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Chapter 11
Countability and Separation Axioms: Hausdorff, Regular and Normal Spaces

11.1 Some Counter Examples Regarding First and Second Countable Spaces

11.1.1 The Product of Two Lindelöf Spaces Need Not be Lindelöf

Example 18. The space \( \mathbb{R}_\ell \) is \( \mathbb{R} \) with the lower limit topology, that is, the topology generated by sets of the form \([a, b)\). \( \mathbb{R}_\ell \) is Lindelöf. However, the Sorgenfrey plane \( \mathbb{R}_\ell^2 \) is not.

The space \( \mathbb{R}_\ell^2 \) has as a basis all sets of the form \([a, b) \times [c, d)\). Consider the subspace \( L = \{(x, -x) : x \in \mathbb{R}_\ell \} \). It is easy to check that \( L \) is closed in \( \mathbb{R}_\ell^2 \). Let us cover \( \mathbb{R}_\ell^2 \) by the open set \( \mathbb{R}_\ell^2 \setminus L \) and by all basis elements of the form \([a, b) \times [c, d)\). Each of these open sets intersects \( L \) in at most one point. Since \( L \) is uncountable, no countable subcollection covers \( \mathbb{R}_\ell^2 \). See Figure 5.

![Fig. 11.1 Image of Set](image_url)

Indeed, if \((x, y)\) is a limit point of \( L \), then every nbhd of \((x, y)\) intersects \( L \). Hence, every basis element \([a, b) \times [c, d)\) for which \((x, y) \in [a, b) \times [c, d)\) intersects \( L \setminus \{(x, y)\} \), but then if \( y < -x \), the basis element \([x, x+1) \times [y, -x)\) does not intersect \( L \) but contains \((x, y)\) and similarly if \( y > x \). Thus, a limit point of \( L \) in \( \mathbb{R}_\ell^2 \) cannot lie outside of \( L \). On the other hand, for each \( x \in \mathbb{R}_\ell \), \( \{(x, -x)\} = L \cap ([x, x+1) \times [-x, -x+1)) \) so that each point of \( L \) is isolated. Hence, the set of limit points of \( L \) is empty and therefore \( L \) contains all of its limit points. Thus, \( L \) is closed, every point of the space \( L \) is isolated points and, hence, \( L \) has the discrete topology as a subspace of \( \mathbb{R}_\ell \).
11.1.2 A Subspace of a Lindelöf Space Need Not be Lindelöf.

Example 19. The ordered square \( I^2_o \) is compact; therefore it is trivially Lindelöf. However, the subspace \( A = I \times (0,1) \) is not Lindelöf. For \( A \) is the union of the disjoint sets \( U_x = \{x\} \times (0,1) \), each of which is open in \( A \). This collection of sets is uncountable, and no proper subcollection covers \( A \).

11.2 The Separation Axioms

There are three principle separation axioms: Hausdorff, Regular and Normal.

Definition (Regular and Normal Spaces). A space \( X \) is said to be regular or a regular space if for a point \( x \) and closed set \( B \) not containing \( x \), there exist disjoint open sets containing \( x \) and \( B \) respectively. A space \( X \) is said to be normal if for each pair of disjoint closed sets \( A \) and \( B \), there exist disjoint open sets containing \( A \) and \( B \), respectively.

Fig. 11.2 The Three Principle Separation Axioms

If \( X \) is, in addition, \( T_1 \), then it is clear that (normal \( \implies \) regular \( \implies \) Hausdorff). (We need to include the condition that one-point sets be closed in order for this to be the case.) The idea is that in a normal space, we may “fatten up” disjoint closed sets into disjoint open sets, whereas in a regular space we may separate closed sets from points not in it by disjoint open sets.

Definition (\( T_3 \) and \( T_4 \) Spaces). A \( T_1 \) space \( X \) that is regular is said to satisfy the \( T_3 \) axiom and be a \( T_3 \) space. A \( T_1 \) space \( X \) that is normal is said to satisfy the \( T_4 \) axiom and be a \( T_4 \) space.

