## Contents

21 More on Metric Spaces and Function Spaces .......................................................... 3
  21.1 Completion of a Metric Space ............................................................................... 3
  21.2 Equicontinuity ................................................................................................. 4

22 Pointwise and Compact Convergence; The Compact-Open Topology ...................... 5
  22.1 Topology of Pointwise Convergence .................................................................. 5
  22.2 Topology of Compact Convergence and Compactly Generated Spaces ............... 5
    22.2.1 Topology of Compact Convergence ......................................................... 5
    22.2.2 Uniform Convergence on Compact Subsets and the Topology of Compact Convergence. ................................................................. 5
    22.2.3 Compactly Generated Spaces ................................................................... 6
      22.2.3.1 Every First-Countable Space and Every Locally Compact Space is Compactly Generated. .......................................................... 6
    22.2.4 \( C(X, Y) \) is Closed in \( Y^X \) in the Topology of Compact Convergence. .................................................................................. 6
    22.2.5 Comparison of The Three Important Topologies on \( Y^X \) When \((Y,d)\) is a Metric Space .......................................................... 7
  22.3 Compact-Open Topology .................................................................................... 7
    22.3.1 When \( Y \) is Metric, the Topology of Compact Convergence and the Compact-Open Topology Coincide on \( C(X,Y) \). ......................................................... 7
    22.3.2 Continuity of the Evaluation Map \( e: X \times C(X,Y) \to Y \) When \( X \) is LCH and \( C(X,Y) \) Given the Compact-Open Topology. .................................................................. 8
    22.3.3 Continuity of Functions Induced From Maps from Products. .......................... 8
  22.4 Exercises ......................................................................................................... 9
    22.4.1 Exercises on the Topology of Compact Convergence and the Uniform Topology ................................................................. 9
    22.4.2 The Compact-Open Topology on \( C(X,Y) \) is Hausdorff/Regular if \( Y \) is Hausdorff/Regular. ................................................................. 10
    22.4.3 If \( Y \) LCH, Composition \( C(X,Y) \times C(Y,Z) \to C(X,Z) \) is Continuous WRT the Compact-Open Topology. ......................................................... 10
    22.4.4 If \( p: A \to B \) is a Quotient Map and \( X \) LCH, \( id_X \times p: X \times A \to X \times B \) is a Quotient Map. II ................................................................. 10
    22.4.5 Locally Compact and Second-Countable Implies \( \sigma \)-Compact. ....................... 10
    22.4.6 If \( X \) is \( \sigma \)-Compact, \( Y \) (Complete) Metric, the Topology of Compact Convergence on \( Y^X \) is (Complety) Metrizable. ................................................................. 10

23 Ascoli’s Theorem .................................................................................................. 11
  23.1 Proof of Ascoli’s Theorem ................................................................................. 11
  23.2 Exercises ......................................................................................................... 12
    23.2.1 Which Are Pointwise Bounded and Which are Equicontinuous? ....................... 12
    23.2.2 An Equivalent Condition for \( \mathcal{F} \subseteq C(X,K^d) \) to be Precompact in the Topology of Compact Convergence when \( X \) LCH. ................................................................. 13
    23.2.3 Arzela’s Theorem ..................................................................................... 13

Index ......................................................................................................................... 15
Chapter 21
More on Metric Spaces and Function Spaces

21.1 Completion of a Metric Space

Recall that the space of bounded functions is complete with respect to the sup metric \( \| \cdot \|_\infty \) since a Cauchy sequence is bounded.

**Theorem 21.88.** Let \((X,d)\) be a metric space. Then there is an isometric embedding of \(X\) into a complete metric space.

**Proof.** Let \(\mathcal{B}(X,\mathbb{R})\) be the set of all bounded functions mapping \(X\) into \(\mathbb{R}\). Fix \(x_0 \in X\) and for each \(a \in X\) define \(\varphi_a(x) = d(x,a) - d(x,x_0)\). Then \(\forall a \in X\), \(\varphi_a\) is bounded since \(d(x,a) \leq d(x,b) + d(a,x_0)\) and \(d(x,x_0) \leq d(x,a) + d(a,x_0)\) by the triangle inequality so that \(|d(x,a) - d(x,x_0)| \leq d(a,x_0)\) and this is true for all \(x \in X\), for each \(a \in X\).

Define \(\Phi: X \to \mathcal{B}(X,\mathbb{R})\) by \(\Phi(a) = \varphi_a\). We claim \(\Phi\) is an isometric embedding of \((X,d)\) into \((\mathcal{B}(X,\mathbb{R}),\rho)\) where \(\rho\) is the sup norm. That is, we show that \(\forall a,b \in X\), \(\rho(\varphi_a,\varphi_b) = d(a,b)\). By definition, \(\rho(\varphi_a,\varphi_b) = \sup \{|\varphi_a(x) - \varphi_b(x)| : x \in X\} = \sup \{|d(x,a) - d(x,b)| : x \in X\}\) and hence, as we saw before, this means that \(\rho(\varphi_a,\varphi_b) \leq d(a,b)\). But this inequality cannot be strict, for when \(x = a\), \(|d(x,a) - d(x,b)| = d(a,b)\). ■

**Definition.** Let \(X\) be a metric space. If \(h: X \to Y\) is an isometric embedding of \(X\) into a complete metric space \(Y\), then the subspace \(h[X]\) of \(Y\) is a complete metric space called the **completion** of \(X\).

