ is of the form $\alpha^* \land \beta^*$. $\phi^*$ witnesses $\phi$ in $A$ at $s$ iff $\alpha^*$ witnesses $\alpha$ in $A$ at $s$ and $\beta^*$ witnesses $\beta$ in $A$ at $s$.

3. Suppose $\phi$ is $\alpha \lor \beta$. $\phi^*$ is of the form $\alpha^* \lor \beta^*$. $\phi^*$ witnesses $\phi$ in $A$ at $s$ with respect to $\phi^*$ iff $\alpha^*$ witnesses $\alpha$ in $A$ at $s$ or $\beta^*$ witnesses $\beta$ in $A$ at $s$.

4. Suppose $\phi$ is $\forall y \psi$. $\phi^*$ has the form $\forall y \psi^*$. $\phi^*$ witnesses $\phi$ in $A$ at $s$ iff $\psi^*$ witnesses $\psi$ in $A$ at $s(y[a])$ for all $a \in |A|$.

5. Suppose $\phi$ is $\exists z \theta$. $\phi^*$ has the form $f(y_1, \ldots, y_n) \downarrow \land \theta^*(z[f(y_1, \ldots, y_n)])$. $\phi^*$ witnesses $\phi$ in $A$ at $s$ iff either
   - $s(y_i)$ fails to be in the bounded part of $A$ for some $i$ or
   - $f^A_L(s(y_1), \ldots, s(y_n))$ is defined and is equal to some $a \in |A|$ such that $\theta^*$ witnesses $\theta$ in $A$ at $s(z[a])$.

**Lemma 2.17** Assume $A$ is a ranked partial structure for $L$, $\phi$ is a formula of $L$, and $\phi^*$ is a formula of $L$ which is a Skolem form of $\phi$. If $s$ is an assignment in $|A|$ such that $\phi^*$ witnesses $\phi$ in $A$ at $s$ then $A \models \phi[s]$.

**Proof.** We argue by induction on the complexity of $\phi$ simultaneously for all $\phi^*$ and $s$. The argument is straightforward for the cases when $\phi$ is an atomic formula, the negation of an atomic formula, a conjunction, a disjunction, or of the form $\forall y \theta$.

Suppose $\phi$ has the form $\exists z \theta$.

Assume $\phi^*$ witnesses $\phi$ in $A$ at $s$. Also suppose $s$ maps the free variables of $\phi$ into $R_{\xi_i}$. We will find $a \in R_i$ such that $A \models \theta[s(z[a])]$.

$\phi^*$ must have the form

$$f(y_1, \ldots, y_n) \downarrow \land \theta^*(z[f(y_1, \ldots, y_n)])$$

for some Skolem form $\theta^*$ of $\theta$. $\theta^*$ witnesses $\theta$ in $A$ at $s(z[a])$ where $a = f^A_L(s(y_1), \ldots, s(y_n))$. Since $s(y_1), \ldots, s(y_n) \in R_{\xi_i}, a \in R_i$. By the induction hypothesis, $A \models \theta[s(z[a])]$. \hfill $\Box$

**Definition 2.18** Assume $A$ is a ranked partial structure for $L$, $\phi$ is a formula of $L$, and $\phi^*$ is a Skolem form of $\phi$ over $L$. Also assume that $A^*$ is an expansion of $A$ to a language $L^*$ which contains the function symbols which occur in $\phi^*$. $A^*$ is a *Skolem expansion* of $A$ for $\phi$ with respect to $\phi^*$ provided $\phi^*$ witnesses $\phi$ in $A^*$ at $s$ whenever $A \models \phi[s]$.
Theorem 2.19 Assume $\mathcal{A}$ is a ranked partial structure for $\mathcal{L}$, $\phi$ is a formula of $\mathcal{L}$, and $\phi^*$ is a Skolem form for $\phi$ over $\mathcal{L}$. If $\mathcal{A}^*$ is a Skolem expansion of $\mathcal{A}$ for $\phi$ with respect to $\phi^*$ then for any assignment $s$ in $[\mathcal{A}]$, $\phi^*$ witnesses $\phi$ in $\mathcal{A}$ at $s$ iff $\mathcal{A} \models \phi[s]$.

Proof. By the definition of Skolem expansion and lemma 2.17. □

Definition 2.20 Assume $\mathcal{A}$ is a ranked partial structure for $\mathcal{L}$ and $\exists z \theta$ is a formula of $\mathcal{L}$. Let $y_1, \ldots, y_n$ be an enumeration of the free variables of $\exists z \theta$. An $n$-ary partial operation $f$ on $[\mathcal{A}]$ is a Skolem function for $\exists z \theta$ in $\mathcal{A}$ with respect to $y_1, \ldots, y_n$ provided that if

- $a_1, \ldots, a_n$ are any elements of the bounded part of $\mathcal{A}$ and
- $\mathcal{A} \models \exists z \theta[y_1, \ldots, y_n|a_1, \ldots, a_n]$ then

- $f(a_1, \ldots, a_n)$ is defined,
- $f(a_1, \ldots, a_n)$ is an element of $R_i$ whenever $a_1, \ldots, a_n \in R_{< i}$, and
- $\mathcal{A} \models \theta[y_1, \ldots, y_n, z|a_1, \ldots, a_n, f(a_1, \ldots, a_n)]$.

Suppose $\mathcal{A}$, $\exists z \theta$, and $y_1, \ldots, y_n$ are as in the definition above. If the rank ordering of $\mathcal{A}$ is a well ordering then, using the axiom of choice, there is a Skolem function for $\phi$ in $\mathcal{A}$ with respect to $y_1, \ldots, y_n$.

Theorem 2.21 Assume $\mathcal{A}$ is a ranked partial structure for $\mathcal{L}$, $\phi$ is a formula of $\mathcal{L}$, and $\phi^*$ is a Skolem form for $\phi$ over $\mathcal{L}$. If the rank ordering of $\mathcal{A}$ is well-founded then there is a Skolem expansion of $\mathcal{A}$ for $\phi$ with respect to $\phi^*$.

Proof. Straightforward induction on formulas where the interpretation of each Skolem function symbol is an appropriate Skolem function. □

The condition that the rank ordering be a well-ordering is needed in the previous theorem. For example, consider the formula $\exists x \ x = x$ in a ranked partial structure in which the intersection of the $R_i$ is empty e.g. a structure with rank ordering the set of natural numbers with the reverse of the usual ordering where $R_i = \{i, i+1, i+2, \ldots\}$ for each $i \in \omega$. The Skolem form of $\exists x \ x = x$, $c, f(c) \land c = c$, is not satisfiable in any expansion since the interpretation of $c$ would need to be in each $R_i$. In the final section we will discuss an alternative but less elegant semantics for the existential quantifier which is equivalent to the semantics we have adopted when the rank ordering is a well-ordering and for which the conclusion of the theorem holds in general.
**Corollary 2.22** Assume $A$ is a ranked partial structure for $\mathcal{L}$ with well-founded rank ordering, $\phi$ is a formula of $\mathcal{L}$, and $\phi^*$ is a Skolem form for $\phi$ over $\mathcal{L}$. If $s$ is an assignment in $|A|$ then $A \models \phi [s]$ iff there is an expansion $A^*$ of $A$ such that $\phi^*$ witnesses $\phi$ in $A^*$ at $s$.

**Proof.** By the previous theorem and lemma 2.17. □

**Remark 2.23** While the example preceding definition 2.16 shows that under the assumptions of the theorem we cannot expect $\phi^*$ to be satisfied in $A^*$ with $s$ whenever $A \models \phi [s]$ (implying condition 3 after definition 2.6 cannot hold in general), the other direction is true:

$$A^* \models \phi^* [s] \Rightarrow A \models \phi [s]$$

We argue by induction on the complexity of $\phi$ simultaneously for all $\phi^*$ and $s$ where $\phi^*$ is a Skolem form of $\phi$ and $s$ is an assignment in $|A|$.

The argument is straightforward for the cases when $\phi$ is an atomic formula, the negation of an atomic formula, a conjunction, a disjunction, or of the form $\forall y \theta$.

Suppose $\phi$ has the form $\exists z \theta$.

Assume that $s$ maps the free variables of $\phi$ into $R_{<q}$. We will show there exists $a \in R_i$ such that $A \models \theta [s(z|a)]$. $\phi^*$ must have the form

$$f(y_1, \ldots, y_n) \downarrow \land \theta^* (z|f(y_1, \ldots, y_n))$$

for some Skolem form $\theta^*$ of $\theta$. Assuming $A \models f(y_1, \ldots, y_n) \downarrow [s], (s(y_1), \ldots, s(y_n))$ is in the domain of $f^A|\mathcal{L}$ and $f^A|\mathcal{L}(s(y_1), \ldots, s(y_n)) \in R_i$. Since $\theta^*$ is $\Pi_1$, part 1 of the substitution lemma implies that $A \models \theta^* [s(z|a)]$ where $a = f^A|\mathcal{L}(s(y_1), \ldots, s(y_n))$.

By the induction hypothesis, $A \models \theta [s(z|a)]$.

The following lemma shows that while condition 3 doesn’t hold in general we have come close to satiating it.

**Lemma 2.24** Assume $A$ is a ranked partial structure for $\mathcal{L}$ with rank ordering $(I, \prec)$, $\phi$ is a formula of $\mathcal{L}$, and $\phi^*$ is a Skolem form of $\phi$ over $\mathcal{L}$. Suppose in addition that $B$ is a Skolem expansion of $A$ for $\phi$ with respect to $\phi^*$.

1. If $B^-$ is the localization of $B$ to some initial segment $J$ of $(I, \prec)$ such that the cardinality of $I \setminus J$ is at least the complexity of $\phi$ then

$$A \models \phi [s] \Rightarrow B^- \models \phi^* [s]$$

for all assignments $s$ in $|B^-|$.

2. If we add to part 1 the assumption that the rank ordering does not have a maximal element then

$$A \models \phi [s] \iff B \models \phi^* [s].$$

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for all $s$.

Proof. We prove part 1 by induction on the complexity of $\phi$ simultaneously for all $\phi^*$, $J$ and $B$ as in the assumption of the lemma. When $\phi$ is an atomic formula or the negation of an atomic formula, the conclusion of part 1 follows from the localization lemma. The cases when $\phi$ is a disjunction, conjunction, or of the form $\forall y \theta$ are straightforward.

Suppose $\phi$ has the form $\exists x \theta$. $\phi^*$ is $f(y_1, \ldots, y_n) \land \theta^*(z|f(y_1, \ldots, y_n))$ where $\theta^*$ is a Skolem form of $\theta$. Let $\phi^*$ and $B$ be is in the assumption of the lemma. Moreover, let $B^-$ and $J$ be as in the hypothesis of part 1 and suppose that $A \models \phi[s]$ where $s$ is an assignment in $[B^-]$. Let $k$ be the complexity of $\phi$ and choose $i \in I \setminus J$ such that there are at least $k - 1$ elements $i'$ of $I$ with $i \prec i'$. Let $a_m = s(y_m)$ for $m = 1, \ldots, n$. Since $s$ is an assignment in $[B^-]$, $a_1, \ldots, a_n \in R_{\prec i}$. Since $A \models \exists x \theta(y_1, \ldots, y_n[a_1, \ldots, a_n], f^B\xi(a_1, \ldots, a_n))$ is defined and is an element of $R_i$. In addition, $A \models \theta(y_1, \ldots, y_n, z[a_1, \ldots, a_n, f^B\xi(a_1, \ldots, a_n)]$. Letting $B_i$ be the localization of $B$ to $\{i' \in I \mid i' \prec i\}$, the induction hypothesis implies that $B_i \models \theta^*(y_1, \ldots, y_n, z[a_1, \ldots, a_n, f^B\xi(a_1, \ldots, a_n)]$. By part 1 of the substitution lemma, $B_i \models \theta^*(z|f(y_1, \ldots, y_n))[y_1, \ldots, y_n[a_1, \ldots, a_n]$. By the localization lemma, $B^- \models \theta^*(z|f(y_1, \ldots, y_n))[y_1, \ldots, y_n[a_1, \ldots, a_n]$. By part 2 can be proved by an induction similar to that for part 1.

