Contents

Chapter 21
More on Metric Spaces and Function Spaces

21.1 Completion of a Metric Space

Recall that the space of bounded functions is complete with respect to the sup metric ρ (or $\| \cdot_1 - \cdot_2 \|_u$) since a Cauchy sequence is bounded.

Theorem 21.88. *Let (X, d) be a metric space. Then there is an isometric embedding of X into a complete metric space.*

Proof. Let $\mathscr{B}(X, \mathbf{R})$ be the set of all bounded functions mapping X into $\mathbf{R}$. Fix $x_0 \in X$ and for each $a \in X$ define $\varphi_a(x) = d(x, a) - d(x, x_0)$. Then $\forall a \in X$, φ_a is bounded since $d(x, a) \leq d(x, b) + d(a, x_0)$ and $d(x, x_0) \leq d(x, a) + d(a, x_0)$ by the triangle inequality so that $|d(x, a) - d(x, x_0)| \leq d(a, x_0)$ and this is true for all $x \in X$, for each $a \in X$.

Define $\Phi \colon X \to \mathscr{B}(X, \mathbf{R})$ by $\Phi(a) = \varphi_a$. We claim Φ is an isometric embedding of (X, d) into $(\mathscr{B}(X, \mathbf{R}), \rho)$ where ρ is the sup norm. That is, we show that $\forall a, b \in X$, $\rho(\varphi_a, \varphi_b) = d(a, b)$. By definition, $\rho(\varphi_a, \varphi_b) = \sup \{|\varphi_a(x) - \varphi_b(x)| : x \in X\} = \sup \{|d(x, a) - d(x, b)| : x \in X\}$ and hence, as we saw before, this means that $\rho(\varphi_a, \varphi_b) \leq d(a, b)$. But this inequality cannot be strict, for when $x = a$, $|d(x, a) - d(x, b)| = d(a, b)$. ∎

Definition. Let X be a metric space. If $h \colon X \to Y$ is an isometric embedding of X into a complete metric space Y, then the subspace $\overline{h[X]}$ of Y is a complete metric space called the ***completion*** of X.

Exercise (Uniqueness of the Completion). *Let $h \colon X \to Y$ and $h' \colon X \to Y'$ be isometric embeddings of the metric space (X, d) in the complete metric spaces (Y, D) and (Y', D'), respectively. Then there is an isometry of $(\overline{h[X]}, D)$ with $(\overline{h'[X]}, D')$ that equals $h'h^{-1}$ on the subspace $h[X]$.*

21.2 Equicontinuity

Definition (Equicontinuity). Let X be a space and (Y,d) a metric space. Let $\mathscr{F} \subseteq C(X,Y)$. If $x_0 \in X$, the set $\mathscr{F}$ of functions is said to be ***equicontinuous at*** x_0 if $\forall \varepsilon > 0$, there is a nbhd U of x_0 such that $\forall x \in U$, $\forall f \in \mathscr{F}$,

$$d(f(x), f(x_0)) < \varepsilon.$$

(So this is no misnomer, we're really saying that the family of functions $\mathscr{F}$ are all "equally continuous" on X.) If the set $\mathscr{F}$ is equicontinuous at x_0 for each $x_0 \in X$, then $\mathscr{F}$ is simply said to be an ***equicontinuous*** family of functions.

Definition (Pointwise Bounded). If X is a space, (Y,d) a metric space, a subset $\mathscr{F} \subseteq C(X,Y)$ is said to be ***pointwise bounded*** under d if $\forall x \in X$, the subset

$$\mathscr{F}_a \stackrel{\text{def}}{=} \{f(a) \colon f \in \mathscr{F}\}$$

of Y is bounded under d.

Lemma 30. *Let X be a space and (Y,d) a metric space. If $\mathscr{F} \subseteq C(X,Y)$ is totally bounded under the uniform metric corresponding to d (i.e., $\overline{\rho}(f,g) = \sup\{\min\{d(f(x),g(x)),1\} \colon x \in X\}$) then $\mathscr{F}$ is equicontinuous under d.*

Proof. Suppose $\mathscr{F}$ is totally bounded. Then $\forall \varepsilon > 0$, $\mathscr{F}$ can be covered finitely many sets of diameter $\leq \varepsilon$ wrt $\overline{\rho}$. Given $0 < \varepsilon < 1$ and $x_0 \in X$, we find a nbhd U of x_0 such that $\forall x \in U$, $\forall f \in \mathscr{F}$, $d(f(x), f(x_0)) < \varepsilon$. Since everything was appropriately arbitrary, this will prove our contention.

Set $\delta = \varepsilon/3$ and cover $\mathscr{F}$ by finitely many δ-balls $B(f_1,\delta), \dots, B(f_n,\delta)$ in $C(X,Y)$ in the $\overline{\rho}$ metric. Each f_i is continuous and there are only finitely many of them, hence, we may therefore choose a nbhd U of x_0 such that $\forall x \in U$, $d(f_i(x), f_i(x_0)) < \delta$ for $i = 1, 2, \dots, n$.