Remark (Warning). Munkres takes as part of his definition of regularity and normality the assumption that the space is also \( T_1 \). We need not do this.

We can rephrase our definitions as follows:

Definition (Regular and Normal Spaces). Let \( X \) be a topological space.

(a) \( X \) is said to be regular or a regular space if each point and closed set disjoint from it can be separated by open sets in \( X \).

(b) \( X \) is said to be normal or a normal space if each disjoint pair of closed sets can be separated by open sets in \( X \).

11.2.1 Useful Equivalent Formulations of Normal and Regular Spaces: Having “Enough” Neighborhoods.

Lemma 16. Let \( X \) be a topological space.

(1) \( X \) is regular iff for each \( x \in X \), for all nbhds \( U \) of \( x \), there exists a nbhd \( V \) of \( x \) such that:

\[ \{x\} \subseteq V \subseteq \overline{V} \subseteq U. \]
11.3 $R_\ell$ is Normal but the Sorgenfrey Plane $R^2_\ell$ is Not Normal.

This example shows that a regular space need not be normal and that the product of normal spaces need not be normal.

Example 20 ($R_\ell$ is Normal but the Sorgenfrey Plane $R^2_\ell$ is Not Normal). Clearly $R_\ell$ is $T_1$, since the topology of $R_\ell$ is finer than that of $R$. To check normality, suppose that $A$ and $B$ are disjoint closed sets in $R_\ell$. For each $a \in A$, choose a basis element $[a, x_a)$ not intersecting $B$; and for each point $b \in B$, choose a basis element $(b, x_b]$ not intersecting $A$. The open sets $U = \bigcup_{a \in A} [a, x_a)$ and $V = \bigcup_{b \in B} (b, x_b]$ are disjoint open nbhds of $A$ and $B$, respectively.

As we know, since normality implies regularity, $R_\ell$ is regular and hence so too is $R^2_\ell$. We claim, however, that it is not normal. Suppose for the sake of a contradiction that $R^2_\ell$ is normal. Let $L = \{(x, -x) : x \in R_\ell\}$. Then $L$ is closed in $R^2_\ell$ and $L$ has the discrete topology—that is, all singletons are open so that every subset of $L$ is both open and closed—we saw this in the comment following an example above. Hence, every subset $A$ of $L$ is closed in $R^2_\ell$ since $L$ itself is. Thus, $L \setminus A$ is also closed in $R^2_\ell$; this means that for every nonempty proper subset $A$ of $L$, one can find disjoint open sets $U_A$ and $V_A$ containing $A$ and $L \setminus A$, respectively, from our assumption that $R^2_\ell$ was normal.

Let $D$ denote the set of points of $R^2_\ell$ having rational coordinates; it is dense in $R^2_\ell$. We define a map $\theta : \mathcal{P}(L) \to \mathcal{P}(D)$ by setting

$$\theta(A) = D \cap U_A \quad \text{if} \quad \emptyset \not\subseteq A \not\subseteq L,$$

$$\theta(\emptyset) = \emptyset,$$

$$\theta(L) = D.$$  

We show that $\theta : \mathcal{P}(L) \to \mathcal{P}(D)$ is injective.

Let $A$ be a proper nonempty subset of $L$. Then $\theta(A) = D \cap U_A$ is neither empty (since $U_A$ is open and $D$ dense in $R^2_\ell$) nor all of $D$ (since $D \cap V_A \neq \emptyset$). It remains to show that if $B$ is another proper nonempty subset of $L$, then $\theta(A) \neq \theta(B)$. One of the sets $A, B$ contains a point not in the other, say $x \in A, x \not\in B$. Then $x \in L \setminus B$, so that $x \in U_a \cap V_B$ (where $V_B$
is an open nbhd of \( L \setminus B \); since the latter set is open and nonempty, it must contain points of \( D \). These points belong to \( U_A \) and not to \( U_B \); therefore \( D \cap U_A \neq D \cap U_B \), as desired, so that \( \theta \) is injective.