The complexity of $\phi$ is a conservative estimate for required size of $I \setminus J$ in part 1 of the preceding lemma. The argument shows that the maximum size of a sequence of nested existential quantifiers in $\phi$ would suffice. A better estimate is the maximum of the heights of the Skolem terms, but the proof is somewhat delicate and uses a modified semantics for $B$ where the universal quantifiers range over $|B^-|$ (the generalized notion of term height in definition 5.3 below can sometimes provide an even better estimate).

3 Negation

One of the key issues in the semantics of ranked partial structures is the relationship of a formula and its negation. In contrast to first-order semantics where for a sentence $\phi$ exactly one of $\phi$ and $\neg \phi$ is satisfied, there are instances of ranked partial structures where both are satisfied and others where neither is satisfied i.e. $\phi \land \neg \phi$ can be satisfiable and $\phi \lor \neg \phi$ is not valid in RPS in general (the obvious meaning of “valid in RPS” will be given as a formal definition in the next section).

Definition 3.1 Assume $A$ is a ranked partial structure for $L$ and $\phi$ is a formula of $L$. For $s$ an assignment in $|A|$, $A$ determines $\phi$ at $s$ if $A \models (\phi \lor \neg \phi)[s]$ i.e. either $A \models \phi[s]$ or $A \models \neg \phi[s]$. $A$ determines $\phi$ if $A$ determines $\phi$ at $s$ for all assignments $s$ in $|A|$. A completely determines $\phi$ at $s$ if $A$ determines $\phi$ at $s$ and
\( \mathcal{A} \models (\phi \land \neg \phi)[s] \) i.e. either \( \mathcal{A} \models \phi[s] \) or \( \mathcal{A} \not\models \neg \phi[s] \). \( \mathcal{A} \) completely determines \( \phi \) if \( \mathcal{A} \) completely determines \( \phi \) at \( s \) for all assignments \( s \) in \( \mathcal{A} \).

Building ranked partial structures which determine given formulas is closely tied to the cut rule (this will be investigated in a later paper). A ranked partial structure often fails to determine some formulas as the example after lemma 2.12 illustrates.

Notice that \( \mathcal{A} \) determines \( \phi \) iff \( \mathcal{A} \models \forall x_1 \cdots \forall x_m (\phi \lor \neg \phi) \) where \( x_1, \ldots, x_m \) lists the free variables in \( \phi \). Also notice that every atomic formula is determined in any ranked partial structure.

We now consider the possibility that both a formula and its negation are satisfied. As mentioned earlier, we will see that a sentence is satisfiable in a partial structure iff it is satisfied in a ranked partial structure of each finite height. This means that for any sentence \( \phi \) there is an \( n \) such that \( \phi \land \neg \phi \) is not satisfied in any ranked partial structure of height \( n \). We will now give an upper bound for \( n \).

**Lemma 3.2** Assume \( \mathcal{A} \) is a ranked partial structure for \( \mathcal{L} \) with rank ordering \( (I, \prec) \). If \( t \) is a term of height \( h \) and \( s \) is an assignment which maps the variables which occur in \( t \) into \( R_i \) for at least \( h + 1 \) distinct \( i \) then

\[
\mathcal{A} \models t[s] \iff \mathcal{A}|\mathcal{L} \models t[s]
\]

**Proof.** Let \( (I, \prec) \) be the rank ordering of \( \mathcal{A} \) and suppose \( i_0 \prec i_1 \prec \cdots \prec i_h \) and \( s \) maps the variables of \( t \) into \( R_{i_0} \). By induction on \( k = 0, \ldots, h \) we can show that if \( u \) is any subterm of \( t \) of height \( k \) and \( s \) maps the variables of \( u \) into \( R_{i_0} \) then \( u^{(\mathcal{A},s)} \) is defined and is an element of \( R_{i_0} \). This establishes the forward direction of the conclusion of the lemma. The reverse direction follows from part 2 of lemma 2.5. \( \Box \)

**Definition 3.3** Assume \( \mathcal{L} \) is a first-order language. We define the **height** of a formula of \( \mathcal{L} \) by induction on the complexity of the formula by the following clauses.

- The height of an atomic formula is the maximum of the heights of the terms which occur in the formula.
- The height of \( \phi \land \psi \) and of \( \phi \lor \psi \) is the maximum of the height of \( \phi \) and the height of \( \psi \).
- The height of \( \forall x \phi \) and \( \exists x \phi \) is \( h + 1 \) where \( h \) is the height of \( \phi \).
- The height of \( \neg \phi \) is the same as the height of \( \phi \).
Lemma 3.4 Assume \( A \) is a ranked partial structure for \( \mathcal{L} \) with rank ordering \((I, \prec)\) and \( \alpha \) is an atomic formula of \( \mathcal{L} \). If \( s \) is an assignment in \(|A|\) which maps the variables which occur in \( \alpha \) into \( R_i \) for at least \( h+1 \) distinct \( i \) where \( h \) is the height of \( \alpha \) then \( A \) completely determines \( \alpha \) at \( s \) i.e.

\[
A \models -\alpha[s] \iff A \not\models \alpha[s]
\]

Proof. This follows from the previous lemma which is equivalent to the case when \( \alpha \) has the form \( \forall \).

\[\Box\]

Theorem 3.5 Assume \( A \) is a ranked partial structure for \( \mathcal{L} \). If \( \phi \) is a formula of height \( h \) and \( s \) is an assignment in \(|A|\) such that there are at least \( h+1 \) distinct \( i \) with the property that \( s \) maps the free variables of \( \phi \) into \( R_i \) then \( A \not\models (\phi \land -\phi)[s] \).

Proof. The previous lemma gives the case when \( \phi \) is atomic. We can now argue by induction on the complexity of \( \phi \) simultaneously for all \( s \) to establish the lemma in full generality.

In contrast to the previous theorem, often a sentence and its negation will both be satisfied in ranked partial structures whose rank ordering is small enough. This statement will be made precise below in the rather technical theorem 3.9. We will consider a simple example here.

Consider the sentence \( \sigma = \forall x \exists y \forall z P(x, y, z) \). Trivially, \( A \models \exists y \forall z P(x, y, z) \) for any \( A \) of rank 1 and any \( s \). Therefore, \( \sigma \) is satisfied in any ranked partial structure of rank 1. \( \neg \sigma \) is also satisfied in every ranked partial structure of rank 1. To see this, consider the negation normal form of \( \neg \sigma \), \( \exists x \forall y \exists z \neg P(x, y, z) \).

Since \( A \models \exists z \neg P(x, y, z) \) trivially for any \( A \) of rank 1 and any assignment \( s \) in \( A \), \( \exists x \forall y \exists z \neg P(x, y, z) \) is satisfied in every ranked partial structure of rank 1. This implies that \( \neg \sigma \) is satisfied in every ranked partial structure of rank 1.

In the previous example we have used the fact that \( A \models \exists x \phi[s] \) whenever there is no \( i \) such that \( s \) maps the free variables of \( \exists x \phi \) into \( R_{<i} \) i.e. \( s \) does not map the free variables into the bounded part of \( A \). In that case, there is a maximal element of the rank ordering and the set of values \( s \) takes on the free variables of \( \exists x \phi \) is not contained in any \( R_i \) unless \( i \) is the maximal element. We will now draw what are, in a sense, the strongest consequences of this fact.

To help motivate the following definition, imagine there is an assignment \( s \) in a ranked partial structure whose rank ordering is a natural number and \( \eta(x) \) is the least \( i \) such that \( s(x) \) is in \( R_i \) for each free variable \( x \) of \( \phi \). Our definition of the forcing height of \( \phi \) relative to \( \eta \) is an attempt to find a lower bound for the largest integer \( n \) such that \( \phi \) is satisfied in every ranked partial structure of height \( n \) with any assignment \( s \) which maps the variables into the rank hierarchy according to \( \eta \) as just described.

**Definition 3.6** Assume \( \mathcal{L} \) is a first-order language. If \( \phi \) is a formula of \( \mathcal{L} \) in negation normal form and \( \eta \) is a function whose domain is a set of variables
containing the free variables of $\phi$ and whose range is contained in $\omega$ then we define the forcing height of $\phi$ relative to $\eta$ by induction on the complexity of $\phi$ simultaneously for all $\eta$ by the following clauses.

- If $\phi$ is an atomic formula or the negation of an atomic formula then the forcing height of $\phi$ relative to $\eta$ is 0.
- The forcing height of $\phi \land \psi$ relative to $\eta$ is the minimum of the forcing height of $\phi$ relative to $\eta$ and the forcing height of $\psi$ relative to $\eta$.
- The forcing height of $\phi \lor \psi$ relative to $\eta$ is the maximum of the forcing height of $\phi$ relative to $\eta$ and the forcing height of $\psi$ relative to $\eta$.
- The forcing height of $\forall x \phi$ relative to $\eta$ is the minimum as $n$ ranges over all natural numbers of the forcing height of $\phi$ relative to $\eta(x | n)$.
- The forcing height of $\exists x \phi$ relative to $\eta$ is the maximum of $n$ and the forcing height of $\phi$ relative to $\eta(x | n)$ where $n$ is the least strict upper bound of the set of numbers $\eta(y)$ where $y$ is a free variable of $\exists x \phi$.

If $\phi$ is a sentence we define the forcing height of $\phi$ to be the forcing height of $\phi$ relative to the empty function.

Clearly, the forcing height of $\phi$ relative to $\eta$ depends only on the values of $\eta$ on the free variables of $\phi$.

Notice that the forcing height of any $\Pi_1$ formula in negation normal form is 0.

**Lemma 3.7** Assume $\mathcal{L}$ is a language.

1. The forcing height of $\phi$ relative to $\eta$ is monotone in $\eta$ i.e. if $\eta(x) \leq \eta'(x)$ for all free variables $x$ of $\phi$ then the forcing height of $\phi$ relative to $\eta$ is less than or equal to the forcing height of $\phi$ relative to $\eta'$.

2. The forcing height of $\forall x \phi$ relative to $\eta$ is equal to the forcing height of $\phi$ relative to $\eta(x | 0)$.

**Proof.** 1 is proved by induction on the complexity of $\phi$ simultaneously for all $\eta$. 2 follows immediately from 1. \qed

Part 2 of the lemma provides an algorithm for computing the forcing height.

**Definition 3.8** A ranked partial structure $\mathcal{A}$ for $\mathcal{L}$ with rank ordering $(I, \prec)$ is proper if $R^A_j$ is a proper subset of $R^A_i$ whenever $j \prec i$. 