Let $f \in \mathscr{F}$. Then f belongs to one of these balls, say $B(f_i, \delta_i)$. Then $\forall x \in U$,

$$\begin{aligned} \min\{d(f(x), f_i(x)), 1\} &< \delta, \\ d(f_i(x), f_i(x_0)) &< \delta, \\ \min\{d(f_i(x_0), f(x_0)), 1\} &< \delta. \end{aligned}$$

The first and third inequalities hold because $\overline{\rho}(f, f_i) < \delta$ and the second holds because $x \in U$. Since $\delta = \varepsilon/3$ and $\varepsilon < 1$, $\delta < 1$, hence, the first and third reduce to only the metric d. Then the triangle inequality implies that $\forall x \in U$, $d(f(x), f(x_0)) < \varepsilon$, as desired. ■

Chapter 22
Pointwise and Compact Convergence; The Compact-Open Topology

22.1 Topology of Pointwise Convergence

Definition (Topology of Pointwise Convergence; Point-Open Topology.). Let X be a set and Y a space. For each $x \in X$ and open set U of Y, let

$$S(x,U) = \left\{ f \in Y^X : f(x) \in U \right\}$$

(that is, the set of functions $f: X \to Y$ such that $f(x) \in U$). The sets $S(x,U)$ are a subbasis for a topology on Y^X called the ***topology of pointwise convergence*** (or the ***point-open topology***).

Proposition 39. *The topology of pointwise convergence is the product topology.*

After all, putting $J = X$, the set $S(\alpha, U)$ of all functions $\mathbf{x}: J \to Y$ such that $\mathbf{x}(\alpha) \in U$ is just the subset $\pi_\alpha^{-1}[U]$ of Y^J, which is the standard subbasis element for the product topology.

22.2 Topology of Compact Convergence and Compactly Generated Spaces

22.2.1 Topology of Compact Convergence

Definition (Topology of Compact Convergence). Let X be a topological space and (Y,d) a metric space. For each $\varepsilon > 0$, $f \in Y^X$ and compact subspace C of X, let

$$B_C(f,\varepsilon) = \left\{ g \in Y^X : \sup \left\{ d(f(x), g(x)) : x \in C \right\} < \varepsilon \right\}.$$

(that is, $B_C(f,\varepsilon)$ is the subset of Y^X comprised of functions whose pointwise distance under d from f on the compact set C has supremum $< \varepsilon$). Then

$$\mathscr{S} = \{B_C(f,\varepsilon) : f \in Y^X,\ \varepsilon > 0,\ \text{and } C \subseteq X \text{ is compact}\}$$

comprises a *basis* for a topology on Y^X called the ***topology of compact convergence*** (or sometimes the "topology of uniform convergence on compact sets").

Remark (Verification of Basis Contention). Clearly every element of Y^X belongs to some $B_C(f,\varepsilon)$. Notice that if $g \in B_C(f,\varepsilon)$, then if $\delta = \varepsilon - \sup\{d(f(x), g(x)) : x \in C\}$, $B_C(g,\delta) \subseteq B_C(f,\varepsilon)$. Now, if $B_C(f,\varepsilon) \cap B_K(g,\varepsilon') \neq \emptyset$, we may pick $h \in B_C(f,\varepsilon) \cap B_K(g,\varepsilon')$ and put $\delta = \min\{\varepsilon - \sup\{d(f(x), h(x)) : x \in C\}, \varepsilon' - \sup\{d(g(x), h(x)) : x \in K\}\}$. Since a finite union of compact sets is compact, it follows that $B_{C\cup K}(h,\delta) \subseteq B_C(f,\varepsilon) \cap B_K(g,\varepsilon')$. Hence, this is indeed an honest-to-God basis for a topology.

22.2.2 Uniform Convergence on Compact Subsets and the Topology of Compact Convergence.

It is easily verified that:

Theorem 22.89. *A sequence $f_n\colon X \to Y$ of functions converges to a function $f\colon X \to Y$ in the topology of compact convergence **iff** for each compact subspace C of X, the sequence $f_n|\,C$ converges uniformly to $f|\,C$.*

22.2.3 Compactly Generated Spaces

Definition (Compact Generated Space). A space X is said to be ***compactly generated*** if it satisfies the following condition: A set $A \subseteq X$ is open in X **iff** for each compact subspace C of X, $A \cap C$ is open in C. By complementation, this is equivalent to saying that: A set $B \subseteq X$ is closed in X **iff** for each compact subspace C of X, $B \cap C$ is closed in C.

Thus, the topology of a compactly generated space is totally determined by the topological properties in compact subspaces. More precisely, for any fixed $A \subseteq X$ we might "realize" A as a subspace of a compact subspace C by its intersection with C. With this understanding, a compactly generated space is a space whose topology is completely determined by the realization of sets as subspaces of compact sets.

