Chapter 5 Applications of the Exponential and Natural Logarithm Functions

Solutions to Practice Problems 5.2

1. Let us compute the balance after 4 years for each type of interest.

8% compounded semiannually: Use the formula at the beginning of this section. Here $P = 1000$, $r = .08$, $m = 2$ (semiannually means there are two interest periods per year), and $t = 4$. Therefore,

$A = 1000 \left(1 + \frac{.08}{2}\right)^{2 \times 4} = 1000(1.04)^8 \approx \$1368.57.$

7 1/2% compounded continuously: Use the formula $A = Pe^{rt}$, where $P = 1000$, $r = .0775$, and $t = 4$. Then

$A = 1000e^{(.0775) \times 4} = 1000e^{3.1} \approx \$1363.43.$

Therefore, 8% compounded semiannually is better.

2. If the $150,000 had been compounded continuously for 10 years at interest rate $r$, the balance would be $150,000e^{.10}$. The question is at what value of $r$ will the balance be $400,000? We need to solve an equation for $r$.

$150,000e^{.10} = 400,000$

$e^{.10} \approx 1.1052$

$r \times 10 = \ln 1.1052$

$r \approx \frac{\ln 2.67}{10} \approx .098.$

Therefore, the investment earned 9.8% interest per year.

5.3 Applications of the Natural Logarithm Function to Economics

In this section we consider two applications of the natural logarithm to the field of economics. Our first application is concerned with relative rates of change and the second with elasticity of demand.

Relative Rates of Change The logarithmic derivative of a function $f(t)$ is defined by the equation

$$\frac{df}{dt} \ln f(t) = \frac{f'(t)}{f(t)}. \quad (1)$$

The quantity on either side of equation (1) is often called the relative rate of change of $f(t)$ per unit change of $t$. Indeed, this quantity compares the rate of change of $f(t)$ (that is, $f'(t)$) with $f(t)$ itself. The percentage rate of change is the relative rate of change of $f(t)$ expressed as a percentage.

A simple example will illustrate these concepts. Suppose that $f(t)$ denotes the average price per pound of sirloin steak at time $t$ and that $g(t)$ denotes the average price of a new car (of a given make and model) at time $t$, where $f(t)$ and $g(t)$ are given in dollars and time is measured in years. Then the ordinary derivatives $f'(t)$ and $g'(t)$ may be interpreted as the rate of change of the price of a pound of sirloin steak and of a new car, respectively, where both are measured in dollars per year. Suppose that, at a given time $t_0$, we have $f(t_0) = 5.25$ and $g(t_0) = 12,000$. Moreover, $f'(t_0) = .75$ and $g'(t_0) = 1500$. Then, at time $t_0$, the price per pound of steak is increasing at a rate of $.75 per year, while the price of a new car is increasing at a rate of $1500 per year. Which price is increasing more quickly? It is not meaningful to say that the car price is increasing faster simply because $1500 is larger than $.75. We must take into account the vast difference between the actual cost of a car and the cost of steak. The usual basis of comparison of price increases is the percentage rate of increase. In other words, at $t = t_0$, the price of sirloin steak is increasing at the percentage rate

$$\frac{f'(t_0)}{f(t_0)} = \frac{.75}{5.25} \approx .143 = 14.3\%$$
per year, but at the same time the price of a new car is increasing at the percentage rate
\[
g'(t_0) = \frac{1500}{12,000} = .125 = 12.5\%
\]
per year. Thus the price of sirloin steak is increasing at a faster percentage rate than the price of a new car.

Economists often use relative rates of change (or percentage rates of change) when discussing the growth of various economic quantities, such as national income or national debt, because such rates of change can be meaningfully compared.

**EXAMPLE 1**

**Gross domestic product** A certain school of economists modeled the Gross Domestic Product of the United States at time \( t \) (measured in years from January 1, 1990) by the formula
\[
f(t) = 3.4 + .04t + .13e^{-t},
\]
where the Gross Domestic Product is measured in trillions of dollars. (See Fig. 1.) What was the predicted percentage rate of growth (or decline) of the economy at \( t = 0 \) and \( t = 1 \)?

