Math 254 (Instructor Sergei Chmutov)

# Calculus of differential forms

**Definition.** *A differential k-form* is an expression written as a sum of terms which are a function times a *wedge product* of $k$ differentials of coordinates.

**Example of a 2-form.** $\omega = (x^2 - y)\, dx \wedge dy \;+\; \dfrac{\sin z}{x-y}\, dx \wedge dz \;+\; ye^{-x \ln z}\, dy \wedge dz.$

- *Wedge product* is similar to an ordinary product, but it is **anticommutative** $\quad dx \wedge dy = -dy \wedge dx$.
  In particular, this implies that the wedge product of two equal differentials is zero $dx \wedge dx = 0$
- *Differential* of a $k$-form $\omega = f dx_1 \wedge dx_2 \wedge \ldots \wedge dx_k$ is equal to the $(k+1)$-form $d\omega = d(f) \wedge dx_1 \wedge dx_2 \wedge \ldots \wedge dx_k$.
  Examples.
  (*i*) If $\omega^0 = f(x,y,z)$ is a 0-form, then $d\omega^0 = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = f'_x dx + f'_y dy + f'_z dz$.
  (*ii*) If $\omega^1 = M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz$, then

$$\begin{aligned}
d\omega^1 &= dM \wedge dx + dN \wedge dy + dP \wedge dz \\
&= \big(M'_x dx + M'_y dy + M'_z dz\big) \wedge dx + \big(N'_x dx + N'_y dy + N'_z dz\big) \wedge dy + \big(P'_x dx + P'_y dy + P'_z dz\big) \wedge dz \\
&= \cancel{M'_x dx \wedge dx} + M'_y dy \wedge dx + M'_z dz \wedge dx \\
&\quad + N'_x dx \wedge dy + \cancel{N'_y dy \wedge dy} + N'_z dz \wedge dy \\
&\quad + P'_x dx \wedge dz + P'_y dy \wedge dz + \cancel{P'_z dz \wedge dz} \\
&= \big(-M'_y + N'_x\big)\, dx \wedge dy + \big(-M'_z + P'_x\big)\, dx \wedge dz + \big(-N'_z + P'_y\big)\, dy \wedge dz
\end{aligned}$$

(*iii*) If $\omega^2 = M(x,y,z)dy \wedge dz + N(x,y,z)dz \wedge dx + P(x,y,z)dx \wedge dy$, then

$$\begin{aligned}
d\omega^2 &= dM \wedge dy \wedge dz + dN \wedge dz \wedge dx + dP \wedge dx \wedge dy \\
&= \big(M'_x dx + M'_y dy + M'_z dz\big) \wedge dy \wedge dz + \big(N'_x dx + N'_y dy + N'_z dz\big) \wedge dz \wedge dx \\
&\quad + \big(P'_x dx + P'_y dy + P'_z dz\big) \wedge dx \wedge dy \\
&= M'_x dx \wedge dy \wedge dz + \cancel{M'_y dy \wedge dy \wedge dz} + \cancel{M'_z dz \wedge dy \wedge dz} + \cancel{N'_x dx \wedge dz \wedge dx} + N'_y dy \wedge dz \wedge dx \\
&\quad + \cancel{N'_z dz \wedge dz \wedge dx} + \cancel{P'_x dx \wedge dx \wedge dy} + \cancel{P'_y dy \wedge dx \wedge dy} + P'_z dz \wedge dx \wedge dy \\
&= \big(M'_x + N'_y + P'_z\big)\, dx \wedge dy \wedge dz
\end{aligned}$$

# Differential forms and vector fields in $\mathbb{R}^3$

| differential forms | | vector fields |
|---|---|---|
| $\omega^0_f = f(x,y,z)$ | $\Leftarrow$ | function $f(x,y,z)$ |
| $\omega^3_f = f dx \wedge dy \wedge dz$ | $\Leftarrow$ | |
| $\omega^1_{\mathbf{F}} = M dx + N dy + P dz$ | $\Leftarrow$ | vector field $\mathbf{F} = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + P(x,y,z)\mathbf{k}$ |
| $\omega^2_{\mathbf{F}} = M dy \wedge dz + N dz \wedge dx + P dx \wedge dy$ | $\Leftarrow$ | |

**Differentials:**

$$\begin{aligned}
d\omega^0_f &= \omega^1_{grad(f)} \\
d\omega^1_{\mathbf{F}} &= \omega^2_{curl(\mathbf{F})} \\
d\omega^2_{\mathbf{F}} &= \omega^3_{div(\mathbf{F})}
\end{aligned}$$

**Integrals:**

dim = 0 : $\displaystyle\int_a \omega^0_f = f(a)$ , where $a$ is a point.

dim = 1 : $\displaystyle\int_C \omega^1_{\mathbf{F}} = \oint_C \mathbf{F} \cdot d\mathbf{r}$ , where $C$ is a curve.

dim = 2 : $\displaystyle\int_\Sigma \omega^2_{\mathbf{F}} = \iint_\Sigma \mathbf{F} \cdot \mathbf{n}\, dS$ , where $\Sigma$ is a surface, and $\mathbf{n}$ is the unit normal vector directed upward.

dim = 3 : $\displaystyle\int_D \omega^3_f = \iiint_D f(x,y,z)\, dV$ , where $D$ is a solid region.

**General Stokes's Theorem**

$$\int_{\partial M} \omega = \int_M d\omega$$

where $\omega$ is a $k$-form; $M$ is an oriented $(k+1)$-dimensional manifold; the orientation on $\partial M$ is induced from $M$

**The Fundamental Theorem of Line Integrals**

$$\int_{\partial C} \omega_f^0 = \int_C \omega_{grad(f)}^1$$

$$f(b) - f(a) = \oint_C grad(f) \cdot d\mathbf{r}$$

**Stokes's Theorem**

$$\int_{C=\partial\Sigma} \omega_{\mathbf{F}}^1 = \int_{\Sigma} \omega_{curl(\mathbf{F})}^2$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{\Sigma} curl(\mathbf{F}) \cdot \mathbf{n} dS$$

**The Divergence Theorem**

$$\int_{\Sigma=\partial D} \omega_{\mathbf{F}}^2 = \int_D \omega_{div(\mathbf{F})}^3$$

$$\iint_{\Sigma} \mathbf{F} \cdot \mathbf{n} dS = \iiint_D div(\mathbf{F}) dV$$