Now we show that there exists an injective map \( \phi: \mathcal{P}(D) \to L \), yielding a contradiction since the composite \( \phi \circ \theta: \mathcal{P}(L) \to L \) is injective, which is impossible for any set (cf. Munkres Theorem 7.8). Because \( D \) is countably infinite and \( L \) has the cardinality of \( \mathbb{R} \), it suffices to define an injective map \( \psi: \mathcal{P}(N) \to \mathbb{R} \). But this is easy. For each \( S \subseteq N \), we let \( \psi \) assign \( S \) to the decimal \( .a_1a_2\ldots \), where \( a_i = 0 \) if \( i \in S \) and \( a_i = 1 \) if \( i \notin S \). That is,

\[
\psi(S) = \sum_{i=1}^{\infty} a_i/10^i.
\]

This yields the desired contradiction.

We showed only that there must exist some proper nonempty subset \( A \) of \( L \) such that the sets \( A \) and \( B = L \setminus A \) are not contained in disjoint open sets of \( \mathbb{R}^2 \). But we did not actually find such a set \( A \). In fact, if we put \( A = D \), the set of points of \( L \) having rational coordinates, then \( A \) is such a set. The proof of this is not easy, however.

### 11.4 Topological Groups

A topological group is a group \( (G, \cdot) \) endowed with a topology such that multiplication \( \cdot: G \times G \to G \) is continuous and the operation of inverse \( x \mapsto x^{-1} \) is continuous from \( G \) to \( G \). (N.B., Munkres assumes topological groups to be \( T_1 \).

**Exercise.** A group \( H \) equipped with a topology is topological group \( \iff \) the map \( (x, y) \mapsto xy^{-1} \) is continuous from \( H \times H \) to \( H \).

**Exercise.** If \( H \) is a subgroup of a topological group \( G \), then \( H \) and \( \overline{H} \) are topological groups.

**Exercise.** Let \( G \) be a topological group. For any \( a \in G \), the map \( x \mapsto ax \) and its inverse \( y \mapsto a^{-1}y \) are continuous—hence, \( \forall a \in G \), \( x \mapsto ax \) is a homeomorphism.

**Proof.** For \( a \in G \), the map \( x \mapsto ax \) is continuous because it is the composition of the the maps \( x \mapsto (a, x) \) from \( G \to \{a\} \times G \) and \( (w, x) \mapsto wx \) from \( \{w\} \times G \to G \). The first map is continuous as its component maps are \( x \mapsto a \) and \( x \mapsto x \) the first of which is constant and, hence, continuous from \( G \to G \) and the second of which is the identity and, hence, continuous from \( G \to G \). The second map is continuous because multiplication is continuous from \( G \times G \to G \). The reasoning is analogous for \( y \mapsto a^{-1}y \).

**Exercise.** A topological group \( G \) is \( T_1 \) \( \iff \) \( \{e\} \) is closed in \( G \).

**Proof.** (\( \Rightarrow \)) This is essentially a definition. (\( \Leftarrow \)) By the preceding exercise, there exists a homeomorphism mapping \( 1 \) to any element \( a \in G \). Hence, \( \forall a \in G \), \( \{a\} \) is the image of the closed set \( \{e\} \) under a homeomorphism of \( G \) so that \( \forall a \in G \), \( \{a\} \) is closed. Hence, \( G \) is \( T_1 \).

**Exercise.** Let \( G \) be a topological group and \( H \) a subgroup of \( G \). If \( x \in G \), define \( xH = \{xh : h \in H\} \); this set is called a left coset of \( H \) in \( G \) (a right coset being defined in the obvious way). Let \( G/H \) denote the collection of left cosets of \( H \) in \( G \); it is a partition of \( G \). Give \( G/H \) the quotient topology.

(a) \( \forall a \in G \), the map \( x \mapsto ax \) induces a homeomorphism of \( G/H \) carrying \( xH \) to \( (ax)H \).