**Solution** Since
\[
f'(t) = .04 - .13e^{-t},
\]
we see that
\[
f'(0) = \frac{.04 - .13}{3.4 + .13} = \frac{-0.09}{3.53} \approx -2.5%;
\]
\[
f'(1) = \frac{.04 - .13e^{-1}}{3.4 + .04 + .13e^{-1}} \approx \frac{-0.00738}{3.4873} \approx -0.2%.
\]

So on January 1, 1990, the economy is predicted to contract or decline at a relative rate of 2.5% per year; on January 1, 1991, the economy is predicted to be still contracting, but only at a relative rate of 0.2% per year.

**EXAMPLE 2**

**Value of an investment** The value in dollars of a certain business investment at time \( t \) may be approximated empirically by the function \( f(t) = 750,000e^{.6\sqrt{t}} \). Use a logarithmic derivative to describe how fast the value of the investment is increasing when \( t = 5 \) years.

**Solution** We have
\[
\frac{f'(t)}{f(t)} = \frac{d}{dt} \ln f(t) = \frac{d}{dt} \left( \ln 750,000 + \ln e^{.6\sqrt{t}} \right)
\]
\[
= \frac{d}{dt} (\ln 750,000 + .6 \sqrt{t})
\]
\[
= (.6) \left( \frac{1}{2} \right) t^{-1/2} = \frac{.3}{\sqrt{t}}.
\]

When \( t = 5 \),
\[
\frac{f'(5)}{f(5)} = \frac{.3}{\sqrt{5}} \approx .134 = 13.4%.
\]

Thus, when \( t = 5 \) years, the value of the investment is increasing at the relative rate of 13.4% per year. So when \( t = 5 \), we should expect the investment to grow by 13.4% in 1 year. (See Fig. 2.) Indeed, if we compute the increase in the investment
from $t = 5$ to $t = 6$ and divide by its value at $t = 5$ to get the percent increase, we find

$$[\text{percentage rate of change}] = \frac{f(6) - f(5)}{f(5)} \approx \frac{32.6 - 28.7}{28.7} \approx 0.136$$

or 13.6%. This percentage is close to 13.4%, the relative rate of change of $f$ at $t = 5$. 

In certain mathematical models, it is assumed that for a limited period of time the percentage rate of change of a particular function is constant. The following example shows that such a function must be an exponential function.

**EXAMPLE 3** If the function $f(t)$ has a constant relative rate of change $k$, show that $f(t) = Ce^{kt}$ for some constant $C$.

**Solution** We are given that

$$\frac{f'(t)}{f(t)} = k.$$ 

Hence $f'(t) = kf(t)$. But this is just the differential equation satisfied by the exponential function. Therefore, we must have $f(t) = Ce^{kt}$ for some constant $C$.

**Elasticity of Demand** In Section 2.7 we considered demand equations for monopolists and for entire industries. Recall that a demand equation expresses, for each quantity $x$ to be produced, the market price that will generate a demand of exactly $x$. For instance, the demand equation

$$p = 150 - .01x$$

(2)

says that to sell $x$ units the price must be set at $150 - .01x$ dollars. To be specific, to sell 6000 units the price must be set at $150 - .01(6000) = 90$ per unit.

Equation (2) may be solved for $x$ in terms of $p$ to yield

$$x = 100(150 - p).$$

(3)

This last equation gives quantity in terms of price. If we let the letter $q$ represent quantity, equation (3) becomes

$$q = 100(150 - p).$$

(3a)

This equation is of the form $q = f(p)$, where in this case $f(p)$ is the function $f(p) = 100(150 - p)$. In what follows it will be convenient to always write our demand functions so that the quantity $q$ is expressed as a function $f(p)$ of the price $p$.

Usually, raising the price of a commodity lowers demand. Therefore, the typical demand function $q = f(p)$ is decreasing and has a negative slope everywhere. (See Fig. 3.)

Recall that the derivative $f'(p)$ compares the change in the quantity demanded with the change in price. By way of contrast, the concept of elasticity is designed to compare the relative rate of change of the quantity demanded with the relative rate of change of price.

