A set $V$ together with the operations of addition, denoted $\oplus$, and scalar multiplication, denoted $\circ$, is said to form a **vector space** if the following axioms are satisfied:

1. $x \oplus y = y \oplus x$ for any $x$ and $y$ in $V$.
2. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ for any $x$, $y$, $z$ in $V$.
3. There exist an element $0$ in $V$ defined by equation $x \oplus 0 = x$ for arbitrary $x$ in $V$.
4. For each $x$ from $V$, there exist an element $-x$ in $V$ defined by equation $x \oplus (-x) = 0$.
5. $\alpha \circ (x \oplus y) = \alpha \circ x \oplus \alpha \circ y$ for each scalar $\alpha$ and any $x$ and $y$ in $V$.
6. $(\alpha + \beta) \circ x = \alpha \circ x \oplus \beta \circ x$ for any scalars $\alpha$ and $\beta$ and any $x$ in $V$.
7. $(\alpha \beta) \circ x = \alpha \circ (\beta \circ x)$ for any scalars $\alpha$ and $\beta$ and any $x$ in $V$.
8. $1 \circ x = x$ for all $x$ in $V$.

1. (p.122 # 10)

   Let $S$ be the set of all ordered pairs of real numbers. Define scalar multiplication and addition on $S$ by

   $\alpha \circ (x_1, x_2) = (\alpha x_1, \alpha x_2)$

   $(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, 0)$

   Show that $S$ is not a vector space. Which of the eight axioms fail to hold?

**Solution.**

I am going to prove the axiom A3 fails by showing that the zero vector does not exist.

Let $(a, b)$ be an arbitrary element of $S$. I’ll show that it **can not** play the role of zero vector. That is the element $0 = (a, b)$ does not satisfy the equation (A3) $x \oplus 0 = x$ for all $x$. Indeed, let us take $x = (0, 1)$. Then by the definition of operations in $S$ we have

$$x \oplus 0 = (0, 1) \oplus (a, b) = (a, 0) \neq (0, 1) = x.$$

If the zero vector does not exist, then the axiom A4 also fails. In fact the axiom A6 fails as well.
2. (p.122 # 13)

Let \( R \) be the set of all real numbers. Define scalar multiplication by
\[
\alpha \circ x = \alpha x \quad \text{(the usual multiplication of real numbers)}
\]
and define addition by
\[
x \oplus y = \max(x, y) \quad \text{(the maximum of the two numbers)}
\]
Is \( R \) a vector space with these operations? Prove your answer.

Solution.
Here also axiom A3 fails.

Indeed, suppose that a number \( a \) represents the zero vector \( 0 \). Then the axiom A3 says that
\[
x \oplus 0 = x
\]
for all \( x \). However, if we choose \( x = a - 1 \) then \( x \oplus 0 = \max(a - 1, a) = a \neq a - 1 = x \).
This means that \( a \) cannot represent the zero vector. In other words the zero vector does not exist and \( R \) is not a vector space.

3. (p.132 # 9ace)

Determine whether the following are spanning sets for \( \mathbb{R}^2 \).

(a) \( \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\} \)

(c) \( \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\} \)

(e) \( \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\} \)

Solution.

(a) The vectors \( v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \) span \( \mathbb{R}^2 \). Indeed the determinant of the matrix
\[
A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}
\]
is equal to 1, which is non-zero. Therefore the matrix \( A \) is invertible and the equation \( A \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = b \) is solvable for all vectors \( b \).

(c) The vectors \( v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \) span \( \mathbb{R}^2 \). Indeed,
\[
(v_1 + 2v_2)/7 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e_1 \quad \text{and} \quad v_1 + 2(v_1 + 2v_2)/7 = (9/7)v_1 + (4/7)v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e_2 .
\]
Since the standard vectors \( e_1 \) and \( e_2 \) span \( \mathbb{R}^2 \), the vectors \( v_1, v_2, v_3 \) also span \( \mathbb{R}^2 \).

(e) Similarly to the part (a) the determinant \[
\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 3 \neq 0.
\]
Thus the vectors \( \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \) span \( \mathbb{R}^2 \).
4. (p.132 # 10bc)

Which of the following are spanning sets for \( \mathbb{R}^3 \). Justify your answers.

\[(b) \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\} \quad (c) \left\{ \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\} \]

Solution.

\[(b) \] The vectors \( \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \) span \( \mathbb{R}^3 \) because
\[
e_1 = \mathbf{v}_1, \quad e_3 = \mathbf{v}_3 - \mathbf{v}_1, \quad \text{and} \quad e_2 = \mathbf{v}_2 - \mathbf{v}_3 + \mathbf{v}_1.
\]

\[(c) \] The vectors \( \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \) do not span \( \mathbb{R}^3 \) because the determinant
\[
\begin{vmatrix}
2 & 3 & 2 \\
1 & 2 & 2 \\
-2 & -2 & 0
\end{vmatrix} = 0.
\]

5. (p.132 # 11b)

Given
\[
\mathbf{x}_1 = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -9 \\ -2 \\ 5 \end{pmatrix}
\]

Is \( \mathbf{y} \in \text{Span}(\mathbf{x}_1, \mathbf{x}_2) \)? Prove your answers.

Solution.

The vector \( \mathbf{y} \) belongs to \( \text{Span}(\mathbf{x}_1, \mathbf{x}_2) \) because \( \mathbf{y} = 3\mathbf{x}_1 - 2\mathbf{x}_2 \).