22.2.3.1 Every First-Countable Space and Every Locally Compact Space is Compactly Generated.

It turns out *a lot* of interesting spaces are compactly generated!

Lemma 31. *If X is locally compact, or if X is first-countable, then X is compactly generated.*

Proof. Suppose X is locally compact. Clearly if A is open in X, then $A \cap C$ is open for any compact subset of X. Thus, suppose $A \cap C$ is open in C for every compact subspace C of X. We show A is open in X. For each $x \in A$, by local compactness, choose an open nbhd U of x contained in a compact subspace $C \supseteq U$ of X. Since $A \cap C$ is open in C by hypothesis and U is open in X, $A \cap U = (A \cap C) \cap U$ is an open nbhd of x that is open in U and, hence, also x. Thus, since $x \in A$ was arbitrary, we may clearly write A as a union of such open sets so that A is open in X.

Suppose now that X is first-countable. Suppose $B \cap C$ is closed in C for each compact subset C of X. We show $\overline{B} = B$. Towards this end, fix $x \in \overline{B}$; we shall show $x \in B$. Since X has a countable nbhd base at x, there is a sequence (x_n) of points of B converging to x. The subspace

$$C = \{x\} \cup \{x_n : x \in \mathbf{N}\}$$

is compact since any nbhd of x contains all but finitely many elements of C because X is first-countable and therefore has a countable nbhd basis at x. Hence, $B \cap C$ is by assumption closed in C. Since $B \cap C$ contains x_n for each $n \in \mathbf{N}$, and $B \cap C$ is closed in X, $x \in B \cap C$. Hence, $x \in B$, as desired. ■

22.2.3.2 A Function is Continuous on a Compactly Generated Space if its Restriction to Each Compact Subspace is Continuous.

Lemma 32. *Let X be a compactly generated space and $f\colon X \to Y$ be a function. If for each compact subspace C of X, the restricted function $f|\,C\colon C \to Y$ is continuous, then f is continuous.*

Proof. Let V be an open subset of Y; we show $f^{-1}[V]$ is open in X. For any subspace C of X,

$$f^{-1}[V] \cap C = (\,f|\,C)^{-1}\,[V].$$

Hence, if C is compact, this set is open in C because $f|\,C$ is continuous. Since X is compactly generated, it follows that $f^{-1}[V]$ is open in X. ■

22.2.4 $C(X, Y)$ is Closed in Y^X in the Topology of Compact Convergence.

Theorem 22.90. *Let X be a compactly generated space and (Y, d) a metric space. Then $C(X, Y)$ is closed in Y^X in the topology of compact convergence.*

Proof. Let $f \in Y^X$ be a limit point of $C(X, Y)$; we wish to show f is continuous so that $f \in C(X, Y)$. It suffices to show that $f|\,C$ is continuous for each compact subspace C of X by the preceding lemma, **Lemma 32**. For each $n \in \mathbf{N}$, consider the nbhd of f (and basis element) $B_C(f, 1/n) = \big\{g \in Y^X :\ \sup\{d(f(x), g(x)) : x \in C\} < \varepsilon\big\}$; it intersects $C(X, Y)$

by definition of f being a limit point of $C(X,Y)$, so we may choose $f_n \in (B_C(f,1/n) \cap C(X,Y))$ for each $n \in \mathbf{N}$. The sequence of function $f_n|\,C\colon C \to Y$ converges uniformly to the function $f|\,C$, so that by the uniform limit theorem, $f|\,C$ is continuous. Since this is true for each compact subspace of X and X is compactly generated, $f\colon X \to Y$ is continuous. ■

Corollary 26. *Let X be a compactly generated space and (Y,d) a metric space. If a sequence of continuous functions $f_n\colon X \to Y$ converges to f in the topology of compact convergence, the f is continuous.*

22.2.5 Comparison of The Three Important Topologies on $\boldsymbol{Y^X}$ When $\boldsymbol{(Y,d)}$ is a Metric Space.

When (Y,d) is a metric space, we now have three important topologies. The product topology (i.e., the topology of pointwise convergence), the uniform topology and the topology of compact convergence. We recall that the *uniform metric* $\overline{\rho}$ on Y^X inducing the *uniform topology* is defined on Y^X by $\overline{\rho}(\mathbf{y},\mathbf{y}') = \sup\{\min\{d(y_x,y'_x),1\}\colon x \in X\}$.

We have the following theorem whose proof is straightforward.

Theorem 22.91. *Let X be a topological space and (Y,d) a metric space. Then we have the following inclusions of topologies on the function space Y^X:*

$$\text{(uniform)} \supseteq \text{(compact convergence)} \supseteq \text{(pointwise convergence)}.$$

If X is compact, (uniform) = (compact convergence). If X is discrete, (compact convergence) = (pointwise convergence).