Let us be more explicit. Consider a particular demand function $q = f(p)$. From our interpretation of the logarithmic derivative in (1), we know that the relative rate of change of the quantity demanded with respect to $p$ is

$$\frac{(d/dp)f(p)}{f(p)} = \frac{f'(p)}{f(p)}.$$
Similarly, the relative rate of change of price with respect to $p$ is

$$\frac{(d/dp)p}{p} = \frac{1}{p}. $$

Hence the ratio of the relative rate of change of the quantity demanded to the relative rate of change of price is

$$\frac{\text{relative rate of change of quantity}}{\text{relative rate of change of price}} = \frac{f'(p)/f(p)}{1/p} = \frac{pf'(p)}{f(p)}. $$

Since $f'(p)$ is always negative for a typical demand function, the quantity $pf'(p)/f(p)$ will be negative for all values of $p$. For convenience, economists prefer to work with positive numbers, and therefore the *elasticity of demand* is taken to be this quantity multiplied by $-1$.

The elasticity of demand $E(p)$ at price $p$ for the demand function $q = f(p)$ is defined to be

$$E(p) = \frac{-pf'(p)}{f(p)}. $$

**EXAMPLE 4**

**Elasticity of demand** The demand function for a certain metal is $q = 100 - 2p$, where $p$ is the price per pound and $q$ is the quantity demanded (in millions of pounds).

(a) What quantity can be sold at $30$ per pound?
(b) Determine the function $E(p)$.
(c) Determine and interpret the elasticity of demand at $p = 30$.
(d) Determine and interpret the elasticity of demand at $p = 20$.

**Solution**

(a) In this case, $q = f(p)$, where $f(p) = 100 - 2p$. When $p = 30$, we have $q = f(30) = 100 - 2(30) = 40$. Therefore, 40 million pounds of the metal can be sold. We also say that the demand is 40 million pounds.

(b) $E(p) = \frac{-pf'(p)}{f(p)} = \frac{-p(-2)}{100 - 2p} = \frac{2p}{100 - 2p}$.

(c) The elasticity of demand at price $p = 30$ is $E(30)$.

$$E(30) = \frac{2(30)}{100 - 2(30)} = \frac{60}{40} = \frac{3}{2}. $$

When the price is set at $30$ per pound, a small increase in price will result in a relative rate of decrease in quantity demanded of about $\frac{3}{2}$ times the relative rate of increase in price. For example, if the price is increased from $30$ by $1\%$, the quantity demanded will decrease by about $1.5\%$.

(d) When $p = 20$, we have

$$E(20) = \frac{2(20)}{100 - 2(20)} = \frac{40}{60} = \frac{2}{3}. $$

When the price is set at $20$ per pound, a small increase in price will result in a relative rate of decrease in quantity demanded of only $\frac{2}{3}$ of the relative rate of increase of price. For example, if the price is increased from $20$ by $1\%$, the quantity demanded will decrease by $\frac{2}{3}$ of $1\%$.  

Economists say that demand is *elastic* at price $p_0$ if $E(p_0) > 1$ and *inelastic* at price $p_0$ if $E(p_0) < 1$. In Example 4, the demand for the metal is elastic at $\$30$ per pound and inelastic at $\$20$ per pound.

The significance of the concept of elasticity may perhaps be best appreciated by studying how revenue, $R(p)$, responds to changes in price. Let us illustrate this response with a concrete example.

**EXAMPLE 5**

Figure 4 shows the elasticity of demand for the metal in Example 4.

![Figure 4](image)

(a) For what values of $p$ is demand elastic? Inelastic?
(b) Find and plot the revenue function for $0 < p < 50$.
(c) How does revenue respond to an increase in price when demand is elastic or respectively, inelastic?

**Solution** (a) In Example 4(b), we found the elasticity of demand to be

$$E(p) = \frac{2p}{100 - 2p}.$$

Let us solve $E(p) = 1$ for $p$.

$$\frac{2p}{100 - 2p} = 1$$

$$2p = 100 - 2p$$

$$4p = 100$$

$$p = 25.$$

By definition, demand is elastic at price $p$ if $E(p) > 1$ and inelastic if $E(p) < 1$. From Figure 4, we see that demand is elastic if $25 < p < 50$ and inelastic if $0 < p < 25$.

