Midterm 3

Math 184 - Summer 2011

Last Name:

First Name:

Student Number:

Section (circle one): 921 (Warren Code) or 922 (Marc Carnovale)

Read all of the following information before starting the exam:

• Calculators are not permitted on this exam. (Note that if a question seems to require a calculator, you should recheck your work for errors). You may leave your answer in a calculator-ready form: a calculator that can do arithmetic and exponents, but not one that can solve equations or take derivatives for you.

• Show all your work in order to receive full credit. If multiple steps are involved in arriving at an answer, a response consisting solely of that answer will not receive credit.

• Circle or otherwise indicate your final answers.

• This test has 7 problems worth a total of 50 points. Make sure that you have all pages of this exam.

• Good luck!
1. (8 points) Find the derivatives in the following cases.

a. Let \( f(x) = \left( \frac{1}{2x + 1} \right)^x \). Find \( f'(x) \).

b. Let \( g(x) = e^{\cos(x) \sin(x)} \). Find \( g'(x) \).
2. (6 points) A loan shark lends you $5000 at an interest rate of 200% per year, to be compounded continuously. Recall that the formula for continuous interest is \( f(t) = Pe^{rt} \), where \( P \) is the amount initially lent, \( r \) is the interest rate, and \( t \) is the amount of time that has passed since the loan was made.

a. How much time must pass before you owe him three times the amount that he lent you?

b. Does your answer to Part a. depend on the size of the initial loan? Why or why not?
3. (10 points) The Bizarro Company has an unusual widget that satisfies the demand relationship

\[ q = 100pe^{-p} \]

where \( q \) is the number of widgets demanded (in thousands) when the price is \( p \) dollars per unit. (Recall that the price elasticity of demand is given by \( E(p) = -\frac{p}{q} \cdot q'(p) \).)

(a) Find the price elasticity of demand \( E(p) \) explicitly as a function of \( p \).
(b) For which values of price \( p \) will decreasing the price slightly result in an increase in revenue for the company?
4. (10 points) Consider the cost function \( C(q) = 8q^2 + 16q + 8 \), where \( C \) is measured in thousands of dollars.

(a) What is the marginal cost at production level \( q = 5 \)?

(b) Use the marginal cost calculated above to estimate the cost of raising the production level from \( q = 5 \) to \( q = 7 \).
(c) Let \( R(q) = -7q^2 + 61q - 22 \) denote the revenue in thousands of dollars from the production of \( q \) units. What is/are the break-even point(s)?

(d) Compute and compare the marginal revenue and marginal cost at the highest break-even point. Should the company increase production beyond the break-even point? Justify your answer using marginals.
5. (6 points) In this problem, you are asked to show that the identity \( \cos^2(x) + \sin^2(x) = 1 \) holds as a result of the formulas for the derivatives of \( \cos \) and \( \sin \). (So do not use the fact that \( \cos^2 x + \sin^2 x = 1 \); this will be the result). You may find it useful to recall that \( \sin(0) = 0 \), \( \cos(0) = 1 \).

Let \( f(x) = \cos^2(x) + \sin^2(x) \).

a. Use the Chain Rule to show that \( f'(x) = 0 \).

b. Assume that if \( f' = 0 \) (that is, \( f \) has derivative equal to zero for all values of \( x \)), then \( f \) is a constant function. Calculate \( f(0) \) and use it with part (a) to explain why \( \cos^2 x + \sin^2 x = 1 \).
6. (5 points) Consider the following two situations:

A. A business determines their production costs to be \( C(q) = 2q + 5 \), where \( q \) is the number of items produced and \( C \) is measured in thousands of dollars.

B. The position of a car along a highway is given by \( s(t) = 2t + 5 \), where \( t \) is measured in minutes and \( s \) is measured in kilometres.

In situation A, my friend calculates the average cost of producing 5 items to be 3 (thousands of dollars per item), while in situation B another friend calculates the average velocity (the average speed needed to travel a given distance in a given time) of the car over the first 5 minutes to be 2 (kilometres per minute). Does this make sense? Explain why or why not. Sketch one or more graphs to support your explanation.
7. (5 points) Define the relative growth rate of the function $f$ over the time interval $T$ to be the relative change in $f$ over an interval of length $T$:

$$ R_T(t) = \frac{f(t+T) - f(t)}{f(t)} $$

For example, when $T = 2$, $R_2$ measures the relative change in $f$ from $t$ up to $t + 2$, for each $t$. $R_2(1)$ tells you the relative change of $f$ over the interval $[1, 3]$ (has length 2, starts at $t = 1$), $R_2(35)$ tells you the relative change of $f$ over the interval $[35, 37]$. By comparison, if $T = 4$, $R_4(1)$ tells you the relative change of $f$ over the interval $[1, 5]$ (has length 4, starts at 1).

Show that for the exponential function $g(t) = y_0 e^{kt}$, where $y_0$ and $k$ are fixed constants, the relative growth rate $R_T$ is constant for any $T$; that is, choose any $T$ and show that $R_T$ is constant for all $t$. 
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