22.3 Compact-Open Topology

The definitions of the uniform topology and the topology of compact convergence made specific use of the metric d for the space Y whereas the topology of pointwise convergence did not; in fact, it is defined for any space Y. It is natural to ask whether either of these topologies can be extended to the case where Y is an arbitrary topological space. There is no satisfactory answer for the space Y^X of *all* functions mapping X into Y. But for the subspace $C(X,Y)$ of continuous functions, we can say something. It turns out the *compact-open topology* on $C(X,Y)$ coincides with the topology of compact convergence when Y is metric and is therefore *independent of the choice of metric on Y*. This makes the compact-open topology very natural.

Definition (Compact-Open Topology). Let X and Y be topological spaces. For each compact subspace C of X and each open subset U of Y, we define

$$S(C,U) \overset{\text{def}}{=} \{f \in C(X,Y)\colon f[C] \subseteq U\}$$

(that is, the set of continuous functions taking the compact subspace $C \subseteq X$ to the compact subspace $f[C]$ contained in the open subset $U \subseteq Y$). Then

$$\mathscr{S} = \{S(C,U)\colon C \subseteq X \text{ is compact and } U \subseteq Y \text{ is open}\}$$

comprises a subbasis for a topology on $C(X,Y)$ called the ***compact-open topology***. While this extends naturally to all of Y^X, it is only interesting on $C(X,Y)$.

Remark. For the topology of pointwise convergence, we defined $S(x,U) = \{f \in Y^X\colon f(x) \in U\}$. Obviously if we restrict this to $C(X,Y)$, $S(x,U)$ is an element of the above subbasis, hence, the compact-open topology is finer than the topology of pointwise convergence.

22.3.1 When $\boldsymbol{Y}$ is Metric, the Topology of Compact Convergence and the Compact-Open Topology Coincide on $\boldsymbol{C(X,Y)}$.

Theorem 22.92. *Let X be a space and (Y,d) a metric space. The compact-open topology and the topology of compact convergence coincide on $C(X,Y)$.*

Proof. If $A \subseteq Y$ and $\varepsilon > 0$, let $U(A,\varepsilon) = \bigcup_{x\in A} B(x,\varepsilon) = \{x \in X : d(x,A) < \varepsilon\}$ be the ε-nbhd of A. If A is compact and V an open nbhd of A, then $\exists \varepsilon > 0$ such that $A \subset U(A,\varepsilon) \subseteq V$. Indeed, as we have seen, the function $d(\cdot, X \setminus V) : X \to \mathbf{R}_{\geq 0}$ is continuous and thus attains its extrema on the compact set A and since $d(a, X\setminus V) \neq 0$ for any $a \in A$, $\inf\{d(a, X\setminus V) : a \in A\} > 0$ and therefore $\varepsilon = \inf\{d(a, X \setminus V) : a \in A\}$ furnishes a $\varepsilon > 0$ such that $U(A,\varepsilon) \subseteq V$.

We first prove that the topology of compact convergence is finer (larger) than the compact-open topology. Consider the subbasis element $S(C,U)$, and let $f \in S(C,U)$. Because f is continuous, $f[C]$ is compact and contained in U. Therefore, we can choose a ε-nbhd of $f[C]$ contained in U, namely $U(f[C],\varepsilon)$. But then for every function $g \in C(X,Y)$ such that $\sup\{d(f(x),g(x)) : x \in C\} < \varepsilon$, $g[C] \subseteq U(f[C],\varepsilon)$, so that, as desired, $B_C(f,\varepsilon) \subseteq S(C,U)$.

Conversely, we prove the compact-open topology is finer (larger) than the topology of compact convergence. Let $f \in C(X,Y)$. Any open set containing f in the topology of compact convergence contains some basis element $B_C(f,\varepsilon)$. We shall find a basis element for the compact-open topology that contains f and lies in $B_C(f,\varepsilon)$.

Each point of $x \in X$ has a nbhd V_x of x such that $f[\overline{V_x}]$ lies in an open set $U_x \subseteq Y$ with $\operatorname{diam} U_x < \varepsilon$—this is immediate by one of the equivalent definitions of continuity. Cover C by finitely many such sets V_x, say for $x = x_1, \ldots, x_n$. Let $C_x = \overline{V_x} \cap C$. Then each C_x is compact as C_x is closed in the subspace topology of C and, hence, is a compact subspace of C so that by transitivity of compactness, is a compact subspace of X. Then the basis element

$$\bigcap_{i=1}^{n} S(C_{x_i}, U_{x_i})$$

contains f and lies in $B_C(f,\varepsilon)$, as desired. ■

Corollary 27. *Let Y be a metric space. The compact convergence topology on $C(X,Y)$ is independent of the metric on Y. Therefore, if X is compact, the uniform topology on $C(X,Y)$ does not depend on the metric Y.*

22.3.2 Continuity of the Evaluation Map $e : X \times C(X,Y) \to Y$ When X is LCH and $C(X,Y)$ Given the Compact-Open Topology.