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Theorem 3.9 Assume \( \mathcal{A} \) is a proper ranked partial structure for \( \mathcal{L} \) whose rank ordering is a natural number \( n \). Also, suppose that \( \phi \) is a formula of \( \mathcal{L} \) which is in negation normal form, \( \eta \) maps a set of variables containing the free variables of \( \phi \) into \( n \), and \( s \) is an assignment in \( |\mathcal{A}| \) such that \( s(x) \notin R_{\prec \eta(x)} \) for each free variable \( x \) of \( \phi \). If the height of \( \mathcal{A} \) is less than or equal to the forcing height of \( \phi \) relative to \( \eta \) then \( \mathcal{A} \models \phi[s] \).

Proof. Argue by induction on the complexity of \( \phi \) simultaneously for all \( \eta \) and \( s \).

Assume \( s \) is an assignment in \( |\mathcal{A}| \) such that \( s(x) \notin R_{\prec \eta(x)} \) for each \( x \) which occurs free in \( \phi \). Also assume that the height of \( \mathcal{A} \) is at most the forcing height of \( \phi \).

For the case when \( \phi \) is atomic or the negation of an atomic formula the lemma is vacuously true since there are no ranked partial structures of height 0. The cases when \( \phi \) is a conjunction, disjunction, or of the form \( \forall x \psi \) are straightforward.

Assume \( \phi \) has the form \( \exists y \psi \). Suppose \( s \) maps the free variables of \( \exists y \psi \) into \( R_{\prec i} \). We must find \( a \in R_i \) such that \( \mathcal{A} \models \psi[s(y|a)] \). Let \( n \) be the least strict upper bound of the natural numbers \( \eta(x) \) where \( x \) is a free variable in \( \exists y \psi \). Since \( s(x) \notin R_{\prec \eta(x)} \) and \( s(x) \in R_i \) for each free variable \( x \) of \( \exists y \psi \), we see that \( n \leq i \). Therefore, \( n \) is less than the height of \( \mathcal{A} \) which, in turn, is at most the forcing height of \( \exists y \psi \). By definition, the forcing height of \( \exists y \psi \) must be the forcing height of \( \psi \) with respect to \( \eta(y|n) \). Since \( \mathcal{A} \) is proper, there exists \( a \) in \( R_i - R_{\prec i} \). Since \( n \leq i \), \( a \notin R_n \). By the induction hypothesis, \( \mathcal{A} \models \psi[s(y|a)] \). □

Notice that the sentence \( \forall x \exists y \forall z \mathcal{P}(x, y, z) \) used in the example following theorem 3.5 and the negation normal form of its negation both have forcing height 1. So we could use the theorem to conclude that both of these sentences are satisfied in all ranked partial structures of height 1.

4 The Logic of Ranked Partial Structures

We now define the logic of ranked partial structures, \( \text{RPS} \), in the obvious way.

Definition 4.1 Assume \( \Gamma \) is a set of formulas of the language \( \mathcal{L} \) and \( \phi \) and \( \psi \) are formulas of \( \mathcal{L} \). We say the \( \Gamma \) implies \( \phi \) in \( \text{RPS} \), also denoted by \( \Gamma \models_{\text{RPS}} \phi \) if for every ranked partial structure \( \mathcal{A} \) for \( \mathcal{L} \) and every assignment \( s \) in \( |\mathcal{A}| \)

\[
\mathcal{A} \models \Gamma[s] \Rightarrow \mathcal{A} \models \phi[s]
\]

Define \( \phi \models_{\text{RPS}} \psi \) just in case \( \{\phi\} \models_{\text{RPS}} \psi \). When \( \emptyset \models_{\text{RPS}} \phi \) we write \( \models_{\text{RPS}} \phi \) and say that \( \phi \) is valid in \( \text{RPS} \). We say that \( \phi \) and \( \psi \) are equivalent in \( \text{RPS} \), written \( \phi \equiv_{\text{RPS}} \psi \), if both \( \phi \models_{\text{RPS}} \psi \) and \( \psi \models_{\text{RPS}} \phi \).

Lemma 4.2 Assume \( \phi_i \equiv_{\text{RPS}} \psi_i \) for \( i = 1, 2 \).
1. \( \phi_1 \land \phi_2 \equiv_{RPS} \psi_1 \land \psi_2 \).
2. \( \phi_1 \lor \phi_2 \equiv_{RPS} \psi_1 \lor \psi_2 \).
3. \( \forall x \phi_1 \equiv_{RPS} \forall x \psi_1 \).
4. \( \exists x \phi_1 \equiv_{RPS} \exists x \psi_1 \) if and only if \( \exists x \phi_1 \) and \( \exists x \psi_1 \) are both valid in RPS or \( \exists x \phi_1 \) and \( \exists x \psi_1 \) have the same free variables.

Proof. Parts 1 through 3 and the backward direction of part 4 are immediate. For the forward direction of part 4, argue by contradiction. Since \( \exists x \phi_1 \equiv_{RPS} \exists x \psi_1 \), we see that \( \exists x \phi_1 \) is valid in RPS iff \( \exists x \psi_1 \) is. So, since we are arguing by contradiction, neither formula is valid. Without loss of generality, \( \exists x \phi_1 \) has a free variable \( y \) which is not a free variable of \( \exists x \psi_1 \). Let \( A \) be a ranked partial structure and \( s \) an assignment in \( |A| \) such that \( A \models \exists x \psi_1 [s] \). By the localization lemma, any end extension \( B \) of \( A \) has the property that \( B \models \exists x \psi_1 [s] \). So, we may assume that the rank ordering of \( A \) has a maximal element \( i \) and that \( R_{<i} \not= R_i \). Let \( a \) be an element of \( R_i - R_{<i} \). \( A \models \exists x \phi_1 [s(y[a])] \) while \( A \not\models \exists x \psi_1 [s(y[a])] \). This contradicts the assumption that \( \exists x \phi_1 \equiv_{RPS} \exists x \psi_1 \). \( \square \)

In a moment, we will see that this lemma does not extend to the operation of negation.

Lemma 4.3 Assume \( \phi \) and \( \psi \) are formulas of \( L \).

1. \( \neg \neg \phi \equiv_{RPS} \phi \).
2. \( \neg (\phi \land \psi) \equiv_{RPS} \neg \phi \lor \neg \psi \).
3. \( \neg (\phi \lor \psi) \equiv_{RPS} \neg \phi \land \neg \psi \).
4. \( \forall x \phi \equiv_{RPS} \exists x \neg \phi \).
5. \( \neg \exists x \phi \equiv_{RPS} \forall x \neg \phi \).
6. \( \phi \equiv_{RPS} \phi^{nf} \).

Proof. Parts 1 through 5 are immediate from the definition of satisfaction in ranked partial structures. Part 6 is equivalent to lemma 2.8. \( \square \)

Lemma 4.4 Assume \( \phi \) and \( \psi \) are formulas of a language \( L \) and \( x \) is a variable which does not occur free in \( \psi \).

1. \( \forall x \phi \land \psi \equiv_{RPS} \forall x (\phi \land \psi) \).
2. \( \forall x \phi \lor \psi \equiv_{RPS} \forall x (\phi \lor \psi) \).
3. If every free variable of \( \psi \) is also free in \( \exists x \phi \) then \( \exists x \phi \land \psi \equiv_{RPS} \exists x (\phi \land \psi) \).

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4. If every free variable of \( \psi \) is also free in \( \exists x \phi \) then \( \exists x \phi \lor \psi \equiv_{\text{RPS}} \exists x(\phi \lor \psi) \).

Proof. Straightforward. \( \square \)

Parts 3 and 4 of the lemma require some conditions on the free variables of \( \psi \). For example, \( \exists x(P_1(x) \lor P_2(y)) \not\equiv_{\text{RPS}} \exists xP_1(x) \lor P_2(y) \). To see this, let \( \mathcal{A} \) be a ranked partial structure of rank 1 where the interpretations of both \( P_1 \) and \( P_2 \) are empty. \( \mathcal{A} \models \exists x(P_1(x) \lor P_2(y)) [s] \) trivially and \( \mathcal{A} \not\models \exists xP_1(x) \lor P_2(y) [s] \) for any \( s \).

We can modify this example to show that lemma 4.2 does not extend to negation as follows. The formulas \( \forall x(P_1(x) \land P_2(y)) \) and \( \forall xP_1(x) \land P_2(y) \) are equivalent in \( \text{RPS} \) by the lemma. However, the negation normal forms of the negations of these formulas are \( \exists x(\neg P_1(x) \lor \neg P_2(y)) \) and \( \exists x\neg P_1(x) \lor \neg P_2(y) \). By an argument similar to that above, these sentences are not equivalent in \( \text{RPS} \).

The previous lemma allows us to find a formula equivalent in \( \text{RPS} \) to a given formula which is in an approximation to prenex form.

**Definition 4.5** Assume \( \mathcal{L} \) is a language. Define the collections of formulas in \( \Sigma_n \) universal normal form and \( \Pi_n \) universal normal form, denoted \( \Sigma_n^{\text{vnf}} \) and \( \Pi_n^{\text{vnf}} \) respectively, simultaneously by induction on \( n \in \omega \). \( \Sigma_0^{\text{vnf}} \) and \( \Pi_0^{\text{vnf}} \) are both the collection of quantifier free formulas which are in negation normal form. \( \Sigma_n^{\text{vnf}} \) is the closure of \( \Pi_n^{\text{vnf}} \) under conjunction, disjunction, and existential quantification. \( \Pi_n^{\text{vnf}} \) is the closure of \( \Sigma_n^{\text{vnf}} \) under universal quantification.

**Theorem 4.6** Assume \( \mathcal{L} \) is a language.

1. Every formula in \( \Sigma_n \) is equivalent in \( \text{RPS} \) to a formula in \( \Sigma_n^{\text{vnf}} \) with the same free variables.

2. Every formula in \( \Pi_n \) is equivalent in \( \text{RPS} \) to a formula in \( \Pi_n^{\text{vnf}} \) with the same free variables.

Proof. Since every formula is equivalent in \( \text{RPS} \) to its negation normal form, it suffices to prove the lemma for formulas in negation normal form. This can be established by induction on formula complexity using the first two parts of the previous lemma. \( \square \)

Notice that \( \Sigma_n^{\text{vnf}} \subseteq \Sigma_{n+1}^{\text{vnf}} \) and \( \Pi_n^{\text{vnf}} \subseteq \Pi_{n+1}^{\text{vnf}} \) for all \( n \in \omega \).

We close this section by establishing compactness for \( \text{RPS} \) and showing that the logic \( \text{RPS} \) is computably enumerable but not computable.

The statement of the compactness theorem for \( \text{RPS} \) is more complicated than the corresponding theorem for first-order logic because the failure of a formula to be satisfied is not equivalent to the negation being satisfied. Also,
we will include information of where the interpretations of the variables lie in the rank hierarchy.

Assume $\Phi$ and $\Psi$ are sets of formulas of $\mathcal{L}$, $(I, \prec)$ is a linear ordering, $V$ is a set of variables, and $I_x$ is a nonempty end segment of $(I, \prec)$ for each variable $x$ in $V$.

$$(\Phi, \Psi, (I, \prec), I_x \ (x \in V))$$
is realizable if there is a structure $\mathcal{A}$ and an assignment $s$ such that

- $\mathcal{A} \models \phi [s]$ for all $\phi \in \Phi$,
- $\mathcal{A} \not\models \psi [s]$ for all $\psi \in \Psi$,
- $(I, \prec)$ is a subordering of the rank ordering of $\mathcal{A}$, and
- for each variable $x \in V$

$$s(x) \in R_i \text{ iff } i \in I_x$$

for all $i \in I$.