**Exercise (Uniqueness of the Completion).** Let \(h: X \to Y\) and \(h': X \to Y'\) be isometric embeddings of the metric space \((X,d)\) in the complete metric spaces \((Y,D)\) and \((Y',D')\), respectively. Then there is an isometry of \((h[X],D)\) with \((h'[X],D')\) that equals \(h'h^{-1}\) on the subspace \(h[X]\).
21.2 Equicontinuity

**Definition (Equicontinuity).** Let \( X \) be a space and \((Y,d)\) a metric space. Let \( F \subseteq C(X,Y) \). If \( x_0 \in X \), the set \( F \) of functions is said to be **equicontinuous at** \( x_0 \) if \( \forall \varepsilon > 0 \), there is a nbhd \( U \) of \( x_0 \) such that \( \forall x \in U, \forall f \in F, \)

\[ d(f(x), f(x_0)) < \varepsilon. \]

(So this is no misnomer, we’re really saying that the family of functions \( F \) are all “equally continuous” on \( X \).) If the set \( F \) is equicontinuous at \( x_0 \) for each \( x_0 \in X \), then \( F \) is simply said to be an **equicontinuous** family of functions.

**Definition (Pointwise Bounded).** If \( X \) is a space, \((Y,d)\) a metric space, a subset \( F \subseteq C(X,Y) \) is said to be **pointwise bounded** under \( d \) if \( \forall x \in X \), the subset \( F_a = \{ f(a) : f \in F \} \) of \( Y \) is bounded under \( d \).

**Lemma 30.** Let \( X \) be a space and \((Y,d)\) a metric space. If \( F \subseteq C(X,Y) \) is totally bounded under the uniform metric corresponding to \( d \) (i.e., \( \rho(f,g) = \sup \{ \min \{d(f(x),g(x)),1\} : x \in X \} \)) then \( F \) is equicontinuous under \( d \).

*Proof.* Suppose \( F \) is totally bounded. Then \( \forall \varepsilon > 0, F \) can be covered finitely many sets of diameter \( \leq \varepsilon \) wrt \( \rho \). Given \( 0 < \varepsilon < 1 \) and \( x_0 \in X \), we find a nbhd \( U \) of \( x_0 \) such that \( \forall x \in U, \forall f \in F, d(f(x), f(x_0)) < \varepsilon. \) Since everything was appropriately arbitrary, this will prove our contention.

Set \( \delta = \varepsilon/3\) and cover \( F \) by finitely many \( \delta \)-balls \( B(f_1, \delta), \ldots, B(f_n, \delta) \) in \( C(X,Y) \) in the \( \rho \) metric. Each \( f_i \) is continuous and there are only finitely many of them, hence, we may therefore choose a nbhd \( U \) of \( x_0 \) such that \( \forall x \in U, d(f_i(x), f_i(x_0)) < \delta \) for \( i = 1, 2, \ldots, n \).

Let \( f \in F \). Then \( f \) belongs to one of these balls, say \( B(f_i, \delta_i) \). Then \( \forall x \in U, \)

\[ \min \{d(f(x), f_i(x)), 1\} < \delta, \]

\[ d(f_i(x), f_i(x_0)) < \delta, \]

\[ \min \{d(f_i(x_0), f(x_0)), 1\} < \delta. \]

The first and third inequalities hold because \( \rho(f, f_i) < \delta \) and the second holds because \( x \in U \). Since \( \delta = \varepsilon/3 \) and \( \varepsilon < 1, \delta < 1 \), hence, the first and third reduce to only the metric \( d \). Then the triangle inequality implies that \( \forall x \in U, d(f(x), f(x_0)) < \varepsilon, \) as desired. \( \blacksquare \)
Chapter 22
Pointwise and Compact Convergence; The Compact-Open Topology

22.1 Topology of Pointwise Convergence

Definition (Topology of Pointwise Convergence; Point-Open Topology.). Let \( X \) be a set and \( Y \) a space. For each \( x \in X \) and open set \( U \) of \( Y \), let
\[
S(x,U) = \{ f \in Y^X : f(x) \in U \}
\]
(that is, the set of functions \( f : X \to Y \) such that \( f(x) \in U \)). The sets \( S(x,U) \) are a subbasis for a topology on \( Y^X \) called the topology of pointwise convergence (or the point-open topology).

Proposition 39. The topology of pointwise convergence is the product topology.

After all, putting \( J = X \), the set \( S(\alpha,U) \) of all functions \( x : J \to Y \) such that \( x(\alpha) \in U \) is just the subset \( \pi_{\alpha}^{-1}[U] \) of \( Y^J \), which is the standard subbasis element for the product topology.

22.2 Topology of Compact Convergence and Compactly Generated Spaces

22.2.1 Topology of Compact Convergence

Definition (Topology of Compact Convergence). Let \( X \) be a topological space and \( (Y,d) \) a metric space. For each \( \varepsilon > 0 \), \( f \in Y^X \) and compact subspace \( C \) of \( X \), let
\[
B_C(f,\varepsilon) = \{ g \in Y^X : \sup \{ d(f(x),g(x)) : x \in C \} < \varepsilon \}
\]
(that is, \( B_C(f,\varepsilon) \) is the subset of \( Y^X \) comprised of functions whose pointwise distance under \( d \) from \( f \) on the compact set \( C \) has supremum \( < \varepsilon \)). Then
\[
\mathcal{S} = \{ B_C(f,\varepsilon) : f \in Y^X, \varepsilon > 0, \text{ and } C \subseteq X \text{ is compact} \}
\]
comprises a basis for a topology on \( Y^X \) called the topology of compact convergence (or sometimes the “topology of uniform convergence on compact sets”).