Definition (Evaluation Map). Given spaces X and Y, we define the ***evaluation map*** $e : X \times C(X,Y) \to Y$ by

$$e(x,f) = f(x).$$

Theorem 22.93. *Let X be an LCH space, Y a space and give $C(X,Y)$ the compact-open topology, then the evaluation map $e : X \times C(X,Y) \to Y$ is continuous.*

Proof. Fix $(x,f) \in X \times C(X,Y)$ and $V \subseteq Y$ an open nbhd of $e(x,f) = f(x)$. Since X is LCH, it has a basis of precompact sets. Thus, since f is continuous, there is a precompact open nbhd U of x such that $f[\overline{U}] \subseteq V$. Consider the basis element $U \times S(\overline{U}, V) \subseteq X \times C(X,Y)$. This is an open set containing (x,f). If $(x',f') \in U \times S(\overline{U},V)$, then $e(x',f') = f'(x') \in V$. ■

22.3.3 Continuity of Functions Induced From Maps from Products.

Definition (Induced Maps Into $C(X,Y)$). Let X, Y and Z be space. Given a continuous function $f : X \times Z \xrightarrow{cont} Y$, there is a corresponding function $F : Z \to C(X,Y)$ defined by

$$(F(z))(x) = f(x,z).$$

That is,

$$F(z) = f(\cdot, z).$$

Conversely, given $F : Z \to C(X,Y)$, this equation defines a corresponding function $f : X \times Y \to Z$. We say that F is the map of Z into $C(X,Y)$ that is ***induced*** by f.

Theorem 22.94. *Let X, Y and Z be spaces and give $C(X,Y)$ the compact-open topology. If the map $f : X \times Z \to Y$ is continuous, then the induced function $F : Z \to C(X,Y)$ is continuous. If, in addition, X is LCH, then the map $f : X \times Z \to Y$ is continuous **iff** the induced function $F : Z \to C(X,Y)$ is continuous.*

Proof. ($\Longleftarrow$) Suppose first that X is LCH and F is continuous. Then $f = e \circ (\mathrm{id}_X \times F)$ where

$$X \times Z \xrightarrow{\mathrm{id}_X \times F} X \times C(X,Y) \xrightarrow{e} Y$$

and hence is a composite of continuous functions. Thus, f is continuous.

($\Longrightarrow$) Now suppose f is continuous. We wish to show $F\colon Z \to C(X,Y)$ is continuous. Fix $z_0 \in Z$ and $S(C,U)$ a subbasis element containing $F(z_0) = f(\cdot, z_0) \in C(X,Y)$. To prove continuity of F at z_0, we shall show that there exists a nbhd W of z_0 in Z such that $F[W] \subseteq S(C,U)$ and hence as everything was sufficiently arbitrary, that F is continuous.

To say $F(z_0) \in S(C,U)$ means that $(F(z_0))(x) = f(x, z_0) \in U$ for all $x \in C$. That is, $f[C \times \{z_0\}] \subseteq U$. Continuity of f implies that $f^{-1}[U]$ is an open nbhd in $X \times Y$ of $C \times \{z_0\}$. Then $f^{-1}[U] \cap (C \times Z)$ is an open subspace of $C \times Z$ containing the slice $C \times \{z_0\}$. The ***first tube lemma*** implies that there is a nbhd W of z_0 in Z such that the entire tube $C \times W \subseteq f^{-1}[U]$. Then for $z \in W$ and $x \in C$, we have $f(x,z) \in U$. Hence, $F[W] \subseteq S(C,U)$ as desired. ■

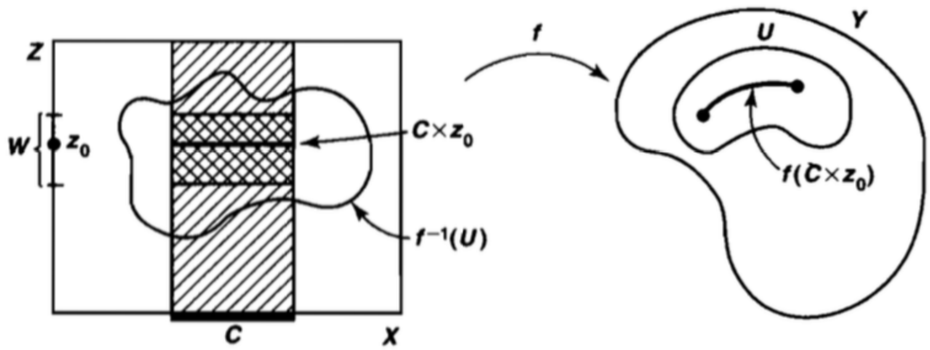

Fig. 22.1 Proof Idea

Example 29. Recall that if f and g are continuous maps of X into Y, then we say that f and g are *homotopic* if there is a continuous map $h\colon X \times [0,1] \to Y$ such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$ for each $x \in X$ and we call the map h a *homotopy* between f and g.