$(\Phi, \Psi, (I, \prec), I_x \ (x \in V))$ is finitely realizable if $(\Phi_0, \Psi_0, I_0, I_x \cap I_0 \ (x \in V_0))$ is realizable whenever $\Phi_0, \Psi_0, I_0,$ and $V_0$ are finite subsets of $\Phi, \Psi, I,$ and $V$ respectively and $I_0$ is the subordering of $(I, \prec)$ with universe $I_0$.

**Theorem 4.7 (compactness)** Assume $\mathcal{L}$, $\Phi$, $\Psi$, $(I, \prec)$, $V$, and $I_x \ (x \in V)$ are as above. $(\Phi, \Psi, (I, \prec), I_x \ (x \in V))$ is realizable iff it is finitely realizable.

*Proof*. We may view a ranked partial structure $\mathcal{A}$ in a language $\mathcal{L}$ as a partial structure in the language obtained by adding a new binary relation symbol $R$ to $\mathcal{L}$ and defining the interpretation of $R$ to be all pairs $(a, i)$ where $a \in R_i$ (the universe of this partial structure will be the universe of $\mathcal{A}$ along with the universe of the rank ordering of $\mathcal{A}$). The theorem now follows from the fact that the logic of partial structures satisfies compactness.

We remark that the previous theorem does not hold in general if we modify the definition of “realizable” by requiring that the rank ordering of $\mathcal{A}$ is $(I, \prec)$ (rather than the condition that $(I, \prec)$ is just a subordering of the rank ordering). However, the modified version of the theorem holds by the localization lemma whenever $\Psi$ is empty.

**Corollary 4.8** Assume $\Gamma$ is a set of formulas of $\mathcal{L}$ and $\phi$ is a formula of $\mathcal{L}$. If $\Gamma \models_{RPS} \phi$ then $\Gamma_0 \models_{RPS} \phi$ for some finite subset $\Gamma_0$ of $\Gamma$.

*Proof*. Immediate from the theorem.
**Theorem 4.9 (completeness)** The relation $\Phi \models_{RPS} \phi$ for $\Phi$ a finite set of formulas and $\phi$ a formula is computably enumerable.

*Proof.* More precisely, the theorem says that the set of *forms* of finite $\Phi$ and $\phi$ such that $\Phi \models \phi$ is computably enumerable.

By an argument similar to that used in the previous theorem, the theorem reduces to the fact that the valid formulas of the logic of partial structures is computably enumerable. \hfill $\square$

Fix a language $\mathcal{L}$ for the rest of this section and let $\mathcal{L}(\mathbb{P})$ be obtained from $\mathcal{L}$ by adding a new unary relation symbol $\mathbb{P}$. We will define a translation $\phi \mapsto \bar{\phi}$ which will reduce $\text{PS}$ to $\text{RPS}$:

\begin{itemize}
  \item \((*)\) $\phi$ is valid in $\text{PS}$ iff $\bar{\phi}$ is valid in $\text{RPS}$
\end{itemize}

One might attempt to find a translation which would satisfy the usual property that every ranked partial structure $\mathcal{A}$ give rises to a partial structure $\bar{\mathcal{A}}$ such that

1. $\bar{\mathcal{A}} \models \phi$ iff $\mathcal{A} \models \bar{\phi}$ for any sentence $\phi$ and
2. every partial structure is of the form $\bar{\mathcal{A}}$ for some ranked partial structure $\mathcal{A}$.

However, we will see in the next section that the set of satisfiable sentences of $\text{RPS}$ is computable and a translation of the form above would imply that the set of satisfiable sentences of $\text{PS}$ is computable. If we modify \((*)\) above to

$\phi$ is satisfiable in $\text{PS}$ iff $\bar{\phi}$ is not valid in $\text{RPS}$

we are lead to modify the first property to

$\bar{\mathcal{A}} \models \phi$ iff $\mathcal{A} \not\models \bar{\phi}$

The reader might think of the relation $\mathcal{A} \not\models \phi[s]$ as an alternative semantics for ranked partial structures in what follows. Notice that

\begin{align*}
  \mathcal{A} \not\models (\phi \land \psi)[s] & \iff \mathcal{A} \not\models \phi[s] \text{ and } \mathcal{A} \not\models \psi[s]. \\
  \mathcal{A} \not\models (\phi \lor \psi)[s] & \iff \mathcal{A} \not\models \phi[s] \text{ or } \mathcal{A} \not\models \psi[s]. \\
  \mathcal{A} \not\models \exists x \phi[x] & \iff \mathcal{A} \not\models \phi[s(a)] \text{ for some } a \in |\mathcal{A}|. \\
  \mathcal{A} \not\models \forall x \phi[x] & \iff \text{ for some } i \in I \text{ with the property that } s \text{ maps the free variables of } \forall x \phi \text{ into } R_i, \mathcal{A} \not\models \phi[s(a)] \text{ for all } a \in R_i.
\end{align*}

Define $\alpha_1$ to be the sentence $\forall x \mathbb{P}(x)$. We see that $\mathcal{A} \not\models \alpha_1$ iff $R_i$ is a subset of $\mathcal{P} \mathcal{A} \mathcal{L}$ for some $i$.

Let $\bar{\alpha}_2$ be the formula $\neg \mathbb{P}(y) \lor \forall z (y = y \land \mathbb{P}(z))$. Notice that $\mathcal{A} \not\models \bar{\alpha}_2[s]$ iff
\[ s(y) \in P^{A|\mathcal{L}} \Rightarrow \text{there is an } i \text{ such that } s(y) \in R_{<i} \text{ and } R_i \subseteq P^{A|\mathcal{L}} \]

Let \( \alpha_2 \) be the sentence \( \exists x (\neg P(x) \land \forall y (x = y \land \alpha_2)) \). We see that \( \mathcal{A} \not\models \neg \alpha_2 \) iff there is a \( a \in |\mathcal{A}| \) and an \( i \) in the rank ordering of \( \mathcal{A} \) such that

- \( a \notin P^{A|\mathcal{L}} \)
- \( a \in R_{<i} \)
- If \( J \) consists of those \( j \) such that \( R_j \subseteq P^{A|\mathcal{L}} \) then \( P^{A|\mathcal{L}} \cap R_i = \bigcup_{j \in J} R_j \) and if \( J \) has a maximal element \( j_{\text{max}} \) then \( R_{<j_{\text{max}}} = R_{j_{\text{max}}} \).

Define \( \alpha \) to be \( \alpha_1 \land \alpha_2 \).

Now let \( \mathcal{A} \) be a ranked partial structure for \( \mathcal{L}(P) \). Let \( \mathcal{N} \) be the partial substructure of the underlying partial structure of \( \mathcal{A} \) whose universe is the union of those \( R_i \) which are contained in \( P^{A|\mathcal{L}} \). Define \( \mathcal{A} \) to be the restriction of \( \mathcal{N} \) to \( \mathcal{L} \).

We will define a preliminary translation \( \phi \mapsto \phi^* \) mapping formulas of \( \mathcal{L} \) to formulas of \( \mathcal{L}(P) \).

For convenience, we will actually define \( \phi^* \) only for \( \phi \) which are in negation normal form. \( \phi^* \) is defined by induction on the complexity of \( \phi \) as follows.

- \( \phi^* \) is \( \phi \) if \( \phi \) is an atomic formula or the negation of an atomic formula.
- \( (\phi \land \psi)^* = \phi^* \land \psi^* \)
- \( (\phi \lor \psi)^* = \phi^* \lor \psi^* \)
- \( (\forall \phi)^* = \exists y (\neg P(y) \land \forall x (y = y \land (\neg P(x) \lor \phi^*))) \) for some variable \( y \) which does not occur in \( \forall x \phi \).
- \( (\exists \phi)^* = \exists x (\forall y (x = x \land P(y)) \land \phi^*) \) where \( y \) is a variable which does not occur in \( \exists x \phi \).

To verify our translation has the right properties we will need

**Lemma 4.10** Assume \( \mathcal{A} \) is a ranked partial structure for \( \mathcal{L} \) and every element of \(|\mathcal{A}|\) is in the bounded part of \( \mathcal{A} \). For any formula \( \phi \) and assignment \( s \) in \( \mathcal{A} \)

\[ \mathcal{A} \models \phi[s] \quad \Rightarrow \quad \mathcal{A}|\mathcal{L} \models \phi[s] \]

and if \( \phi \) is \( \Pi_1 \) then

\[ \mathcal{A} \models \phi[s] \quad \text{iff} \quad \mathcal{A}|\mathcal{L} \models \phi[s] \]

**Proof.** We may assume that \( \phi \) is in negation normal form by lemma 2.8. One can first establish the conclusion for formulas of the form \( t_i \) by induction on the height of \( t \). Then a straightforward induction on the complexity of \( \phi \) establishes the lemma in full generality. \( \square \)
The hypothesis in the previous lemma that every element of |$A$| is in the bounded part of $A$ is equivalent to $A$ having the property that either the rank ordering has no maximal element or the rank ordering has a maximal element $i$ such that $R_i = R_{<i}$.

Generalizations of the lemma can be obtained by combining it with the localization lemma e.g. if $R_i = R_{<i}$ then any sentence satisfied in $A$ is satisfied in the substructure of $A|\mathcal{L}$ with universe $R_i$ (this explains the necessity of the requirement of properness in theorem 3.9).

**Theorem 4.11 (incompleteness)** The set of valid sentences of RPS is not computable.

**Proof.** More precisely, the theorem says that the set of forms of valid formulas of RPS is not computable.

**Claim 1:** Assume $A$ is a ranked partial structure for $\mathcal{L}(P)$. If $A \not\models \neg \alpha$ then for any formula $\phi$ of $\mathcal{L}$ in negation normal form and any assignment $s$ in $|\hat{A}|$

$$A \models \phi[s] \text{ iff } A \not\models \neg \phi^*[s]$$

Let $J$ be the set of all $j$ in the rank ordering such that $R_j \subseteq P^A|\mathcal{L}$. The universe of $\hat{A}$ is $\bigcup_{j \in J} R_j$. Let $B$ be the localization of $A$ to $J$. Notice that $\hat{A} = B|\mathcal{L}$. Since $\hat{A} \not\models \neg \alpha_2$, we know that if $J$ has a maximal element $j_{\max}$ then $R_{j_{\max}} = R_{<j_{\max}}$. Therefore, every element of $B$ is in the bounded part of $B$.

The previous lemma (applied to $B$) and the localization lemma can now be used to verify the claim in the case $\phi$ is atomic or the negation of an atomic formula (recall that atomic formulas are determined). The full claim now follows by induction on the complexity of $\phi$.

**Claim 2:** If $\phi$ is a sentence of $\mathcal{L}$ in negation normal form then $\phi$ is satisfiable in $PS$ iff $\neg(\alpha \land \phi^*)$ is not valid in RPS.

An easy construction shows that any partial structure for $\mathcal{L}$ is of the form $\hat{A}$ for some ranked partial structure $A$ such that $A \not\models \neg \alpha$. Since $\hat{A} \models \neg(\alpha \land \phi^*)$ iff $A \not\models \neg \alpha$ and $A \not\models \neg \phi^*$ this claim follows from claim 1.