(b) Recall that

$$[\text{revenue}] = [\text{quantity}] \cdot [\text{price per unit}].$$

Using the formula for demand (in millions of pounds) from Example 4, we obtain the revenue function

$$R = (100 - 2p) \cdot p = p(100 - 2p) \text{ (in millions of dollars).}$$

This is a parabola opening down with $p$-intercepts at $p = 0$ and $p = 50$. Its maximum is located at the midpoint of the $p$-intercepts, or $p = 25$. (See Fig. 5.)
(c) In part (a), we determined that demand is elastic for $25 < p < 50$. For $p$ in this price range, Figure 5 shows that an increase in price results in a decrease in revenue, and a decrease in price results in an increase in revenue. Hence we conclude that when demand is elastic the change of revenue is in the opposite direction of the change in price. Similarly, when demand is inelastic ($0 < p < 25$), Figure 5 shows that the change of revenue is in the same direction as the change in price.

Example 5 illustrates an important application of elasticity as an indicator of the response of revenue to a change in price. As we now show, this response analysis can be applied in general.

Start by expressing revenue as a function of price:

$$ R(p) = f(p) \cdot p, $$

where $f(p)$ is the demand function. If we differentiate $R(p)$ using the product rule, we find that

$$ R'(p) = \frac{d}{dp} [f(p) \cdot p] = f'(p) \cdot 1 + p \cdot f'(p) \\
= f(p) \left[ 1 + \frac{p f'(p)}{f(p)} \right] \\
= f(p) [1 - E(p)]. \quad (4) $$

Now suppose that demand is elastic at some price $p_0$. Then $E(p_0) > 1$ and $1 - E(p_0)$ is negative. Since $f(p)$ is always positive, we see from (4) that $R'(p_0)$ is negative. Therefore, by the first derivative rule, $R(p)$ is decreasing at $p_0$. So an increase in price will result in a decrease in revenue, and a decrease in price will result in an increase in revenue. On the other hand, if demand is inelastic at $p_0$, then $1 - E(p_0)$ will be positive and hence $R'(p_0)$ will be positive. In this case an increase in price will result in an increase in revenue, and a decrease in price will result in a decrease in revenue. We may summarize this as follows:

The change in revenue is in the opposite direction of the change in price when demand is elastic and in the same direction when demand is inelastic.
EXERCISES 5.3

Determine the percentage rate of change of the functions at the points indicated.

1. \( f(t) = t^2 \) at \( t = 10 \) and \( t = 50 \)
2. \( f(t) = t^{10} \) at \( t = 10 \) and \( t = 50 \)
3. \( f(x) = e^{-3x} \) at \( x = 10 \) and \( x = 20 \)
4. \( f(x) = e^{-0.5x} \) at \( x = 1 \) and \( x = 10 \)
5. \( f(t) = e^{3t} \) at \( t = 1 \) and \( t = 5 \)
6. \( G(s) = e^{-0.5s} \) at \( s = 1 \) and \( s = 10 \)
7. \( f(p) = 1/(p + 2) \) at \( p = 2 \) and \( p = 8 \)
8. \( g(p) = 5/(2p + 3) \) at \( p = 1 \) and \( p = 11 \)
9. **Percentage Rate of Growth** The annual sales \( S \) (in dollars) of a company may be approximated empirically by the formula

\[
S = 50,000 \sqrt{e^{0.1t}},
\]

where \( t \) is the number of years beyond some fixed reference date. Use a logarithmic derivative to determine the percentage rate of growth of sales at \( t = 4 \).

10. **Percentage Rate of Change** The price of wheat per bushel at time \( t \) (in months) is approximated by

\[
f(t) = 4 + 0.001t + 0.01e^{-t}.
\]

What is the percentage rate of change of \( f(t) \) at \( t = 0 \)? \( t = 1 \) ? \( t = 2 \)?

11. An investment grows at a continuous 12% rate per year. In how many years will the value of the investment double?

12. The value of a piece of property is growing at a continuous \( r \)% rate per year and the value doubles in 3 years. Find \( r \).

For each demand function, find \( E(p) \) and determine if demand is elastic or inelastic (or neither) at the indicated price.

13. \( q = 700 - 5p, p = 80 \)
14. \( q = 600e^{-2p}, p = 10 \)
15. \( q = 400(116 - p^2), p = 6 \)
16. \( q = (77/p^3) + 3, p = 1 \)
17. \( q = p^2e^{-1(p+3)}, p = 4 \)
18. \( q = 700/(p + 5), p = 15 \)

19. **Elasticity of Demand** Currently, 1800 people ride a certain commuter train each day and pay \$4 for a ticket. The number of people \( q \) willing to ride the train at price \( p \) is \( q = 600(5 - \sqrt{p}) \). The railroad would like to increase its revenue.
   (a) Is demand elastic or inelastic at \( p = 4 \)?
   (b) Should the price of a ticket be raised or lowered?