Roughly speaking, a homotopy is a "continuous one-parameter family" of maps from X to Y. More precisely, we note that a homotopy h gives rise to a map

$$H\colon [0,1] \to C(X,Y)$$

that assigns, to each parameter $t \in [0,1]$, the corresponding continuous map from X to Y. Assuming that X is LCH, the theorem we just proved shows that h is continuous **iff** H is continuous. This means (in the case X is LCH) that a homotopy h between f and g corresponds precisely to a *path* in the function space $C(X,Y)$ from the point f of $C(X,Y)$ to the point g.

22.4 Exercises

22.4.1 Exercises on the Topology of Compact Convergence and the Uniform Topology.

Exercise. *The set $\mathscr{B}(\mathbf{R},\mathbf{R})$ of bounded functions is closed in $\mathbf{R}^{\mathbf{R}}$ in the uniform topology but not in the topology of compact convergence.*

Exercise. *In which of the three topologies does the sequence of functions $f_n(x)\colon \mathbf{R} \to \mathbf{R}$ defined by $f_n(x) = x/n$ converge?*

Exercise. *Consider the sequence of function $f_n\colon (-1,1) \to \mathbf{R}$ defined by $f_n(x) = \sum_{k=1}^{n} kx^k$. Show that (f_n) does not converge in the uniform topology. Show that (f_n) converges in the topology of compact convergence; conclude that the limit function is continuous.*

22.4.2 The Compact-Open Topology on $C(X,Y)$ is Hausdorff/Regular if Y is Hausdorff/Regular.

Exercise. *If Y is Hausdorff, then $C(X,Y)$ is Hausdorff in the compact-open topology. If Y is regular, $C(X,Y)$ is regular in the compact-open topology.*

22.4.3 If Y LCH, Composition $C(X,Y) \times C(Y,Z) \to C(X,Z)$ is Continuous WRT the Compact-Open Topology.

Exercise. *If Y is a LCH space, then composition of maps*

$$C(X,Y) \times C(Y,Z) \to C(X,Z)$$

is continuous with respect to the compact-open topology (i.e., provided provided the compact-open topology is used throughout). [Hint: If $g \circ f \in S(C,U)$, find V such that $f[C] \subseteq V$, $g[\overline{V}] \subseteq U$.]

22.4.4 If $p\colon A \to B$ is a Quotient Map and X LCH, $id_X \times p\colon X \times A \to X \times B$ is a Quotient Map. II

Exercise. *If $p\colon A \to B$ is a quotient map, X is LCH, then $id_X \times p\colon X \times A \to X \times B$ is a quotient map.*

- ***(Step 1)*** *Let Y be the quotient space induced by $id_X \times p$ and $q\colon X \times A \to Y$ the quotient map. Show that there is a bijective continuous map $f\colon Y \to X \times B$ such that $f \circ q = id_X \times p$.*
- ***(Step 2)*** *Let $g = f^{-1}$; let $G\colon B \to C(X,Y)$ and $Q\colon A \to C(X,Y)$ be the maps induced by g and q, respectively. Show that $Q = G \circ p$.*
- ***(Step 3)*** *Show that Q is continuous; conclude that G is continuous so that g is continuous.*

22.4.5 Locally Compact and Second-Countable Implies σ-Compact.

Exercise. *A locally compact, second-countable space is σ-compact.*

22.4.6 If X is σ-Compact, Y (Complete) Metric, the Topology of Compact Convergence on Y^X is (Completely) Metrizable.

Exercise. *If X is σ-compact, (Y,d) a metric space, then the topology of compact convergence on Y^X is metrizable. If (Y,d) is complete, then the topology of compact convergence on Y^X is completely metrizable.*

Chapter 23
Ascoli's Theorem

23.1 Proof of Ascoli's Theorem

The entirety of this section is devoted the proof of *Ascoli's theorem.*

Theorem 23.95 (*Ascoli's Theorem*). *Let X be a space and (Y, d) a metric space. Give $C(X, Y)$ the topology of compact convergence and let $\mathscr{F} \subseteq C(X, Y)$.*

(a) *If $\mathscr{F}$ is equicontinuous under d and $\forall a \in X$, the set*

$$\mathscr{F}_a = \{f(a) \colon f \in \mathscr{F}\} \subseteq Y$$

is precompact, then $\mathscr{F}$ is contained in a compact subspace of $C(X, Y)$ (in fact, under these assumptions, $\mathrm{Cl}_{C(X,Y)}(\mathscr{F})$ is compact).