For any formula $\phi$ we have

$$\phi \text{ is valid in } PS \text{ iff } (\neg \phi)^{nf} \text{ is not satisfiable in } PS$$

$$\text{iff } \neg (\alpha \land ((\neg \phi)^{nf})^*) \text{ is valid in } RPS$$

The second equivalence follows from the second claim. The theorem now follows since the collection of formulas which are valid in $PS$ is not computable (we will not concern ourselves with the exact requirements of the language for RPS needed here).
5  A Version of the Downward Löwenheim-Skolem Theorem

In this section we present a natural extension of the downward Löwenheim-Skolem Theorem for ranked partial structures. The development is based on familiar lines.

Definition 5.1 Suppose $A$ and $B$ are ranked partial structures for $\mathcal{L}$. $A$ is bounded by $B$ if $A|L$ is a partial substructure of $B|L$, $A$ is closed under the interpretations of the function symbols of $L$ in $B|L$, and the bounded part of $A$ is a subset of the bounded part of $B$.

Lemma 5.2 Assume $B$ is a ranked partial structure for $\mathcal{L}$. If

- $\phi$ is a formula of $\mathcal{L}$ and $\phi^*$ is a formula in a language $\mathcal{L}^*$ extending $\mathcal{L}$ which is a Skolem form of $\phi$ and
- $B^*$ is an expansion of $B$ to $\mathcal{L}^*$ such that $B^*$ is a Skolem expansion of $B$ for $\phi$ with respect to $\phi^*$.

then for every ranked partial structure $A$ which is bounded by $B$ and closed under the interpretations of the function symbols of $\mathcal{L}^*$ in $B^*|\mathcal{L}^*$

$$B \models \phi[s] \Rightarrow A \models \phi[s]$$

whenever $s$ is an assignment in $|A|$.

Proof. For $B$ fixed, we prove the lemma by induction on the complexity of $\phi$.

When $\phi$ has the form $t \uparrow$ or $t \uparrow$ the lemma follows immediately from the definition of satisfaction. The remaining cases where $\phi$ is an atomic formula or the negation of an atomic formula are now also clear.

The cases when $\phi$ is a conjunction, disjunction, or of the form $\forall y \psi$ are straightforward.

Suppose $\phi$ has the form $\exists z \psi$ and $\phi^*$, $\mathcal{L}^*$, and $B^*$ are as in the hypotheses of the lemma. Let $A$ be a ranked partial structure which is bounded by $B$ and closed under the interpretations of the function symbols of $\mathcal{L}^*$ in $B^*|\mathcal{L}^*$ and suppose $s$ is an assignment in $|A|$. Finally, assume that $B \models \exists z \psi[s]$.

In order to show that $A \models \exists z \psi[s]$, suppose $s$ maps the free variables of $\phi$ into $R^A_{\exists z \psi}$. $\phi^*$ has the form $f(y_1, \ldots, y_n) \downarrow \land \psi^*(z|f(y_1, \ldots, y_n))$ where $\psi^*$ is a Skolem form of $\psi$ and $y_1, \ldots, y_n$ list the free variables of $\phi$. Let $a_i = s(y_i)$ for $i = 1, \ldots, n$. Since $a_1, \ldots, a_n$ are in the bounded part of $A$ they must also be in the bounded part of $B$. Therefore, $f^B|\mathcal{L}^*(a_1, \ldots, a_n)$ is defined and has some value $b$. In addition, $B \models \psi[y_1, \ldots, y_n, z|a_1, \ldots, a_n, b]$. Since $A$ is closed under $f^B|\mathcal{L}^*$, $b \in R^A_{\exists z \psi}$. By the induction hypothesis, $A \models \psi[y_1, \ldots, y_n, z|a_1, \ldots, a_n, b]$. Therefore, $A \models \exists z \psi[y_1, \ldots, y_n, a_1, \ldots, a_n]$. $\square$

For an interesting special case of the lemma, suppose $\phi$ is a $\Pi_1$ formula of $\mathcal{L}$ and $B$ is a ranked partial structure for $\mathcal{L}$. If $A$ is bounded by $B$ then
\[ B \models \phi [s] \Rightarrow A \models \phi [s]. \]

whenever \( s \) is an assignment in \([A]\).

When \( B \) has finite height and \( L \) is finite we can find \( A \) as in the previous lemma which is finite i.e. has a finite universe. We now want to estimate the possible size of the universe of \( A \). The universe of \( A \) will come from evaluating terms involving witnessing functions. So, we will want to estimate the sizes of relevant collections of terms.

**Definition 5.3** Assume \( L \) is a language. For \( t \) a term of \( L \) and \( \eta \) a function whose domain is a set of variables containing the variables which occur in \( t \) and whose range is contained in \( \omega \), define the *height of \( t \) relative to \( \eta \)* by induction on the height of \( t \) so that

- if \( x \) is a variable then the height of \( x \) relative to \( \eta \) is \( \eta(x) \),
- if \( c \) is a constant symbol then the height of \( c \) relative to \( \eta \) is 0.
- if \( t \) is \( f(t_1, \ldots, t_k) \) then the height of \( t \) relative to \( \eta \) is \( h + 1 \) where \( h \) is the maximum of the heights of the \( t_i \) relative to \( \eta \).

The height of \( t \) relative to \( \eta \) is monotone in \( \eta \) in the obvious sense.

**Definition 5.4** Suppose \( k, m \in \omega \). For \( h_0, \ldots, h_{k-1}, r_0, \ldots, r_{m-1}, h \in \omega \) define \( \tau(h_0, \ldots, h_{k-1}; r_0, \ldots, r_{m-1}; h) \) to be the number of terms in a language with \( r_j \)-ary function symbols for \( j < m \) whose height relative to the function \( v_i \mapsto h_i \) is at most \( h \). We write \( \tau(r_0, \ldots, r_{m-1}; h) \) for \( \tau(h_0, \ldots, h_{k-1}; r_0, \ldots, r_{m-1}; h) \) when \( k = 0 \).

Note that the terms counted to calculate \( \tau(h_0, \ldots, h_{k-1}; r_0, \ldots, r_{m-1}; h) \) must have all their variables among \( v_0, \ldots, v_{k-1} \).

\[ \tau(r_0, \ldots, r_m; h) \] satisfies the following recurrence relation:

\[ (\dagger) \quad \begin{cases} 
\tau(r_0, \ldots, r_m; 0) = r_0 \\
\tau(r_0, \ldots, r_m; h + 1) = \sum_{i=0}^m r_i \tau(r_0, \ldots, r_m; h)^i 
\end{cases} \]

**Lemma 5.5** Assume \( h_0, \ldots, h_{k-1}, r_0, \ldots, r_{m-1}, h \in \omega \).

1. If \( h_i \geq h'_i \) for \( i < k \), \( r_j \leq r'_j \) for \( j < m \), and \( h \leq h' \) then

\[ \tau(h_0, \ldots, h_{k-1}; r_0, \ldots, r_{m-1}; h) \leq \tau(h'_0, \ldots, h'_{k-1}; r'_0, \ldots, r'_{m-1}; h'). \]

2. If \( h_i = 0 \) then

\[ \tau(h_0, \ldots, h_{k-1}; r_0, \ldots, r_{m-1}; h) = \tau(h_0, \ldots, h_{i-1}, h_{i+1}, \ldots, h_{k-1}; 1 + r_0, r_1, \ldots, r_{m-1}; h). \]
3. \( \tau(r_0, \ldots, r_m; h) \leq (r_0 + \cdots + r_m)^{1+m+m^2+\cdots+m^h}. \)

**Proof.** Parts 1 and 2 are clear. Part 3 can be proved by induction on \( h \) using the recurrence relation (†) above. Alternatively, for \( m \neq 0 \) notice that \( (r_0 + \cdots + r_m)^{1+m+m^2+\cdots+m^h} \) is the number of ways of tagging an \( m \)-branching tree of height \( h \) with \( r_0 + \cdots + r_m \) many labels. It is easy to map the set of such labelled trees onto the set of terms of height at most \( h \) in an appropriate language. \( \square \)

Assume \( \Phi \) is a finite set of formulas of \( \mathcal{L} \) which are in negation normal form and variable normal form. For each \( \phi \in \Phi \) let \( \phi^* \) be a Skolem form of \( \phi \) such that whenever \( \phi \) and \( \psi \) are distinct elements of \( \Phi \) then any function symbol which occurs in both \( \phi^* \) and \( \psi^* \) is in \( \mathcal{L} \). The Skolem signature of \( \Phi \) over \( \mathcal{L} \) is the function which maps each \( i \in \omega \) to the number of \( i \)-ary function symbols in the language consisting of \( \mathcal{L} \) along with all function symbols which occur in some \( \phi^* \).

If the Skolem signature of \( \Phi \) is \( f \) and \( f(i) = 0 \) for \( i \geq m \) we sometimes say that the Skolem signature of \( \Phi \) is \( \langle f(0), \ldots, f(m-1) \rangle \).

**Theorem 5.6** Assume \( \mathcal{B} \) is a ranked partial structure for \( \mathcal{L} \) whose rank ordering is a natural number \( n \) and \( a_i \in R_{h_i} \) for \( i < k \). If \( \Phi \) is a set of formulas of \( \mathcal{L} \) whose Skolem signature over \( \mathcal{L} \) is \( \langle r_0, \ldots, r_m \rangle \) and \( r_0 \neq 0 \) then there exists a ranked partial structure \( \mathcal{A} \) of height \( n \) which is bounded by \( \mathcal{B} \) such that

1. \( a_i \in R_{h_i}^A \) for \( i < k \);

2. if \( \phi \in \Phi \) and \( s \) is an assignment in \( \mathcal{A} \) such that \( \mathcal{B} \models \phi[s] \) then \( \mathcal{A} \models \phi[s] \), and

3. for all \( h < n \)

\[ (*)\quad \text{card}(R_{h}^A) \leq \tau(h_0, \ldots, h_{k-1}; r_0, \ldots, r_m; h) \]

**Proof.** Notice that we are tacitly assuming in the assumption of the theorem that each element of \( \Phi \) is in negation normal form and variable normal form. Let \( \Phi \) and \( \langle r_0, \ldots, r_m \rangle \) be as in the hypothesis. For each \( \phi \in \Phi \) let \( \phi^* \) be a Skolem form of \( \phi \) over \( \mathcal{L} \). The \( \phi^* \) can be chosen in such a way that no function symbol not in \( \mathcal{L} \) appears in more than one of the \( \phi^* \).

By theorem 2.21, there is an expansion \( \mathcal{B}^* \) of \( \mathcal{B} \) in a language \( \mathcal{L}^* \) such that for each \( \phi \in \Phi \), \( \mathcal{B}^* \) is a Skolem expansion of \( \mathcal{B} \) for \( \phi \) with respect to \( \phi^* \). Moreover, we may assume that \( \mathcal{L}^* \) has exactly \( r_i \) function symbols of arity \( i \) for each \( i \leq m \).

Define \( \eta : \{ v_0, \ldots, v_{k-1} \} \rightarrow \omega \) by \( \eta(v_i) = h_i \) and let \( \hat{s} \) be an assignment in \( \mathcal{B}^* \) such that \( s(v_i) = a_i \) for \( i < k \). For \( h \leq n \) let \( X_h \) consist of all defined \( \phi^*(\mathcal{L}^*; \hat{s}) \) where \( t \) is a term of \( \mathcal{L}^* \) which has height at most \( h \) relative to \( \eta \) and all of whose variables are among \( v_0, \ldots, v_{k-1} \). Notice that \( X_h \subseteq R_h^B \). Define \( \mathcal{A} \) to be \( (\mathcal{B}|\mathcal{L}, X_0, \ldots, X_{n-1}) \).