(b) If \( H \) is a closed subgroup in the topology of \( G \), then \( G/H \) is \( T_1 \).

(c) The quotient map \( \pi: G \to G/H \) is an open map.

(d) If \( H \) is a normal subgroup of \( G \), then \( G/H \) is a topological group.

(e) If \( H \) is closed normal subgroup in the topology of \( G \), then \( G/H \) is \( T_1 \).

**Exercise.** The integers \( \mathbb{Z} \) are a normal subgroup of \( (\mathbb{R}, +) \) and the quotient \( \mathbb{R}/\mathbb{Z} \) is a familiar topological group: \( S^1 \). Prove this.

**Exercise.** Let \( G \) be a topological group and \( A, U \subseteq G \) where \( U \) is open. Then \( AU \) and \( UA \) are open in \( G \). If, in addition, \( U \) is an open nbhd of \( e \), then \( A \subseteq UA \) and \( A \subseteq AU \).

**Proof.** Let \( A, U \) be as in the statement. Then \( AU = \bigcup_{a \in A} aU \) which is a union of open sets since the map \( x \mapsto ax \) is a homeomorphism for any \( a \in A \), so that the image of \( U \) under any such map is \( aU \) and hence open. Thus, \( AU \) is open. The proof for \( UA \) is analogous with \( UA = \bigcup_{a \in A} Ua \). If, in addition, \( U \) is an open nbhd of \( e \), then obviously \( A \subseteq UA \) and \( A \subseteq AU \).
**Exercise.** Let $G$ be a topological group. To say that a set $A \subseteq G$ is **symmetric** means that $A^{-1} = A$. If $U$ is symmetric, then $\forall x, y \in G$, $y \in xU$ iff $x \in yG$.

Proof. Easy. ■

**Exercise.** Let $U$ be a set containing $e$. Then $U \cap U^{-1}$ is symmetric.

Proof. $U \cap U^{-1} \neq \emptyset$ as it contains $e$. For every $x \in U \cap U^{-1}$, $x \in U$ and $x \in U^{-1}$. Thus, $x \in U^{-1}$, so we must have $x^{-1} \in U$. But then $x, x^{-1} \in U$ so $x, x^{-1} \in U^{-1}$. Hence, $U \cap U^{-1}$ is symmetric. ■

**Exercise.** Let $G$ be a topological group. Every nbhd $U$ of $e$ contains a symmetric nbhd of $e$—namely, $U \cap U^{-1}$.

Proof. Let $A \subseteq U$ be open such that $e \in A$. Since $x \mapsto x^{-1}$ is a homeomorphism of $G$ with itself, $A \cap A^{-1}$ is open in $G$ and contains $e$. This is symmetric by the previous exercise and clearly $A \cap A^{-1} \subseteq U \cap U^{-1}$. Hence, $U \cap U^{-1}$ is a symmetric nbhd of $e$. ■

**Exercise.** Let $G$ be a topological group.

(a) For each $n \in \mathbb{N}$, every nbhd $U$ of $e$ in $G$, contains a symmetric neighborhood $V$ of $e$ such that $VV \cdots V \subseteq U$.

(b) Every nbhd $U$ of $e$ in $G$ contains a nbhd $V$ such that $VV^{-1} \subseteq U$. In particular, $V$ can be chosen to be symmetric.

(c) Every $T_1$ topological group is Hausdorff. In fact, for any $T_1$ topological group and for any two distinct points $x, y$, there exists a nbhd $V$ of $e$ such that $Vx \capVy = \emptyset$.

(d) Topological groups are regular. In particular, for any point $x$ and closed set $C$, there exists a nbhd $U$ of $e$ such that $(xU) \cap (CU) = \emptyset$.

(e) If $H$ is a closed normal subgroup of $G$, then $G/H$ is regular.