Remark (Verification of Basis Contention). Clearly every element of \( Y^X \) belongs to some \( B_C(f,\varepsilon) \). Notice that if \( g \in B_C(f,\varepsilon) \), then if \( \delta = \varepsilon - \sup \{ d(f(x),g(x)) : x \in C \} \), \( B_C(g,\delta) \subseteq B_C(f,\varepsilon) \). Now, if \( B_C(f,\varepsilon) \cap B_K(g,\varepsilon') \neq \emptyset \), we may pick \( h \in B_C(f,\varepsilon) \cap B_K(g,\varepsilon') \) and put \( \delta = \min \{ \varepsilon - \sup \{ d(f(x),h(x)) : x \in C \}, \varepsilon' - \sup \{ d(g(x),h(x)) : x \in K \} \} \). Since a finite union of compact sets is compact, it follows that \( B_{C \cup K}(h,\delta) \subseteq B_C(f,\varepsilon) \cap B_K(g,\varepsilon') \). Hence, this is indeed an honest-to-God basis for a topology.

22.2.2 Uniform Convergence on Compact Subsets and the Topology of Compact Convergence.

It is easily verified that:
Theorem 22.89. A sequence $f_n : X \to Y$ of functions converges to a function $f : X \to Y$ in the topology of compact convergence if and only if for each compact subspace $C$ of $X$, the sequence $f_n|C$ converges uniformly to $f|C$.

22.2.3 Compactly Generated Spaces

Definition (Compact Generated Space). A space $X$ is said to be compactly generated if it satisfies the following condition: A set $A \subseteq X$ is open in $X$ if and only if for each compact subspace $C$ of $X$, $A \cap C$ is open in $C$. By complementation, this is equivalent to saying that: A set $B \subseteq X$ is closed in $X$ if and only if for each compact subset $C$ of $X$, $B \cap C$ is closed in $C$.

Thus, the topology of a compactly generated space is totally determined by the topological properties in compact subspaces. More precisely, for any fixed $A \subseteq X$ we might “realize” $A$ as a subspace of a compact subspace $C$ by its intersection with $C$. With this understanding, a compactly generated space is a space whose topology is completely determined by the realization of sets as subspaces of compact sets.

22.2.3.1 Every First-Countable Space and Every Locally Compact Space is Compactly Generated.

It turns out a lot of interesting spaces are compactly generated!

Lemma 31. If $X$ is locally compact, or if $X$ is first-countable, then $X$ is compactly generated.

Proof. Suppose $X$ is locally compact. Clearly if $A$ is open in $X$, then $A \cap C$ is open for any compact subset of $X$. Thus, suppose $A \cap C$ is open in $C$ for every compact subset $C$ of $X$. We show $A$ is open in $X$. For each $x \in A$, by local compactness, choose an open nbhd $U$ of $x$ contained in a compact subset $C$ of $X$. Since $A \cap C$ is open in $C$ by hypothesis and $U$ is open in $X$, $A \cap U = (A \cap C) \cap U$ is an open nbhd of $x$ that is open in $U$ and, hence, also in $X$. Thus, since $x \in A$ was arbitrary, we may clearly write $A$ as a union of such open sets so that $A$ is open in $X$.

Suppose now that $X$ is first-countable. Suppose $B \cap C$ is closed in $C$ for each compact subset $C$ of $X$. We show $\overline{B} = B$. Towards this end, fix $x \in \overline{B}$; we shall show $x \in B$. Since $X$ has a countable nbhd base at $x$, there is a sequence $(x_n)$ of points of $B$ converging to $x$. The subspace $C = \{x\} \cup \{x_n : x \in \mathbb{N}\}$ is compact since any nbhd of $x$ contains all but finitely many elements of $C$ because $X$ is first-countable and therefore has a countable nbhd basis at $x$. Hence, $B \cap C$ is by assumption closed in $C$. Since $B \cap C$ contains $x_n$ for each $n \in \mathbb{N}$, and $B \cap C$ is closed in $X$, $x \in B \cap C$. Hence, $x \in B$, as desired. ◼

22.2.3.2 A Function is Continuous on a Compactly Generated Space if its Restriction to Each Compact Subspace is Continuous.

Lemma 32. Let $X$ be a compactly generated space and $f : X \to Y$ be a function. If for each compact subspace $C$ of $X$, the restricted function $f|C : C \to Y$ is continuous, then $f$ is continuous.

Proof. Let $V$ be an open subset of $Y$; we show $f^{-1}[V]$ is open in $X$. For any subspace $C$ of $X$,

$$f^{-1}[V] \cap C = (f|C)^{-1}[V].$$

Hence, if $C$ is compact, this set is open in $C$ because $f|C$ is continuous. Since $X$ is compactly generated, it follows that $f^{-1}[V]$ is open in $X$. ◼

22.2.4 $C(X,Y)$ is Closed in $Y^X$ in the Topology of Compact Convergence.

Theorem 22.90. Let $X$ be a compactly generated space and $(Y,d)$ a metric space. Then $C(X,Y)$ is closed in $Y^X$ in the topology of compact convergence.