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Case 1: \( X_0 \neq \emptyset \).

Clearly, \( \mathcal{A} \) is a ranked partial structure which is bounded by \( \mathcal{B} \) and closed under the interpretations of the function symbols in \( \mathcal{B}^* \). Part 2 of the conclusion follows from lemma 5.2. Part 1 of the conclusion follows from the fact that the rank of \( v_i \) with respect to \( \eta \) is \( h_i \). And part 3 of the conclusion follows immediately from the definition of \( \tau(h_0, \ldots, h_{k-1}; r_0, \ldots, r_m, h) \).

Case 2: \( X_0 = \emptyset \).

In this case, there must be a constant symbol of \( \mathcal{L}^* \) which is undefined in \( \mathcal{B}^* \). Let \( \mathcal{B}^{**} \) be identical to \( \mathcal{B}^* \) except now define such a constant symbol to be an arbitrary element of \( R_0^{**} \). Define \( X_h \) as above with \( \mathcal{B}^* \) replaced by \( \mathcal{B}^{**} \). The conclusion of the lemma follows as in case 1.

The bound obtained from the argument may be inflated in some ways e.g., we may have chosen different Skolem function symbols for identical subformulas of elements of \( \Phi \).

Corollary 5.7 The theorem also holds if the condition \( r_0 \neq 0 \) is replaced by the hypothesis that \( h_i = 0 \) for some \( i < k \).

Proof. The proof is the same except that now case 2 is impossible. \( \square \)

Definition 5.8 A set of formulas \( \Phi \) is satisfiable in a ranked partial structure \( \mathcal{A} \) with rank ordering \( (I, \prec) \) if there is an assignment \( s \) in \( \bigcap_{i \in I} R_i \) such that \( \mathcal{A} \models \Phi[s] \).

If \( \phi \) is a formula of \( \mathcal{L} \) we will sometimes say \( \phi \) is satisfiable in \( \mathcal{A} \) instead of \( \{ \phi \} \) is satisfiable in \( \mathcal{A} \). Notice that the condition that \( \phi \) is satisfiable in \( \mathcal{A} \) is, in general, slightly stronger than saying an existential closure of \( \phi \) is satisfiable in \( \mathcal{A} \).

In light of theorem 3.9, the definition above would not be very useful if we dropped the requirement that \( s \) be an assignment in \( \bigcap_{i \in I} R_i \).

Corollary 5.9 Assume \( \Phi \) is a finite set of formulas in a language \( \mathcal{L} \). If \( \Phi \) is satisfiable in a ranked partial structure of height \( n \) then, letting \( c \) be the sum of the lengths of the elements of \( \Phi \), \( \Phi \) is satisfiable in a ranked partial structure of height \( n \) with the property that \( \text{card}(R_h) < c^{b+h+1} \) for all \( h < n \).

Proof. We may assume that each element of \( \Phi \) is in variable normal form since renaming bound variable preserves length. Notice that each element of \( \Phi^{nmf} \) (the set of formulas \( \phi^{nmf} \) where \( \phi \in \Phi \)) is also in variable normal form. We may assume that the predicate and function symbols of \( \mathcal{L} \) are exactly those which occur in some element of \( \Phi \) augmented by a new constant symbol. Let \( V \) be the set of variables which occur free in some element of \( \Phi \) and let \( (r_0, \ldots, r_m) \) be the Skolem signature of \( \Phi^{nmf} \) over \( \mathcal{L} \). Since \( \mathcal{L} \) has at least one constant symbol, \( r_0 \neq 0 \). We may assume that \( r_m \neq 0 \). By theorem 5.6, there is a ranked partial
structure $\mathcal{A}$ with rank ordering $n$ which satisfies $\Phi$ under some assignment $s$ in $R_0$ such that, letting $k$ be the cardinality of $V$ and setting $h_i = 0$ for $i < k$, $\text{card}(R_h) \leq \tau(h_0, \ldots, h_{k-1}; r_0, \ldots, r_m; h)$. We have

$$\text{card}(R_h) \leq \tau(h_0, \ldots, h_{k-1}; r_0, \ldots, r_m; h)$$

$$= \tau(k + r_0, r_1, \ldots, r_m; h)$$

(part 2 of lemma 5.5)

$$\leq (k + r_0 + r_1 + \cdots + r_m)^{1+m^2+\cdots+m^h}$$

(part 3 of lemma 5.5)

for all $h < n$. Since both $k + r_0 + \cdots + r_m$ and $m$ are easily seen to be at most $c$, the conclusion of the corollary follows.

A double exponential bound in the corollary is the best possible. Suppose $\mathcal{L}$ is a language with no predicate symbols and $r_j$ $j$-ary function symbols for $j = 0, \ldots, m$. Let $\theta$ be the sentence which naturally expresses that the functions are total and act freely on the universe. If $\mathcal{A}$ is a ranked partial structure with rank ordering a positive natural number $n$ then $\text{card}(R_h) \geq \tau(r_0, \ldots, r_m; h)$ for $h < n$. Even if $r_0 = 0$ we see that $R_h$ has at least $\tau(0, r_1, \ldots, r_m; h) = \tau(1, r_1, \ldots, r_m; h)$ elements since $R_0 \neq \emptyset$. The last part of the following lemma shows that if $\mathcal{L}$ has at least one binary function symbol then there is a double exponential lower bound for the size of $R_h$.

**Lemma 5.10** Assume $r_0, \ldots, r_m, h \in \omega$.

1. $\tau(0, r_1, \ldots, r_m; h) = 0$.

2. $\tau(r_0, r_1; h) = r_0 \sum_{i=0}^h r_1^i$.

3. $\tau(1, 0, 1; h) \geq 2^{2^h-1}$ for $h \geq 1$.

**Proof.** Parts 1 and 2 are clear. Part 3 can be established by induction using the recurrence relation (\dagger).

**Corollary 5.11** The set of formulas which are satisfiable in some ranked partial structure is computable.

**Proof.** By the localization lemma, a formula is satisfiable iff it is satisfiable in a ranked partial structure of height 1. Corollary 5.9 gives an upper bound for the size of one such structure if any exist.

**6 Completeness for the Logic of Partial Structures through Ranked Partial Structures**

In this section we will draw connections between the logic of partial structures and the logic of ranked partial structures. Our main goal is to establish
Theorem 6.1 Assume $\Gamma$ is a set of formulas of $\mathcal{L}$. $\Gamma$ is satisfiable in some partial structure iff $\Gamma$ is satisfiable in a ranked partial structure of height $n$ for each positive natural number $n$.

We will prove the theorem later in this section.

Corollary 6.2 Assume $\Gamma$ is a set of formulas of $\mathcal{L}$ and $\phi$ is a formula of $\mathcal{L}$. $\Gamma \models_{PS} \phi$ iff there is a natural number $n$ such that $\Gamma \cup \{ \neg \phi \}$ is not satisfiable in any ranked partial structure of height $n$.

Proof. Immediate from the theorem. $\square$

We can derive the fact that the (forms of) valid formulas of the logic of partial structures form a computably enumerable set by combining the theorem with corollary 5.9: to see that a given formula is valid, find a natural number $n$ such that there is no ranked partial structure of height $n$ whose universe has size at most $\text{length}(\neg \phi)^{\text{length}(\neg \phi)^n}$ in which $\neg \phi$ is satisfiable.

The following lemma shows that we can view any partial structure as a ranked partial structure by letting each $R_i$ be the universe of the partial structure.

Lemma 6.3 Assume $\mathcal{M}$ is a partial structure for the language $\mathcal{L}$ and suppose $(I, \prec)$ is a linear ordering with at least two elements. Let $\mathcal{A}$ be the ranked partial structure with underlying partial structure $\mathcal{M}$ and rank ordering $(I, \prec)$ where $R_i = |\mathcal{M}|$ for each $i \in I$. If $\phi$ is a formula and $s$ is an assignment in $|\mathcal{A}|$ then

$$\mathcal{M} \models \phi[s] \iff \mathcal{A} \models \phi[s]$$

Proof. The reverse direction of the conclusion is by lemma 4.10. The forward direction can be established by induction on the complexity of $\phi$ simultaneously for all $s$. $\square$

We remark that if $\mathcal{N}$ is a partial structure for $\mathcal{L}$ and $\mathcal{N} \models \phi[s]$ then any ranked partial structure $\mathcal{A}$ which is an approximation to $\mathcal{N}$ and is closed under some set of Skolem functions for $\phi$ will also have the property that $\mathcal{A} \models \phi[s]$. We will establish a more general fact later.

Proof of theorem 6.1 The forward direction follows from the previous lemma.

For the reverse direction, assume that $\Gamma$ is a set of formulas which is satisfiable in a ranked partial structure of height $n$ for each integer $n$. By the compactness theorem (theorem 4.7) and the localization lemma, $\Gamma$ is satisfiable in a ranked partial structure $\mathcal{A}$ of height $\omega$. By lemma 4.10, $\Gamma$ is satisfiable in $\mathcal{A}[\mathcal{L}]$. $\square$

We will give another proof in the case $\Gamma$ is finite which does not rely on compactness for $PS$. The idea of the proof is to piece together ranked partial
structures of finite height which satisfy $\Gamma$ to produce a ranked partial structure of height $\omega$ which satisfies $\Gamma$ by using König’s lemma (in fact, the weak version of König’s lemma). To insure finite branching we use the bounds from the version of the Löwenheim-Skolem theorem in the previous section.

**Lemma 6.4** Assume $\mathcal{A}$ is a ranked partial structure for $\mathcal{L}$ and $\phi$ is a $\Sigma_1$ formula of $\mathcal{L}$ of height $h$. If $s$ is an assignment in $|\mathcal{A}|$ which maps the free variables of $\phi$ into $R_i$ for at least $h + 1$ different $i$ then

$$\mathcal{A} \models \phi[s] \implies \mathcal{B} \models \phi[s]$$

for any end extension $\mathcal{B}$ of $\mathcal{A}$.

**Proof.** Notice that the negation normal form of $\phi$ has the same height as $\phi$. So, by lemma 2.8, we may assume that $\phi$ is in negation normal form. We argue by induction on the height of $\phi$.

Assume $i_0 < i_1 < \cdots < i_h$ are elements of the rank ordering, $s$ maps the free variables of $\phi$ into $R_{i_0}$, and $\mathcal{A} \models \phi[s]$. Let $\mathcal{B}$ be an end extension of $\mathcal{A}$.

Suppose $\phi$ is $t_{i_0}$. The height of $t$ is $h$. Lemma 3.2 implies that $\mathcal{A}[\mathcal{L}] \models t_{i_0}[s]$. Therefore, $\mathcal{B}[\mathcal{L}] \models t_{i_0}[s]$. By part 1 of lemma 2.5, $\mathcal{B} \models t_{i_0}[s]$.