20. **Elasticity of Demand** An electronic store can sell \( q = 10,000/(p + 50) - 30 \) cellular phones at a price \( p \) dollars per phone. The current price is \$150.
   (a) Is demand elastic or inelastic at \( p = 150 \)?
   (b) If the price is lowered slightly, will revenue increase or decrease?

21. **Elasticity of Demand** A movie theater has a seating capacity of 3000 people. The number of people attending a show at price \( p \) dollars per ticket is \( q = (18,000/p) - 1500 \). Currently, the price is \$6 per ticket.
   (a) Is demand elastic or inelastic at \( p = 6 \)?
   (b) If the price is lowered, will revenue increase or decrease?

22. **Elasticity of Demand** A subway charges 65 cents per person and has 10,000 riders each day. The demand function for the subway is \( q = 2000\sqrt{90 - p} \).
   (a) Is demand elastic or inelastic at \( p = 65 \)?
   (b) Should the price of a ride be raised or lowered to increase the amount of money taken in by the subway?

23. **Elasticity of Demand** A country that is the major supplier of a certain commodity wishes to improve its balance of trade position by lowering the price of the commodity. The demand function is \( q = 1000/p^2 \).
   (a) Compute \( E(p) \).
   (b) Will the country succeed in raising its revenue?

24. Show that any demand function of the form \( q = a/p^m \) has constant elasticity \( m \).
25. Show that \( E(x) = x \cdot C'(x)/C(x) \).
26. Show that \( E_c \) is equal to the marginal cost divided by the average cost.
27. Let \( C(x) = (1/10)x^2 + 5x + 300 \). Show that \( E_c(50) < 1 \). (Hence when producing 50 units, a small relative increase in production results in an even smaller relative increase in total cost. Also, the average cost of producing 50 units is greater than the marginal cost at \( x = 50 \).)
28. Let \( C(x) = 1000e^{0.2x} \). Determine and simplify the formula for \( E_c(x) \). Show that \( E_c(60) > 1 \) and interpret this result.

## Solutions to Practice Problems 5.3

1. The demand function is \( f(p) = 60,000e^{-0.5p} \).
   
   \[
   f'(p) = -30,000e^{-0.5p} 
   \]
   \[
   E(p) = \frac{-p f'(p)}{f(p)} = \frac{-p(-30,000)e^{-0.5p}}{60,000e^{-0.5p}} = \frac{p}{2} 
   \]
   \[
   E(2.5) = \frac{2.5}{2} = 1.25. 
   \]

2. The demand is elastic, because \( E(2.5) > 1 \).
3. Since demand is elastic at \$2.50, a slight change in price causes revenue to change in the opposite direction. Hence revenue will decrease.

### 5.4 Further Exponential Models

After jumping out of an airplane, a skydiver falls at an increasing rate. However, the wind rushing past the skydiver's body creates an upward force that begins to counterbalance the downward force of gravity. This air friction finally becomes so great that the skydiver's velocity reaches a limiting speed called the **terminal velocity**. If we let \( v(t) \) be the downward velocity of the skydiver after \( t \) seconds of free fall, a good mathematical model for \( v(t) \) is given by

\[
v(t) = M(1 - e^{-kt}),
\]

where \( M \) is the terminal velocity and \( k \) is some positive constant. (See Fig. 1.) When \( t \) is close to zero, \( e^{-kt} \) is close to 1, and the velocity is small. As \( t \) increases, \( e^{-kt} \) becomes small and so \( v(t) \) approaches \( M \).

**EXAMPLE 1**

**Velocity of a skydiver** Show that the velocity given in equation (1) satisfies the equations

\[
v'(t) = k[M - v(t)], \quad v(0) = 0. \tag{2}
\]

**Solution** From (1) we have \( v(t) = M - Me^{-kt} \). Then

\[
v'(t) = Mke^{-kt}.
\]

However,

\[
k[M - v(t)] = k[M - (M - Me^{-kt})] = kMe^{-kt},
\]

so the differential equation \( v'(t) = k[M - v(t)] \) holds. Also,

\[
v(0) = M - Me^0 = M - M = 0.
\]