Proof. (a) It suffices to prove this first open nbhds of $e$. The proof is by induction. Continuity of the multiplication operation and the fact that $ee = e$, for each open nbhd $U$ of $e$ in $G$, the set $\{(x,y) \in G \times G : xy \in U\}$ is an open nbhd of $(e,e)$ in $G \times G$. Hence, there exist open nbhds $V_1$ and $V_2$ of $e$ in $G$ such that the basis element $V_1 \times V_2 \subseteq \{(x,y) \in G \times G : xy \in U\}$. Hence, that $V_1 V_2 \subseteq U$. By a previous exercise, since $e \in V_1 \cap V_2$ and at least one set $V_1$ is open, $V_1 V_2$ is open in $G$. Let $V_3 = V_1 \cap V_2$, which is open as a finite intersection of open sets, and put $V = V_3 \cap V_1$. As we know, $V$ is an open nbhd of $e$ and by the previous exercise, $V$ is symmetric. Moreover, $VV \subseteq U$, as, since $e \in V$, we have $VV \subseteq U \cap U^{-1} \subseteq U$. Thus $G$ is Hausdorff.

(d) Apply the case of $n = 2$ above and observe that, in this case, $V = V^{-1}$ so that $VV^{-1} \subseteq U$.

(e) Let $G$ be a $T_1$ topological group. Let $x, y \in G$ be distinct. It suffices to consider the case that $y = e$ since the map $g \mapsto g^{-1}$ is a homeomorphism of $G$ with itself, let $x \in G$. Let $U$ be an open nbhd of $e$ such that $x \notin U$—this exists since $G$ is $T_1$, so that $\{x\}$ is closed so that $G \setminus \{x\}$ is an open nbhd of $e$. Let $V$ be a symmetric nbhd of $e$ contained in $U$ such that $VV \subseteq U$. Then $Vx \cap Vy = \emptyset$ since otherwise there exist $v_1, v_2 \in V$ such that $v_1 x = v_2$ implying that $x = x^{-1}v_1^{-1} \in VV \subseteq U$ which is impossible. Thus $G$ is Hausdorff.

Let $G$ be a topological group, we shall follow Professor Falkner’s proof on page 165. Suppose $C$ is closed in $G$ and $x \in G \setminus C$. We wish to show that there exist disjoint open nbhds $A$ and $B$ of $x$ and $B$, respectively. Now, $x^{-1}(G \setminus C)$ is an open nbhd of $e$, so by (b) there is an open nbhd $U$ of $e$ such that $UU^{-1} \subseteq x^{-1}(G \setminus C)$. Then $xUU^{-1} \subseteq G \setminus C$. Now $xU$ is a neighborhood of $x$ and $CU$ is an open set (by a previous exercise, since $U$ is open) containing $C$. Thus, to finish, it suffices to show that $(xU) \cap (CU) = \emptyset$. Suppose for the sake of a contradiction that $(xU) \cap (CU) \neq \emptyset$ and let $z \in (xU) \cap (CU)$. Then $z = xu_1 = yu_2$ for some $u_1, u_2 \in U$. Hence, $xu_1u_2^{-1} = y \in C$ but also $xu_1u_2^{-1} \in xUU^{-1} \subseteq G \setminus C$, which is a contradiction. Thus, $(xU) \cap (CU) = \emptyset$. Hence, since $x$ and $C$ were arbitrary, every topological group is regular.