Proof. Let $f \in Y^X$ be a limit point of $C(X,Y)$; we wish to show $f$ is continuous so that $f \in C(X,Y)$. It suffices to show that $f|C$ is continuous for each compact subspace $C$ of $X$ by the preceding lemma, Lemma 32. For each $n \in \mathbb{N}$, consider the nbhd of $f$ (and basis element) $B_C(f, 1/n) = \{g \in Y^X : \sup \{d(f(x), g(x)) : x \in C\} < \varepsilon\}$; it intersects $C(X,Y)$
by definition of \( f \) being a limit point of \( C(X,Y) \), so we may choose \( f_n \in (BC(f, 1/n) \cap C(X,Y)) \) for each \( n \in \mathbb{N} \). The sequence of functions \( f_n|C : C \rightarrow Y \) converges uniformly to the function \( f|C \), so that by the uniform limit theorem, \( f|C \) is continuous. Since this is true for each compact subspace of \( X \) and \( X \) is compactly generated, \( f : X \rightarrow Y \) is continuous. \( \blacksquare \)

**Corollary 26.** Let \( X \) be a compactly generated space and \( (Y,d) \) a metric space. If a sequence of continuous functions \( f_n : X \rightarrow Y \) converges to \( f \) in the topology of compact convergence, the \( f \) is continuous.

### 22.2.5 Comparison of The Three Important Topologies on \( Y^X \) When \( (Y,d) \) is a Metric Space.

When \( (Y,d) \) is a metric space, we now have three important topologies. The product topology (i.e., the topology of pointwise convergence), the uniform topology and the topology of compact convergence. We recall that the uniform topology is defined on \( Y^X \) by \( p(y, y') = \sup \{ \min\{d(y_x, y'_x) : x \in X\} : x \in X \} \).

We have the following theorem whose proof is straightforward.

**Theorem 22.91.** Let \( X \) be a topological space and \( (Y,d) \) a metric space. Then we have the following inclusions of topologies on the function space \( Y^X \):

\[
(\text{uniform}) \supseteq (\text{compact convergence}) \supseteq (\text{pointwise convergence}).
\]

If \( X \) is compact, \( (\text{uniform}) = (\text{compact convergence}) \). If \( X \) is discrete, \( (\text{compact convergence}) = (\text{pointwise convergence}) \).

### 22.3 Compact-Open Topology

The definitions of the uniform topology and the topology of compact convergence made specific use of the metric \( d \) for the space \( Y \) whereas the topology of pointwise convergence did not; in fact, it is defined for any space \( Y \). It is natural to ask whether either of these topologies can be extended to the case where \( Y \) is an arbitrary topological space. There is no satisfactory answer for the space \( Y^X \) of all functions mapping \( X \) into \( Y \). But for the subspace \( C(X,Y) \) of continuous functions, we can say something. It turns out the compact-open topology on \( C(X,Y) \) coincides with the topology of compact convergence when \( Y \) is metric and is therefore independent of the choice of metric on \( Y \). This makes the compact-open topology very natural.

**Definition (Compact-Open Topology).** Let \( X \) and \( Y \) be topological spaces. For each compact subspace \( C \) of \( X \) and each open subset \( U \) of \( Y \), we define

\[
S(C,U) \overset{\text{def}}{=} \{ f \in C(X,Y) : f|C \subseteq U \}
\]

(that is, the set of continuous functions taking the compact subspace \( C \subseteq X \) to the compact subspace \( f[C] \) contained in the open subset \( U \subseteq Y \)). Then

\[
\mathcal{S} = \{ S(C,U) : C \subseteq X \text{ is compact and } U \subseteq Y \text{ is open} \}
\]

comprises a subbasis for a topology on \( C(X,Y) \) called the **compact-open topology**. While this extends naturally to all of \( Y^X \), it is only interesting on \( C(X,Y) \).

**Remark.** For the topology of pointwise convergence, we defined \( S(x,U) = \{ f \in Y^X : f(x) \in U \} \). Obviously if we restrict this to \( C(X,Y) \), \( S(x,U) \) is an element of the above subbasis, hence, the compact-open topology is finer than the topology of pointwise convergence.

### 22.3.1 When \( Y \) is Metric, the Topology of Compact Convergence and the Compact-Open Topology Coincide on \( C(X,Y) \).

**Theorem 22.92.** Let \( X \) be a space and \( (Y,d) \) a metric space. The compact-open topology and the topology of compact convergence coincide on \( C(X,Y) \).
Proof. If \( A \subseteq Y \) and \( \varepsilon > 0 \), let \( U(A, \varepsilon) = \bigcup_{x \in A} B(x, \varepsilon) = \{ x \in X : d(x, A) < \varepsilon \} \) be the \( \varepsilon \)-nbhd of \( A \). If \( A \) is compact and \( V \) an open \( \varepsilon \)-nbhd of \( A \), then \( \exists \varepsilon > 0 \) such that \( A \subseteq U(A, \varepsilon) \subseteq V \). Indeed, as we have seen, the function \( d(\cdot, X \setminus V) : X \to \mathbb{R}_{\geq 0} \) is continuous and thus attains its extrema on the compact set \( A \) and since \( d(a, X \setminus V) \neq 0 \) for any \( a \in A \), \( \inf\{d(a, X \setminus V) : a \in A\} > 0 \) and therefore \( \varepsilon = \inf\{d(a, X \setminus V) : a \in A\} \) furnishes a \( \varepsilon > 0 \) such that \( U(A, \varepsilon) \subseteq V \).