(b) *If, in addition, X is LCH, then $\mathscr{F}$ is equicontinuous under d and $\forall a \in X$, the set $\mathscr{F}_a = \{f(a) \colon f \in \mathscr{F}\} \subseteq Y$ is precompact **iff** $\mathscr{F}$ is contained in a compact subspace of $C(X, Y)$.*

*Proof (**a**).* Throughout, we give $Y^X = \prod_{x \in X} Y$ the product topology, which is the same as the topology of pointwise convergence. Then Y^X is Hausdorff as Y is a metric space and, hence, Hausdorff. The space $C(X, Y)$, which has the topology of compact convergence, cannot be realized as a subspace of Y^X (i.e., the inclusion map is not an embedding). Let $\mathscr{G} = \mathrm{Cl}_{Y^X}(\mathscr{F})$. We shall show **(1)** $\mathscr{G}$ is compact in Y^X, **(2)** that $\mathscr{G}$ is equicontinuous under d and **(3)** that the topology of compact convergence coincides with the topology of pointwise convergence on $\mathscr{G}$.

Step 1. We show $\mathscr{G}$ is a compact subspace of Y^X. Given $a \in X$, let $C_a = \mathrm{Cl}_Y(\mathscr{F}_a)$. By hypothesis, C_a is compact $\forall a \in X$. Thus,

$$\mathscr{F} \subseteq \prod_{a \in X} C_a,$$

since this product by definition consists of all function $f \colon X \to Y$ satisfying the condition $f(a) \in C_a$ for all $a \in X$. By Tychonoff's theorem, $\prod_{a \in X} C_a$ is compact in Y^X. Since $C_a \subseteq Y$ is compact and Y is a metric space, hence, Hausdorff, each C_a is closed so that $\prod_{a \in X} C_a$ is closed and compact. Since the closure of a product equals the product of the closures in the product topology and $\mathscr{F} = \prod_{a \in X} \mathscr{F}_a$, $\mathscr{G} = \prod_{a \in X} C_a$ and hence $\mathscr{G}$ is compact in Y^X.

Step 2. We show $\mathscr{G}$ is equicontinuous under d. Fix $x_0 \in X$ and $\varepsilon > 0$. Since $\mathscr{F}$ is equicontinuous under d, we may choose a nbhd U of x_0 such that $\forall f \in \mathscr{F}$, $\forall x \in U$, $d(f(x), f(x_0)) < \varepsilon/3$. We shall show that $\forall g \in \mathscr{G}$, $\forall x \in U$, $d(g(x), g(x_0)) < \varepsilon$, proving the contention. Fix $g \in \mathscr{G}$ and $x \in U$. Define $V_x = \left\{h \in Y^X \colon d(h(x), g(x)) < \varepsilon\right\} \cap \left\{h \in Y^X \colon d(h(x_0), g(x_0)) < \varepsilon\right\}$. Then V_x is an open nbhd of g in Y^X since it is the intersection of two basis elements of Y^X containing g. Because $g \in \mathrm{Cl}_{Y^X}(\mathscr{F})$, $\exists f \in V_x \cap \mathscr{F}$. By the triangle inequality, $d(g(x), g(x_0)) \leq d(g(x), f(x)) + d(f(x), g(x_0)) \leq d(g(x), f(x)) + d(f(x), f(x_0)) + d(f(x_0), g(x_0)) < \varepsilon$. Hence, as everything was sufficiently arbitrary, $\mathscr{G}$ is equicontinuous under d and hence $\mathscr{G} \subseteq C(X, Y)$.

Step 3. We show the product topology agrees with the compact-open topology on $\mathscr{G}$. In general, the compact convergence topology is finer (larger) than the product topology. We prove that the product topology is finer (larger) than the compact convergence topology for the subset $\mathscr{G} \subseteq C(X, Y)$. It will be easier to work with the compact convergence basis than the subbasis for the compact-convergence topology.

Let $g \in \mathscr{G}$ and let $B_C(g, \varepsilon)$ be a basis element for the compact convergence topology on Y^X containing g. We find a basis element B for the pointwise convergence topology on Y^X containing g such that

$$[B \cap \mathscr{G}] \subseteq [B_C(g, \varepsilon) \cap \mathscr{G}].$$

By equicontinuity of $\mathscr{G}$ and compactness of C we can cover C by finitely many open sets $U_1, \ldots, U_n$ of X containing points $x_1, \ldots, x_n$, respectively, such that for each i, $\forall g \in \mathscr{G}$, $\forall x \in U_i$,

$$d(g(x), g(x_i)) < \varepsilon/3.$$

Define

$$B = \{h \in Y^X : d(h(x_i), g(x_i)) < \varepsilon/3 \text{ for } i = 1, 2, \ldots, n\}$$

a basis element of Y containing g. We assert $B \cap \mathscr{G} \subseteq B_C(g, \varepsilon) \cap \mathscr{G}$.
Fix $h \in B \cap \mathscr{G}$, $\varepsilon > 0$ and $x \in C$. Then $x \in U_i$ for some i—, WLOG say $x \in U_1$. Then since $x \in U_i$ and $g, h \in \mathscr{G}$, $d(h(x), h(x_i)), d(g(x), g(x_i)) < \varepsilon/3$ and since $h \in B$, $d(h(x_i), g(x_i)) < \varepsilon$. Hence, it is clear by the triangle inequality that $d(h(x), g(x)) < \varepsilon$, as desired.

