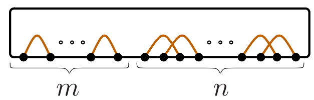
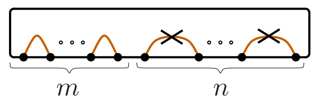
Classification of chord diagrams modulo 2T relation

Definition. An *oriented caravan of m "one-humped camels" and n "two-humped camels"* is the product of m diagrams with one chord and n diagrams with two crossing chords:



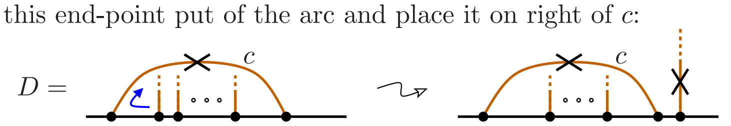
A *non-oriented caravan* is the product of m diagrams with one chord and n diagrams with a single twisted chord:



Theorem. *Any (twisted) chord diagram is equivalent, modulo 2T relations, to a caravan.*

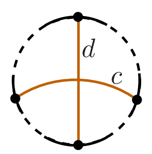
Proof.

Step1. Clearing the twisted chord. Consider a twisted chord c in a chord diagram D, and an arc bounded by c. Take the left-most end-point of chords on this arc. By twisted version of 2T we can take this end-point put of the arc and place it on right of c:

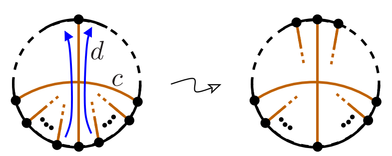


Doing this with all the end-points on this arc we clean up the arc reducing the diagram to a product of a diagram with a single chord and a diagram with one chord less than D. Thus we may assume that there is no any more twisted chord in our chord diagram D.

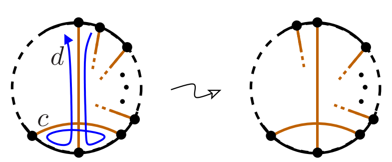
Step 2. Factoring two-humped camel. If the chords of our diagram D are not intersecting, then D is a caravan of one-humped camels and the theorem follows. Suppose there are two intersecting chords c and d:



Consider an arc bounded by c. It contains one end-point of d and possibly the end-points of other chords. We can take them out of the arc sliding them up along d by 2T relation:

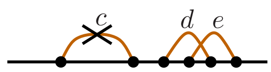


So we may assume that one arc bounded by c contains only one end-point, the one of chord d. Now we are going to clean up the right arc bounded by d. Take the highest end-point on this right arc. Slide it down along d, then to the left along c, and then up back along d again:

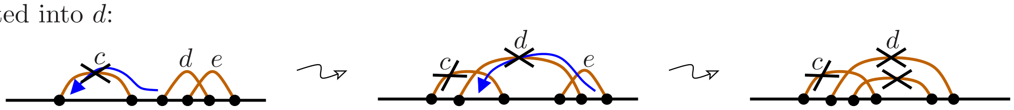


The result will be the moving of the topmost point from the right arc to the left one. Doing this with all end-points on the right arc except the end-point of c, we factor out a two-humped camel. Thus the original diagram was reduced to a caravan with a number of one-humped camels some of which might be twisted and a number of two-humped camels. So the theorem is proved in oriented case. For non-oriented case we prove that the presence of a twisted chords reduces a two-humped camel to the product of two twisted isolated chords.

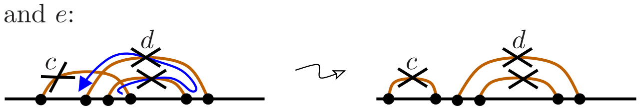
Step 3. Twisted camel kills two-humped camel. Let c be a twisted chord next to a two-humped camel d, e:



Sliding d inside c and then the right end-point of e along d makes all three chords twisted with e nested into d:



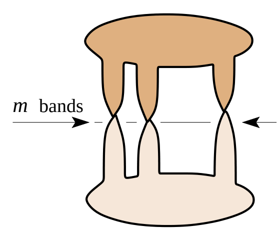
Sliding the right end-point of c along e to the right and then along d to the left separates c from the nested chords d and e:



Finally sliding the left end-point of d along e to the inside and then one more time along e outside makes all chords isolated.

The theorem is completely proved. □

Exercises.

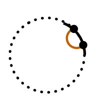


(1) Consider the surface consisting of two discs and m twisted bands between them. Is it orientable? Find the genus and the number of boundary components.
(2) Prove that the Euler characteristics χ and the genus g of a closed orientable surface are related as $\chi = 2 - 2g$.
(3) Prove that the Euler characteristics χ and the number of cross-caps μ of a closed non-orientable surface are related as $\chi = 2 - \mu$.

Closed surfaces

To obtain a closed surface from a surface with boundary we can glue in a disc to each boundary component. Translating this to the language of chord diagram we need to mod out diagrams with isolated non-twisted chords.

1T relation.

 $= 0$.

Theorem. *Any closed surface homeomorphic to the surface obtained by gluing a disc to the boundary of the surface associated with either a caravan of n two-humped camels, or a non-oriented* *caravan of n twisted one-humped camels.*