We first prove that the topology of compact convergence is finer (larger) than the compact-open topology. Consider the subbasis element \( S(C, U) \), and let \( f \in S(C, U) \). Because \( f \) is continuous, \( f[C] \) is compact and contained in \( U \). Therefore, we can choose a \( \varepsilon \)-nbhd of \( f[C] \) contained in \( U \), namely \( U(f[C], \varepsilon) \). But then for every function \( g \in C(X, Y) \) such that \( \sup\{d(f(x), g(x)) : x \in C\} < \varepsilon \), \( g[C] \subseteq U(f[C], \varepsilon) \), so that, as desired, \( B_C(f, \varepsilon) \subseteq S(C, U) \).

Conversely, we prove the compact-open topology is finer (larger) than the topology of compact convergence. Let \( f \in C(X, Y) \). Any open set containing \( f \) in the topology of compact convergence contains some basis element \( B_C(f, \varepsilon) \). We shall find a basis element for the compact-open topology that contains \( f \) and lies in \( B_C(f, \varepsilon) \).

Each point of \( x \in X \) has a \( \varepsilon \)-nbhd \( V_x \) of \( x \) such that \( f[V_x] \) lies in an open set \( U_x \subseteq Y \) with \( \text{diam} U_x < \varepsilon \)—this is immediate by one of the equivalent definitions of continuity. Cover \( C \) by finitely many such sets \( V_x \), say for \( x = x_1, \ldots, x_n \). Let \( C_x = V_x \cap C \). Then each \( C_x \) is compact as \( C_x \) is closed in the subspace topology of \( C \) and, hence, is a compact subspace of \( C \) so that by transitivity of compactness, is a compact subspace of \( X \). Then the basis element

\[
\bigcap_{i=1}^{n} S(C_{x_i}, U_{x_i})
\]

contains \( f \) and lies in \( B_C(f, \varepsilon) \), as desired.

**Corollary 27.** Let \( Y \) be a metric space. The compact convergence topology on \( C(X, Y) \) is independent of the metric on \( Y \). Therefore, if \( X \) is compact, the uniform topology on \( C(X, Y) \) does not depend on the metric \( Y \).

### 22.3.2 Continuity of the Evaluation Map \( e : X \times C(X, Y) \to Y \) When \( X \) is LCH and \( C(X, Y) \) Given the Compact-Open Topology.

**Definition (Evaluation Map).** Given spaces \( X \) and \( Y \), we define the **evaluation map** \( e : X \times C(X, Y) \to Y \) by

\[
e(x, f) = f(x).
\]

**Theorem 22.93.** Let \( X \) be an LCH space, \( Y \) a space and give \( C(X, Y) \) the compact-open topology, then the evaluation map \( e : X \times C(X, Y) \to Y \) is continuous.

**Proof.** Fix \( (x, f) ∈ X \times C(X, Y) \) and \( V ⊆ Y \) an open \( \varepsilon \)-nbhd of \( e(x, f) = f(x) \). Since \( X \) is LCH, it has a basis of precompact sets. Thus, since \( f \) is continuous, there is a precompact open \( \varepsilon \)-nbhd \( U \) of \( x \) such that \( f[U] \subseteq V \). Consider the basis element \( U \times S(U, V) \subseteq X \times C(X, Y) \). This is an open set containing \( (x, f) \). If \( (x', f') ∈ U \times S(U, V) \), then \( e(x', f') = f'(x') ∈ V \).

### 22.3.3 Continuity of Functions Induced From Maps from Products.

**Definition (Induced Maps Into \( C(X, Y) \)).** Let \( X, Y \) and \( Z \) be space. Given a continuous function \( f : X \times Z \to Y \), there is a corresponding function \( F : Z \to C(X, Y) \) defined by

\[
(F(z))(x) = f(x, z).
\]

That is,

\[
F(z) = f(\cdot, z).
\]

Conversely, given \( F : Z \to C(X, Y) \), this equation defines a corresponding function \( f : X \times Y \to Z \). We say that \( F \) is the map of \( Z \) into \( C(X, Y) \) that is **induced** by \( f \).

**Theorem 22.94.** Let \( X, Y \) and \( Z \) be spaces and give \( C(X, Y) \) the compact-open topology. If the map \( f : X \times Z \to Y \) is continuous, then the induced function \( F : Z \to C(X, Y) \) is continuous. Conversely, given \( F : Z \to C(X, Y) \), this equation defines a corresponding function \( f : X \times Y \to Z \). We say that \( F \) is the map of \( Z \) into \( C(X, Y) \) that is **induced** by \( f \).

**Proof.** \( (\Leftarrow) \) Suppose first that \( X \) is LCH and \( F \) is continuous. Then \( f = e \circ (\text{id}_X \times F) \) where
and hence is a composite of continuous functions. Thus, \( f \) is continuous.

( \implies \) Now suppose \( f \) is continuous. We wish to show \( F: Z \to C(X,Y) \) is continuous. Fix \( z_0 \in Z \) and \( S(C,U) \) a subbasis element containing \( F(z_0) = f(\cdot, z_0) \subseteq C(X,Y) \). To prove continuity of \( F \) at \( z_0 \), we shall show that there exists a nbhd \( W \) of \( z_0 \) in \( Z \) such that \( F[W] \subseteq S(C,U) \) and hence as everything was sufficiently arbitrary, that \( F \) is continuous.