(e) As we know, $G/H$ is a quotient group if $H$ is a normal subgroup of $G$. With the quotient topology, the open sets of $G/H$ are precisely those for which its preimage under the canonical quotient map $\pi : G \to G/H$ is surjective. Moreover, by a previous exercise, it is a topological group. Now, suppose $E$ is closed in $G/H$ and $\pi(x) \notin E$. Then $xH = \pi^{-1}(\pi[x])$ is disjoint from $F = \pi^{-1}[E]$, the latter of which is closed by continuity of $\pi$. This follows because, otherwise, there exists $g \in xH \cap F$, but then $\pi(y) = \pi(x) \notin E$, a contradiction. Moreover, it is clear that $FH = F$, since $F = \bigcup_{x \in F} xH$. However, since $H$ is closed, $xH$ is closed. By (d), we may let $V$ be an open nbhd of $e$ such that $(Vx) \cap (VF) = \emptyset$. Hence, $Vx \cap F = \emptyset$. We would like to find disjoint open nbhds of $xH$ and $F$ that are saturated. Right multiplying by $H$ noting $FH = F$, $V(xH) \cap VF = \emptyset$, since otherwise we would have $(vx)h = f$ which is impossible since $Vx \cap F = \emptyset$. Therefore, $V(xH)$ and $V FH = VF$ are saturated open sets. Thus, $\pi(x) \in \pi[V(xH)]$ and $F \subseteq \pi[VF]$ and these sets are disjoint since their preimages $V(xH)$ and $VF$ are disjoint. Moreover, $\pi[V(xH)]$ and $\pi[VF]$ are open the quotient map $\pi : G \to G/H$ is open since $\pi[U] = \bigcup_{h \in H} Uh$ a union of open sets. This proves $G/H$ is regular. ■

**Exercise.** Let $G$ be a topological group.
(a) If \( A \) is a closed subspace of \( G \) and \( B \) a compact subspace of \( G \), then \( AB \) is closed. [Hint: If \( c \notin AB \), find a nbhd \( W \) of \( c \) such that \( WB^{-1} \cap A = \emptyset \).]

(b) If \( H \) is a compact subgroup of \( G \), the quotient map \( p: G \to G/H \) is a closed map.

(c) If \( H \) is a compact subgroup of \( G \) and \( G/H \) is compact, then \( G \) is compact.

**Exercise.** Let \( X \) be a space; let \( G \) be a topological group. An **action** of \( G \) on \( X \) is a continuous map \( \alpha: G \times X \to X \) such that

1. \( e \cdot x = x \) for all \( x \in X \)
2. \( g_1 \cdot (g_2 \cdot x) = (g_1 \cdot g_2) \cdot x \) for all \( x \in X \) and \( g_1, g_2 \in G \).

Define \( x \sim g \cdot x \) for all \( x \) and \( g \); the resulting quotient space is denoted \( X/G \) and is called the **orbit space** of the action \( \alpha \).

**Theorem.** Let \( X \) be a topological space, \( G \) a compact topological group and let \( \alpha \) be an action of \( G \) on \( X \). If \( X \) is Hausdorff, or regular, or normal, or locally compact, or second-countable, then so is \( X/G \).

**Exercise.** Let \( G \) be a (possibly \( T_1 \)) topological group. Let \( C \) be the component of \( G \) containing the identity element \( e \). Then \( C \) is a normal subgroup of \( G \). [Hint: If \( x \in G \), then \( xC \) is the component of \( G \) containing \( x \).]

**Exercise.** If \( G \) is a locally compact topological group and \( H \) a subgroup, then the quotient space \( G/H \) is locally compact.

**Exercise.** Every topological group is completely regular.

[Hint: Let \( V_0 \) be a nbhd of \( e \). Inductively construct a sequence \( (V_n) \) of nbhds of \( e \) such that \( V_n V_n \subseteq V_{n-1} \). Consider the set of all dyadic rationals \( p \) in \((0,1]\). For each dyadic ration \( p \) in \((0,1]\), define an open set \( U(p) \) inductively as follows:

\[ U(1) = V_0, \quad U(1/2) = V_1. \]

Given \( n \in \mathbb{N} \), if \( U(k/2^n) \) is defined for \( 0 < k/2^n \leq 1 \), define \( U(1/2^{n+1}) = V_{n+1} \),

\[ U((2k+1)/2^{n+1}) = V_{n+1} U(k/2^n) \quad \text{for} \quad 0 < k < 2^n. \]

For \( p = 0 \), let \( U(p) = \emptyset \). Show that \( V_n U(k/2^n) \subseteq U((k+1)/2^n) \) for all \( k, n \in \mathbb{N} \). Proceed as in Urysohn’s lemma.]
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