Suppose $\phi$ is $t_{i_1}$. Since $\mathcal{A} \models \phi[s]$, $\mathcal{A}[\mathcal{L}] \models t_{i_1}[s]$ implying $\mathcal{A}[\mathcal{L}] \not\models t_{i_0}[s]$. By lemma 3.2, $\mathcal{A} \not\models t_{i_0}[s]$. The localization lemma implies that $\mathcal{B} \not\models t_{i_0}[s]$. Using part 1 of lemma 2.5, we see that $\mathcal{B}[\mathcal{L}] \not\models t_{i_0}[s]$. Therefore, $\mathcal{B}[\mathcal{L}] \models t_{i_0}[s]$ i.e. $\mathcal{B} \models t_{i_0}[s]$.

The results of the previous two paragraphs along with lemma 3.2 make the general cases when $\phi$ is either atomic or the negation of an atomic formula straightforward.

The case when $\phi$ is a conjunction and the case when $\phi$ is a disjunction are straightforward.

Assume $\phi$ is $\exists z \psi$. The height of $\psi$ is $h - 1$. In order to show that $\mathcal{B} \models \exists z \psi[s]$, suppose $s$ maps the free variables of $\phi$ into $R_{i_1}$, there is an $a \in R_{i_1} \cap R_{i_0}$ such that $\mathcal{A} \models \psi[s(z,a)]$. By the induction hypothesis, $\mathcal{B} \models \psi[s(z,a)]$. This implies that $\mathcal{B} \models \exists z \psi[s]$.

**Lemma 6.5** Assume $\mathcal{A}$ is a ranked partial structure for $\mathcal{L}$ with rank ordering $\omega$ such that $R_n$ is finite for each $n \in \omega$. Let $\mathcal{A}_n$ be the localization of $\mathcal{A}$ to $n$ for each positive $n \in \omega$. If $\phi$ is a formula and $s$ is an assignment in $R_0$ such that $\mathcal{A}_n \models \phi[s]$ for each $n \in \omega$ then $\mathcal{A} \models \phi[s]$.

**Proof.** We may assume that $\phi$ is in negation normal form. By induction on the complexity of $\phi$ we show that

$(*)$ If $s$ is an assignment in $R_n$ and $\mathcal{A}_m \models \phi[s]$ whenever $n \leq m$ then $\mathcal{A} \models \phi[s]$.

When $\phi$ is an atomic formula or the negation of an atomic formula $(*)$ follows from lemma 6.4.
The cases when $\phi$ is a conjunction, disjunction, or of the form $\forall y \psi$ are straightforward from the induction hypothesis (using the localization lemma for the case of disjunction).

Suppose $\phi$ is $\exists y \psi$. Assume that $s$ is an assignment in $R_n$ such that $A_m \models \phi[s]$ whenever $n \leq m$. Suppose $s$ maps the free variables of $\exists y \psi$ into $R_{< i}$. We must find $a \in R_i$ such that $A \models \psi[s(z[a])]$. We may assume that $i < n$. For $m \geq n$, define $X_m$ to be the collection of all $a \in R_i$ such that $A_{m+1} \models \psi[s(z[a])]$. Since $A_m \models \exists x \psi[s]$, each $X_m$ is nonempty. And by the localization lemma, $X_n \supseteq X_{n+1} \supseteq \cdots$. Since the $X_m$ are all finite they must stabilize. Hence there is some $a$ in the intersection of all the $X_n$. By the induction hypothesis, $A \models \psi[s(z[a])]$. Since $i$ was arbitrary with the property that $s$ maps the free variables of $\exists y \psi$ into $R_{< i}$, $A \models \exists y \psi[s]$.

We remark that the assumption that each $R_n$ is finite is needed in general. For example, let $\mathcal{L}$ be the language with a single 2-place predicate symbol $P$. Let $\mathcal{M}$ be the structure for $\mathcal{L}$ whose universe is the set of integers where the interpretation of $P$ is given by

$$(k, m) \in P^A \text{ iff } m < |k|$$

Define $A$ to be the ranked partial structure for $\mathcal{L}$ with $R_n = (-\infty, n]$ for $n \in \omega$ and whose underlying partial structure is $\mathcal{M}$. Letting $A_n$ be as above, we see that $A_n \models \exists x \forall y P(x, y)$ for all $n$ while $A \not \models \exists x \forall y P(x, y)$.

**Second proof of theorem 6.1 for the case $\Gamma$ finite.** As we mentioned in the first proof, the forward direction follows from lemma 6.3.

For the reverse direction, suppose $\Gamma$ is a finite set of formulas which is satisfied in a ranked partial structure of height $n$ for each positive $n \in \omega$. Let $c$ be the sum of the lengths of the elements of $\Gamma$. Corollary 5.9 implies that for each positive $n$, $\Gamma$ is satisfiable in a ranked partial structure of the form $(\mathcal{M}, m_0, \ldots, m_{n-1})$ where

- $m_0, \ldots, m_{n-1}$ are natural numbers,
- $m_h < c^{h+1}$ for $h < n$, and
- the universe of $\mathcal{M}$ is $m_{n-1}$.

Let $T$ be the collection of all pairs $((\mathcal{M}, m_0, \ldots, m_{n-1}), \sigma)$ where $n$ is arbitrary, $(\mathcal{M}, m_0, \ldots, m_{n-1})$ satisfies the conditions above, and $\sigma$ is a partial assignment in $m_0$ whose domain is the set of free variables of the elements of $\Gamma$ such that $(\mathcal{M}, m_0, \ldots, m_{n-1}) \models \phi[\sigma]$ for each $\phi \in \Gamma$. Partially order $T$ so that $(A_1, \sigma_1)$ is less than or equal to $(A_2, \sigma_2)$ iff $A_2$ is an end extension of $A_1$ and $\sigma_1 = \sigma_2$.

By König’s lemma, $T$ has an infinite branch $(A_n, \sigma)$ ($0 < n < \omega$). Let $B$ be the ranked partial structure with rank ordering $\omega$ which is an end extension of each $A_n$. By the previous lemma, $B \models \phi[\sigma]$ for each $\phi \in \Gamma$. By the comments after lemma 4.10, $B \models \phi[\sigma]$ for each $\phi \in \Gamma$. \qed
7 Ranked Structures for First-Order Logic

Suppose $\Phi$ is a set of first-order formulas of $\mathcal{L}$ and $\phi$ is a first-order formula of $\mathcal{L}$. We have

$$\Phi \models \phi \text{ iff } \Phi \cup \Delta \models_{PS} \phi$$

where $\Delta$ consists of the sentences saying the functions of the language are total. For example, when $f$ is an $m$-ary function symbol we could us the sentence $\forall x_1 \cdots \forall x_m \exists y f(x_1, \ldots, x_m) = y$ to express that $f$ is total. By theorem 6.1, we see that $\Phi \cup \Delta \models_{PS} \phi$ iff there exists positive $n \in \omega$ such that $\Phi \cup \Delta \cup \{ \neg \phi \}$ is not satisfiable in any ranked partial structure of height $n$.

Notice that $A \models \forall x_1 \cdots \forall x_m \exists y f(x_1, \ldots, x_m) = y$ iff

(*) $(a_1, \ldots, a_m)$ is in the domain of $f^A$ and $f(a_1, \ldots, a_m) \in R_i$ whenever $a_1, \ldots, a_m \in R_{<i}$

Definition 7.1 Assume $A$ is a ranked partial structure for $\mathcal{L}$. $A$ is a ranked structure for $\mathcal{L}$ iff condition (*) above holds for each function symbol $f$ of $\mathcal{L}$.

We have seen

Theorem 7.2 If $\Phi$ is a set of first-order formulas of $\mathcal{L}$ and $\phi$ is a first-order formula of $\mathcal{L}$ then $\Phi \models \phi$ iff there is an $n \in \omega$ such that $\Phi \cup \{ \neg \phi \}$ is not satisfiable in any ranked structure of height $n$. \qed

Lemma 7.3 Assume $A$ is a ranked structure for $\mathcal{L}$. If $t$ is a term of $\mathcal{L}$ then $A \models t$. \qed

Proof. Immediate. \qed

The description of the semantics for ranked structures is simpler than that for ranked partial structures since we don’t have to be concerned with questions of convergence.

Lemma 7.4 Assume $A$ is a partial structure for $\mathcal{L}$, $s$ is an assignment in $|A|$, and $P(t_1, \ldots, t_m)$ is an atomic formula of $\mathcal{L}$.

1. $A \models P(t_1, \ldots, t_m)[s]$ is equivalent to the condition that if $t_i^{(A|\mathcal{L},s)}$ is defined for $i = 1, \ldots, m$ then $(t_1^{(A|\mathcal{L},s)}, \ldots, t_m^{(A|\mathcal{L},s)}) \in P^A$.

2. $A \models \neg P(t_1, \ldots, t_m)[s]$ is equivalent to the condition that if $t_i^{(A|\mathcal{L},s)}$ is defined for $i = 1, \ldots, m$ then $(t_1^{(A|\mathcal{L},s)}, \ldots, t_m^{(A|\mathcal{L},s)}) \notin P^A$. 

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Proof. 1 follows from the previous lemma. 2 is a slight restatement of the definition for ranked partial structures and does not require the restriction that \( \mathcal{A} \) be a ranked structure. \( \square \)

Another change that could be made if we had restricted our attention to first-order logic would be to simplify the definition of Skolem form in definition 1.4 by deleting the conjunct saying the Skolem terms converges.

8 Variations

In this section we will discuss our choices for the semantics of ranked partial structures and some alternatives, Skolem's approach to provability, and the relationship of our approach to the finitization method employed by Mycielski.

Two important concerns in our choices are elegance and a close relationship to the logic of partial structures that can be used to verify consistency. In particular, we want theorem 6.1 to hold (a set of formulas is satisfiable in a partial structure iff it is satisfiable in a ranked partial structure of each positive integer height). Also, we would like the semantics to be as unrestricted as possible i.e. we would like the set of formulas \( \phi \) such that \( \mathcal{A} \models \phi[s] \) to be as large as possible so that finding ranked partial structures satisfying \( \phi \) will be as easy as possible.

8.1 Semantics for atomic formulas and negations of atomic formulas.

When \( \alpha \) is a negation of an atomic formula or an atomic formula which isn't of the form \( t_1 = t_2 \) we have defined \( \mathcal{A} \models \alpha[s] \) so as to be equivalent to the existence of a partial structure \( \mathcal{N} \) which is approximated by \( \mathcal{A} \) with the property that \( \mathcal{N} \models \alpha[s] \). If we were to define the semantics of equations in the same way, the discussion in remark 2.10 shows that the resulting semantics would be more restrictive than that which we have presented. Moreover, if we restrict our attention to first-order and ranked structures, the semantics of inequalities can also fail the above equivalence e.g. we can have \( \mathcal{A} \models f(x) \neq f(x)[s] \) (this will happen if \( s(x) \) is not in the bounded part of \( \mathcal{A} \)) while there is no first-order structure with \( \mathcal{N} \models f(x) \neq f(x)[s] \).

8.2 Semantics for connectives and quantifiers.

Given the semantics of atomic and negated atomic formulas, we used three principles from section 2 to motivate the full semantics:

1. Conjunction, disjunction and universal quantification are to be interpreted as in first-order logic.

2. The negation of a formula is to be viewed as the dual of the original formula.
3. A formula $\phi$ in negation normal form should be satisfied in a ranked partial structure $\mathcal{A}$ under an assignment $s$ iff there is an expansion of $\mathcal{A}$ which satisfies a Skolem form of (an alphabetic variant of) $\phi$ under $s$.