To say \( F(z_0) \in S(C,U) \) means that \( (F(z_0))(x) = f(x, z_0) \in U \) for all \( x \in C \). That is, \( f[C \times \{z_0\}] \subseteq U \). Continuity of \( f \) implies that \( f^{-1}[U] \) is an open nbhd in \( X \times Y \) of \( C \times \{z_0\} \). Then \( f^{-1}[U] \cap (C \times Z) \) is an open subspace of \( C \times Z \) containing the slice \( C \times \{z_0\} \). The \textbf{first tube lemma} implies that there is a nbhd \( W \) of \( z_0 \) in \( Z \) such that the entire tube \( C \times W \subseteq f^{-1}[U] \). Then for \( z \in W \) and \( x \in C \), we have \( f(x, z) \in U \). Hence, \( F[W] \subseteq S(C,U) \) as desired. \( \blacksquare \)

\[ X \times Z \xrightarrow{id_X \times F} X \times C(X,Y) \xrightarrow{\pi_2} Y \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig22-1}
\caption{Proof Idea}
\end{figure}

\textbf{Example 29.} Recall that if \( f \) and \( g \) are continuous maps of \( X \) into \( Y \), then we say that \( f \) and \( g \) are \textit{homotopic} if there is a continuous map \( h: X \times [0,1] \to Y \) such that \( h(x,0) = f(x) \) and \( h(x,1) = g(x) \) for each \( x \in X \) and we call the map \( h \) a \textit{homotopy} between \( f \) and \( g \).

Roughly speaking, a homotopy is a “continuous one-parameter family” of maps from \( X \) to \( Y \). More precisely, we note that a homotopy \( h \) gives rise to a map

\[ H: [0,1] \to C(X,Y) \]

that assigns, to each parameter \( t \in [0,1] \), the corresponding continuous map from \( X \) to \( Y \). Assuming that \( X \) is LCH, the theorem we just proved shows that \( h \) is continuous \textit{iff} \( H \) is continuous. This means (in the case \( X \) is LCH) that a homotopy \( h \) between \( f \) and \( g \) corresponds precisely to a \textit{path} in the function space \( C(X,Y) \) from the point \( f \) of \( C(X,Y) \) to the point \( g \).

\section{22.4 Exercises}

\subsection{22.4.1 Exercises on the Topology of Compact Convergence and the Uniform Topology.}

\textbf{Exercise.} The set \( \mathcal{B}(\mathbb{R}, \mathbb{R}) \) of bounded functions is closed in \( \mathbb{R}^\mathbb{R} \) in the uniform topology but not in the topology of compact convergence.

\textbf{Exercise.} In which of the three topologies does the sequence of functions \( f_n(x): \mathbb{R} \to \mathbb{R} \) defined by \( f_n(x) = x/n \) converge?

\textbf{Exercise.} Consider the sequence of function \( f_n: (-1,1) \to \mathbb{R} \) defined by \( f_n(x) = \sum_{k=1}^{n} kx^k \). Show that \( f_n \) does not converge in the uniform topology. Show that \( (f_n) \) converges in the topology of compact convergence; conclude that the limit function is continuous.
22.4.2 The Compact-Open Topology on $C(X,Y)$ is Hausdorff/Regular if $Y$ is Hausdorff/Regular.

**Exercise.** If $Y$ is Hausdorff, then $C(X,Y)$ is Hausdorff in the compact-open topology. If $Y$ is regular, $C(X,Y)$ is regular in the compact-open topology.

22.4.3 If $Y$ LCH, Composition $C(X,Y) \times C(Y,Z) \to C(X,Z)$ is Continuous WRT the Compact-Open Topology.

**Exercise.** If $Y$ is a LCH space, then composition of maps

$$C(X,Y) \times C(Y,Z) \to C(X,Z)$$

is continuous with respect to the compact-open topology (i.e., provided the compact-open topology is used throughout). [Hint: If $g \circ f \in S(C,U)$, find $V$ such that $f[C] \subseteq V, g[V] \subseteq U$.]

22.4.4 If $p: A \to B$ is a Quotient Map and $X$ LCH, $\text{id}_X \times p: X \times A \to X \times B$ is a Quotient Map. II

**Exercise.** If $p: A \to B$ is a quotient map, $X$ is LCH, then $\text{id}_X \times p: X \times A \to X \times B$ is a quotient map.

(Step 1) Let $Y$ be the quotient space induced by $\text{id}_X \times p$ and $q: X \times A \to Y$ the quotient map. Show that there is a bijective continuous map $f: Y \to X \times B$ such that $f \circ q = \text{id}_X \times p$.

(Step 2) Let $g = f^{-1}$; let $G: B \to C(X,Y)$ and $Q: A \to C(X,Y)$ be the maps induced by $g$ and $q$, respectively. Show that $Q = G \circ p$.

(Step 3) Show that $Q$ is continuous; conclude that $G$ is continuous so that $g$ is continuous.

22.4.5 Locally Compact and Second-Countable Implies $\sigma$-Compact.

**Exercise.** A locally compact, second-countable space is $\sigma$-compact.

22.4.6 If $X$ is $\sigma$-Compact, $Y$ (Complete) Metric, the Topology of Compact Convergence on $Y^X$ is (Completely) Metrizable.