Step 4. Finally, since $\mathscr{F} \subseteq \mathscr{G} \subseteq C(X, Y)$ and $\mathscr{G}$ is compact in both the product topology and the topology of compact convergence by what we just showed in Step 3, it follows that $\mathscr{G}$ is a compact subspace of $C(X, Y)$ in the topology of compact convergence, proving **(a)**.

*Proof (**b**).* Let $\mathscr{H}$ be a compact subspace of $C(X, Y)$ containing $\mathscr{F}$. We show $\mathscr{H}$ is equicontinuous and $\mathscr{H}_a$ is compact for each $a \in X$. It follows that $\mathscr{F}$ is equicontinuous as $\mathscr{F} \subseteq \mathscr{H}$. Furthermore, since Y is a metric space, it is Hausdorff and, hence, each compact set $\mathscr{H}_a \subseteq Y$ is closed—this implies that the $\overline{\mathscr{F}_a} \subseteq \mathscr{H}_a$ so that $\overline{\mathscr{F}_a}$ is compact as a closed subspace of a compact space.

To show $\mathscr{H}_a$ is compact, consider the composite of the map

$$j_a : C(X, Y) \to X \times C(X, Y)$$

defined by $j(f) = (a, f)$ and the evaluation map

$$e : X \times C(X, Y) \to Y,$$

given by $e(x, f) = f(x)$. The map j_a is continuous as $\pi_i \circ j_a$ is continuous for $i = 1, 2$. The map e is continuous as a consequence of **Theorem 1.92** and **Theorem 1.94** applied in that order. Then $(e \circ j_a)[\mathscr{H}] = \mathscr{H}_a$ so that $\mathscr{H}_a$ is the continuous image of a compact set, hence, compact.

Now we show $\mathscr{H}$ is equicontinuous at $a \in X$ relative to the metric d. Since X is LCH, we may let $A \subseteq X$ be a compact set containing an open nbhd of a. Since X is locally compact, it is compactly generated. It clearly will be sufficient to show that

$$\mathscr{R} = \{f|A : \in \mathscr{H}\} \subseteq C(A, Y)$$

is equicontinuous at a. Give $C(A, Y)$ the compact convergence topology. We assert the restriction map $r : C(X, Y) \to C(A, Y)$ defined by $f \mapsto f|A$ is continuous. If this is the case, then r maps $\mathscr{H}$ onto $\mathscr{R}$, implying $\mathscr{R}$ is compact. Since A is compact, the topology of compact convergence and the uniform topology agree on $C(A, Y)$. Also, $\mathscr{R} \subseteq C(A, Y)$. Thus, $\mathscr{R}$ being compact in $C(A, Y)$ with uniform topology induced by the metric $\overline{\rho}$ induced by d, $\mathscr{R}$ is totally bounded wrt the uniform metric $\overline{\rho}$ on $C(A, Y)$. But then **Lemma 30** implies that $\mathscr{R}$ is equicontinuous under d. ■

23.2 Exercises

23.2.1 Which Are Pointwise Bounded and Which are Equicontinuous?

Exercise. *Which of the following subsets of $C(\mathbf{R}, \mathbf{R})$ are pointwise bounded? Which are equicontinuous?*

- **(a)** $f_n(x) = x + \sin(nx)$.
- **(b)** $g_n(x) = n + \sin x$.
- **(c)** $h_n(x) = |x|^{1/n}$.
- **(d)** $k_n(x) = n \sin(x/n)$.

23.2.2 An Equivalent Condition for $\mathscr{F} \subseteq C(X, \mathrm{K}^d)$ to be Precompact in the Topology of Compact Convergence when X LCH.

Exercise. *If X is LCH, then a subspace $\mathscr{F} \subseteq C(X, \mathbf{K}^d)$ is precompact **iff** $\mathscr{F}$ is pointwise bounded and equicontinuous under either of the standard metrics on $\mathbf{K}^d$.*

23.2.3 Arzela's Theorem

Exercise (Arzela's Theorem). *Let X be a σ-compact Hausdorff space. If a sequence of functions $f_n \colon X \to \mathbf{K}^d$ is pointwise bounded and equicontinuous, then (f_n) has a convergent subsequence in the topology of compact convergence whose limit function is continuous. [Hint: Show $C(X, \mathbf{K}^d)$ is first-countable.]*

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