While we adhered to the first two of these principles, we only approximated the third. When $\phi$ is a sentence of complexity $c$ and $\phi^*$ is a Skolem form of $\phi$, part 1 of lemma 2.24 and remark 2.23 show that

- If $\phi$ is satisfied in a ranked partial structure of height $n + c$ then $\phi^*$ is satisfied in a ranked partial structure of height $n$.
- If $\phi^*$ is satisfied in a ranked partial structure of height $n$ then $\phi$ is also.

Remark 2.23 shows that the semantics based on principles 1-3 is more restrictive than that which we have chosen. Nevertheless, we do think there may be something to gain in looking into the semantics based on 1-3 (possibly with the alternate semantics for atomic and negated atomic formulas above). One can give an elementary description of such semantics along the usual lines by allowing assignments to also take on a special value $\infty$ which is to indicate that the variable represents some object occurring in some approximated structure but which is not in the given rank hierarchy.

An intermediate logic between ours and that based on principles 1-3 can be obtained by strengthening the conditions for the satisfaction of an existential formula by replacing clauses 6 and 11 by

6'. $\mathcal{A} \models \exists x \phi[s]$ iff either $s$ does not map the free variables of $\exists x \phi$ into the bounded part of $\mathcal{A}$ or there is $a \in |\mathcal{A}|$ such that $\mathcal{A} \models \phi[s(x[a])]$ and $a \in R_i$ whenever $s$ maps the free variables into $R_{<i}$.

11'. $\mathcal{A} \models \neg \forall x \phi[s]$ iff either $s$ does not map the free variables of $\neg \forall x \phi$ into the bounded part of $\mathcal{A}$ or there is $a \in |\mathcal{A}|$ such that $\mathcal{A} \models \neg \phi[s(x[a])]$ and $a \in R_i$ whenever $s$ maps the free variables into $R_{<i}$.

With this semantics, we would not require the condition in theorem 2.21 that the rank ordering is a well ordering i.e. Skolem expansions would always exist. These alternate clauses are slightly more complex than those we have chosen. Also, our semantics agrees with that based on 6' and 11' for any ranked partial structure with a well-ordered rank ordering – the case we are most interested in. While the resulting logic is more restrictive than ours, the two are very close (the key difference seems to be that $\exists x x = x$ is not valid under this alternative semantics as can be seen by considering any ranked partial structure where the intersection of the ranks is empty).

8.3 Weakening the semantics of the existential quantifier

We can define other alternative semantics by changing the conditions concerning how quickly existential statements must be witnessed in the rank hierarchy.
We could be more liberal in the clause defining the semantics of the existential quantifier by allowing a witness for the quantified variable need not appear immediately in the rank hierarchy after the interpretations of the free variables but may appear some fixed finite number of stages later. Such considerations lead to logics with a constructive flavor which we intend to investigate in a future paper.

The entire notion of ranked partial structure is inappropriate as a semantic foundation within theories which are too weak to have exponentiation (as the example after corollary 5.9 shows). For such cases, we may want to weaken the closure conditions on the rank hierarchy itself and allow even more time for witnesses for existential quantifiers. In the extreme, we may simply have an enumeration \(a_0, \ldots, a_n\) of the universe of \(\mathcal{A}\) with \(R_i = \{a_0, \ldots, a_i\}\) for \(i \leq n\). For \(h : \omega \rightarrow \omega\) we could define a semantics based on \(h\) for ranked partial structures whose rank ordering is a natural number so that \(\mathcal{A} \models \exists x \phi[s]\) \(\mod(h)\) means that there is some \(a \in R_{f[i]}\) such that \(\mathcal{A} \models \phi[s(x|a)]\) \(\mod(h)\) whenever \(s\) maps the free variables of \(\exists x \phi\) into \(R_{\omega i}\) (\(h\) would also determine how quickly the values of the interpretations of the function symbols at elements of the universe would be required to appear in the rank hierarchy).

8.4 Skolem's approach.

We will now discuss the relationship of Skolem's approach to provability in first-order logic with our method using ranked structures. Fix a finite language which contains at least one constant symbol.

If \(\psi\) is a \(\Pi_1\) sentence in negation normal form, by a ground instance of \(\psi\) we mean any formula which results from instantiating each universal quantifier in \(\psi\) by a closed term i.e. a term without variables.

Theorem 7.2 is closely related to Skolem's approach to provability: to show a formula \(\phi\) is not satisfiable, take a Skolem form \(\phi^*\) of \(\phi\) and find an \(n\) such that the set of all ground instances of \(\phi^*\) involving only terms of height less than \(n\) is not satisfiable in propositional logic. This method works for languages in which \(=\) is not given any special status beyond the other predicate symbols. This is not an obstacle since we can replace \(\phi\) by the conjunction of \(\phi\) with the usual \(\Pi_1\) sentences saying that \(=\) is a congruence. Fix a sentence \(\phi\) in a language \(\mathcal{L}\) which is not satisfiable and let \(n_{Sk}\) be the least \(n\) as above. For simplicity, assume \(=\) does not occur in \(\phi\). Theorem 7.2 says that there is an integer \(n\) such that \(\phi\) is not satisfiable in any ranked structure of height \(n\). Let \(n_{RS}\) be the least \(n\) with this property. The reader may suspect that \(n_{Sk} \approx n_{RS}\). To provide some evidence of this, first recall that the least \(n\) such that \(\phi\) is not satisfied in a ranked structure of height \(n\) is approximately equal to the least \(n\) such that \(\phi^*\) is not satisfied in a ranked structure of height \(n\). So we will take the liberty of assuming that \(\phi\) itself is \(\Pi_1\).

Suppose \(n\) is an integer. We will see by a relatively easy argument that \(A_n \Rightarrow B_n \Rightarrow C_n\) where \(A_n\), \(B_n\), and \(C_n\) are

\(A_n\): The set of ground instances of \(\phi\) obtained by instantiating each universal
quantifier in $\phi$ by a closed term of height less than $n$ is satisfiable in
propositional logic.

$B_n$: $\phi$ is satisfiable in a ranked partial structure of height $n$.

$C_n$: The set of ground instances of $\phi$ in which every term that appears has
height less than $n$ is satisfiable in propositional logic.

Notice that some of the ground instances in $A_n$ may contain terms of height $n$
or greater while that cannot be the case for the ground instances in $C_n$.

Let $S$ be the set of atomic formulas without any occurrence of any variable.
By a truth assignment we will mean a function which maps $S$ into $\{T,F\}$.

Assume $A_n$ holds. Let $\nu$ be a truth assignment which satisfies each ground
instance of $\phi$ as described in $A_n$. To verify $B_n$, we will define a ranked partial
structure $A$ of height $n$ such that $R_i$ consists of all closed terms of height at
most $i$ for $i < n$. For each $k$-ary function symbol $f$ and terms $t_1, \ldots, t_k$ of height
less that $n - 1$ define

$$f^A(t_1, \ldots, t_k) = f(t_1, \ldots, t_k)$$

and leave $f^A(t_1, \ldots, t_k)$ undefined otherwise. For each $k$-ary predicate symbol
$P$ and terms $t_1, \ldots, t_k$ of height less than $n$ define

$$P^A(t_1, \ldots, t_k) \iff \nu(P(t_1, \ldots, t_k)) = T$$

Noting that $A \models t^\uparrow$ whenever $t$ is a closed term of height at least $n$, an easy
argument shows that whenever $\alpha$ is an atomic formula or the negation of an
atomic formula whose free variables are among $x_1, \ldots, x_m$ and $t_1, \ldots, t_m$ are
closed terms of any height then

$$\nu \text{ satisfies } \alpha(x_1, \ldots, x_m, t_1, \ldots, t_m) \Rightarrow A \models \alpha[x_1, \ldots, x_m, t_1, \ldots, t_m]$$

A simply induction on $x_1, \ldots, x_m$ shows that $A \models \phi$.

Assume $B_n$ holds. Let $A$ be a ranked partial structure of height $n$ which
satisfies $\phi$. Define a truth assignment $\nu$ such that $\nu(\alpha) = T$ iff $A \models \alpha$ for each
atomic formula $\alpha$. $\nu$ is easily seen to satisfy all ground instances of $\phi$ in which
all terms have height less than $n$ (notice that if $t$ is a closed term of height $i < n$
then $t^\uparrow \in R_i$).

Let $h$ be the maximum of the heights of the terms which occur in $\phi$. Notice
that $C_{n+h} \Rightarrow A_n$ for all $n$ since any term which occurs in a ground instance of
the type described in $A_n$ has height less than $n + h$. We see that

$$n_{RS} \leq n_{Sk} \leq n_{RS} + h$$

The first inequality follows from the fact that $B_n \Rightarrow C_n$ for all $n$ and the second
inequality follows from the fact that $C_{n+h} \Rightarrow B_n$ for all $n$.  

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8.5 Mycielski’s method of finitization.

We conclude with a discussion of Mycielski’s finitization. We assume the reader is familiar with [5].

Both Mycielski’s method and our approach are attempts to reduce semantics to finite structures. There is a natural translation between the two methods. Notice that Mycielski’s definition of FIN(T) for a set of sentences T in a language L, while given only for the language obtained by adding unary predicates Ω_p for each rational p, can be generalized to any linear ordering (I, <). Denote this theory by FIN(I, <)(T) and the corresponding language by L(I, <). If we make the unimportant additional requirement that the underlying structure of every ranked structure actually be a structure (and not just a partial structure) then there is an obvious correspondence between structures for L(I, <) which satisfy axiom schemes (2)-(4) from [5] and ranked structures for L with rank ordering (I, <): the interpretation of Ω_i in the former corresponds to R_i in the latter. We will write

\[ A \models \phi[s] \]

whenever A is a ranked structure for L, \( \phi \) is a formula of L(I, <), s is an assignment in \(|A|\), and the structure for L(I, <) which corresponds to A as above satisfies \( \phi \) under s. One can easily establish

1. If (I, <) is the rank ordering of A then \( A \models \text{FIN}(I, <)(T) \) iff \( A \models T \) and A determines all formulas.

2. If \( \phi \) contains no function symbols then

\[ A \models \phi[s] \Rightarrow A \models \psi[s] \]

for any regular relativization \( \psi \) of \( \phi \).

The requirement that \( \phi \) contain no function symbols in 2 is necessary since under \( \models \) the underlying structure of A is used to evaluate atomic formulas while under our semantics only \(|A|L\) is used (possibly leading to an atomic formula and its negation being satisfied). So we see that for languages without function symbols \( A \models T \) lies between \( A \models \text{FIN}(I, <)T \) and \( A \models \phi \) for every regular relativization \( \phi \) of an element of T. These inclusions are strict.

We are interested in weak conditions for the consistency of a theory T. The consistency of FIN(T) provides one such condition. The completeness theorem of section 5 provides another weaker (by 1 above) possibility: the condition that for each positive \( n \in \omega \) there is a ranked structure of height \( n \) which satisfies T. 2 suggests a weaker condition yet: for each positive \( n \in \omega \) there is a ranked structure of height \( n \) which satisfies, in the sense of \( \models \), all regular relativizations of elements of T. In fact, the sufficiency of this last condition can be established by the methods of this paper, even for languages with function symbols. Weaker yet, one can use a hybrid semantics in which \( \models \) uses the conditions of our semantics for atomic formulas and negations of atomic formulas.

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Postscript: After submission of this paper, the author was made aware that J.E. Quinsey had used notions similar to those used here in his Ph.D. thesis [7].

References


8. J. Silver, mimeographed notes.