**Exercise.** If $X$ is $\sigma$-compact, $(Y,d)$ a metric space, then the topology of compact convergence on $Y^X$ is metrizable. If $(Y,d)$ is complete, then the topology of compact convergence on $Y^X$ is completely metrizable.
Theorem 23.95 (Ascoli’s Theorem). Let $X$ be a space and $(Y,d)$ a metric space. Give $C(X,Y)$ the topology of compact convergence and let $\mathcal{F} \subseteq C(X,Y)$.

(a) If $\mathcal{F}$ is equicontinuous under $d$ and $\forall a \in X$, the set

$$\mathcal{F}_a = \{f(a): f \in \mathcal{F}\} \subseteq Y$$

is precompact, then $\mathcal{F}$ is contained in a compact subspace of $C(X,Y)$ (in fact, under these assumptions, $\text{Cl}_{C(X,Y)}(\mathcal{F})$ is compact).

(b) If, in addition, $X$ is LCH, then $\mathcal{F}$ is equicontinuous under $d$ and $\forall a \in X$, the set $\mathcal{F}_a = \{f(a): f \in \mathcal{F}\} \subseteq Y$ is precompact if and only if $\mathcal{F}$ is contained in a compact subspace of $C(X,Y)$.

Proof (a). Throughout, we give $Y^X = \prod_{x \in X} Y$ the product topology, which is the same as the topology of pointwise convergence. Then $Y^X$ is Hausdorff as $Y$ is a metric space and, hence, Hausdorff. The space $C(X,Y)$, which has the topology of compact convergence, cannot be realized as a subspace of $Y^X$ (i.e., the inclusion map is not an embedding).

Let $\mathcal{G} = \text{Cl}_{Y^X}(\mathcal{F})$. We shall show (1) $\mathcal{G}$ is compact in $Y^X$, (2) that $\mathcal{G}$ is equicontinuous under $d$ and (3) that the topology of compact convergence coincides with the topology of pointwise convergence on $\mathcal{G}$.

**Step 1.** We show $\mathcal{G}$ is a compact subspace of $Y^X$. Given $a \in X$, let $C_a = \text{Cl}_Y(\mathcal{F}_a)$. By hypothesis, $C_a$ is compact $\forall a \in X$. Thus,

$$\mathcal{F} \subseteq \prod_{a \in X} C_a,$$

since this product by definition consists of all function $f: X \to Y$ satisfying the condition $f(a) \in C_a$ for all $a \in X$. By Tychonoff’s theorem, $\prod_{a \in X} C_a$ is compact in $Y^X$. Since $\prod_{a \in X} C_a \subseteq Y$ is compact and $Y$ is a metric space, hence, Hausdorff, each $C_a$ is closed so that $\prod_{a \in X} C_a$ is closed and compact. Since the closure of a product equals the product of the closures in the product topology and $\mathcal{F} = \prod_{a \in X} \mathcal{F}_a$, $\mathcal{G} = \prod_{a \in X} C_a$ and hence $\mathcal{G}$ is compact in $Y^X$.

**Step 2.** We show $\mathcal{G}$ is equicontinuous under $d$. Fix $x_0 \in X$ and $\varepsilon > 0$. Since $\mathcal{F}$ is equicontinuous under $d$, we may choose a nbhd $U$ of $x_0$ such that $\forall f \in \mathcal{F}, \forall x \in U, d(f(x_0),f(x_0)) < \varepsilon/3$. We shall show that $\forall g \in \mathcal{G}, \forall x \in U, d(g(x_0),g(x)) < \varepsilon$, proving the contention. Fix $g \in \mathcal{G}$ and $x \in U$. Define $V_x = \{h \in Y^X: d(h(x_0),g(x)) < \varepsilon\} \cap \{h \in Y^X: d(h(x_0),g(x_0)) < \varepsilon\}$. Then $V_x$ is an open nbhd of $g$ in $Y^X$ since it is the intersection of two basis elements of $Y^X$ containing $g$. Because $g \in \text{Cl}_{Y^X}(\mathcal{F})$, $\exists f \in V_x \cap \mathcal{F}$. By the triangle inequality, $d(g(x_0),g(x)) \leq d(g(x),f(x)) + d(f(x),g(x_0)) \leq d(g(x),f(x)) + d(f(x),f(x_0)) + d(f(x_0),g(x_0)) < \varepsilon$. Hence, as everything was sufficiently arbitrary, $\mathcal{G}$ is equicontinuous under $d$ and hence $\mathcal{G} \subseteq C(X,Y)$.

**Step 3.** We show the product topology agrees with the compact-open topology on $\mathcal{G}$. In general, the compact convergence topology is finer (larger) than the product topology. We prove that the product topology is finer (larger) than the compact convergence topology for the subset $\mathcal{G} \subseteq C(X,Y)$. It will be easier to work with the compact convergence basis than the subbasis for the compact-convergence topology.

Let $g \in \mathcal{G}$ and let $B_C(g,\varepsilon)$ be a basis element for the compact convergence topology on $Y^X$ containing $g$. We find a basis element $B$ for the pointwise convergence topology on $Y^X$ containing $g$ such that

$$[B \cap \mathcal{G}] \subseteq [B_C(g,\varepsilon) \cap \mathcal{G}].$$
By equicontinuity of \( G \) and compactness of \( C \) we can cover \( C \) by finitely many open sets \( U_1, \ldots, U_n \) of \( X \) containing points \( x_1, \ldots, x_n \), respectively, such that for each \( i \), \( \forall g \in G, \forall x \in U_i, \)

\[
d(g(x), g(x_i)) < \varepsilon/3.
\]

Define

\[
B = \{ h \in Y^X : d(h(x_i), g(x_i)) < \varepsilon/3 \text{ for } i = 1, 2, \ldots, n \}
\]

a basis element of \( Y \) containing \( g \). We assert \( B \cap G \subseteq B_{C}(g, \varepsilon) \cap G \).

Fix \( h \in B \cap G, \varepsilon > 0 \) and \( x \in C \). Then \( x \in U_i \) for some \( i \)—, WLOG say \( x \in U_1 \). Then since \( x \in U_i \) and \( g, h \in G, d(h(x), h(x_i)), d(g(x), g(x_i)) < \varepsilon/3 \) and since \( h \in B, d(h(x_i), g(x_i)) < \varepsilon \). Hence, it is clear by the triangle inequality that \( d(h(x), g(x)) < \varepsilon \), as desired.

**Step 4.** Finally, since \( F \subseteq G \subseteq C(X, Y) \) and \( G \) is compact in both the product topology and the topology of compact convergence by what we just showed in Step 3, it follows that \( G \) is a compact subspace of \( C(X, Y) \) in the topology of compact convergence, proving (a).

**Proof (b).** Let \( H \) be a compact subspace of \( C(X, Y) \) containing \( F \). We show \( H \) is equicontinuous and \( H_a \) is compact for each \( a \in X \). It follows that \( F \) is equicontinuous as \( F \subseteq H \). Furthermore, since \( Y \) is a metric space, it is Hausdorff and, hence, each compact set \( H_a \subseteq Y \) is closed—this implies that the \( F_a \subseteq H_a \) so that \( F_a \) is compact as a closed subspace of a compact space.

To show \( H_a \) is compact, consider the composite of the map

\[
j_a : C(X, Y) \to X \times C(X, Y)
\]

defined by \( j(f) = (a, f) \) and the evaluation map

\[
e : X \times C(X, Y) \to Y,
\]

given by \( e(x, f) = f(x) \). The map \( j_a \) is continuous as \( \pi_i \circ j_a \) is continuous for \( i = 1, 2 \). The map \( e \) is continuous as a consequence of **Theorem 1.92** and **Theorem 1.94** applied in that order. Then \( (e \circ j_a)[H] = H_a \) so that \( H_a \) is the continuous image of a compact set, hence, compact.

Now we show \( H \) is equicontinuous at \( a \in X \) relative to the metric \( d \). Since \( X \) is LCH, we may let \( A \subseteq X \) be a compact set containing an open nbhd of \( a \). Since \( X \) is locally compact, it is compactly generated. It clearly will be sufficient to show that

\[
\mathcal{B} = \{ f|A : \in H \} \subseteq C(A, Y)
\]
is equicontinuous at \( a \). Give \( C(A, Y) \) the compact convergence topology. We assert the restriction map \( r : C(X, Y) \to C(A, Y) \) defined by \( f \mapsto f|A \) is continuous. If this is the case, then \( r \) maps \( H \) onto \( \mathcal{B} \), implying \( \mathcal{B} \) is compact. Since \( A \) is compact, the topology of compact convergence and the uniform topology agree on \( C(A, Y) \). Also, \( \mathcal{B} \subseteq C(A, Y) \). Thus, \( \mathcal{B} \) being compact in \( C(A, Y) \) with uniform topology induced by the metric \( \mathcal{p} \) induced by \( d \), \( \mathcal{B} \) is totally bounded wrt the uniform metric \( \mathcal{p} \) on \( C(A, Y) \). But then **Lemma 30** implies that \( \mathcal{B} \) is equicontinuous under \( d \). \( \blacksquare \)

### 23.2 Exercises

#### 23.2.1 Which Are Pointwise Bounded and Which are Equicontinuous?

**Exercise.** Which of the following subsets of \( C(\mathbb{R}, \mathbb{R}) \) are pointwise bounded? Which are equicontinuous?

(a) \( f_n(x) = x + \sin(nx) \).
(b) \( g_n(x) = n + \sin x \).
(c) \( h_n(x) = |x|^{1/n} \).
(d) \( k_n(x) = n \sin(x/n) \).
23.2.2 An Equivalent Condition for \( \mathcal{F} \subseteq C(X, K^d) \) to be Precompact in the Topology of Compact Convergence when \( X \) LCH.

**Exercise.** If \( X \) is LCH, then a subspace \( \mathcal{F} \subseteq C(X, K^d) \) is precompact iff \( \mathcal{F} \) is pointwise bounded and equicontinuous under either of the standard metrics on \( K^d \).

23.2.3 Arzela’s Theorem

**Exercise (Arzela’s Theorem).** Let \( X \) be a \( \sigma \)-compact Hausdorff space. If a sequence of functions \( f_n : X \to K^d \) is pointwise bounded and equicontinuous, then \( (f_n) \) has a convergent subsequence in the topology of compact convergence whose limit function is continuous. [Hint: Show \( C(X, K^d) \) is first-countable.]
Index

Arzela’s Theorem, 13
Ascoli’s Theorem, 11

compact-open topology, 7
compactly generated, 6
completion, 3

equicontinuous, 4
equicontinuous at $x_0$, 4
evaluation map, 8

First Tube Lemma, 9

homotopic, 9
homotopy, 9

induced, 8

point-open topology, 5
pointwise bounded, 4

quotient map, 10

topology of compact convergence, 5
topology of pointwise convergence, 5
totally bounded, 4