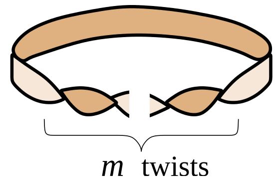
Representation of surfaces by chord diagrams

Exercises.

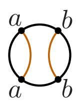


(1) Among the surfaces we studied last week find the one homeomorphic to a band with m twists.

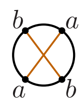
(2) Describe the result of cutting a band with m twists along a middle line parallel to its edge.

Definition. A *chord diagram* of order n (or *degree* n) is an oriented circle with a distinguished set of n disjoint pairs of distinct points, considered up to orientation preserving homeomorphisms of the circle.

This means that only the mutual combinatorial positions of the chord-ends on the circle are important. Their precise geometrical locations on the circle the geometrical form of the chords themselves are irrelevant. Thus we may think about a chord diagram of order n as a word of length $2n$ up to the cyclic equivalence and where each letter occurs precisely twice. Here are two examples:

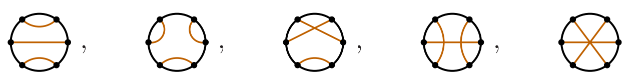


= "*abba*" = "*aabb*",

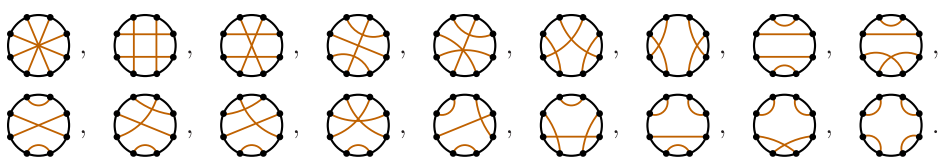


= "*abab*" = "*baba*".

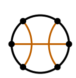
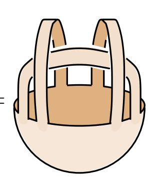
These two chord diagrams are the only chord diagrams for $n = 2$. For $n = 3$ there are five chord diagrams:

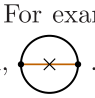


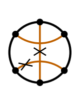
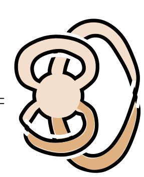
For $n = 4$ there are 18 chord diagrams:



From chord diagrams to surfaces with boundary.
With each chord diagram D we can associate an oriented surface Σ_D by attaching a disc to the circle of D and thickening the chords of D to narrow bands:

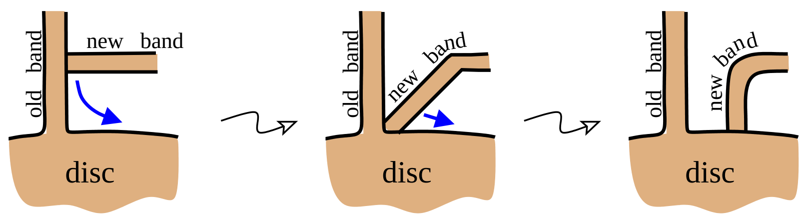
$D =$  $\rightsquigarrow$ $\Sigma_D =$  .

To represent non-orientable surfaces we mark some chords with a twist meaning that the corresponding bands should be twisted. For example, the Möbius band can be represented by the chord diagram with one twisted chord,  . Here is another example:

$D =$  $\rightsquigarrow$ $\Sigma_D =$  .

Theorem. *Any compact surface with boundary can be represented by a (possibly twisted) chord diagram.*

Proof. The idea of the proof is borrowed from [SC]. A *substantial arc* on a surface is a simple curve, that is a homeomorphic image of the closed interval $[0, 1]$, with the end-points on the boundary of the circle and which does not separate the surface. The Jordan Curve Theorem implies that a surface without substantial arcs is homeomorphic to a planar disc, which we may consider as a chord diagrams without chords. If there is a substantial arc we cut our surface along it. More precisely we cut out a very narrow band along the arc. The surface we obtain will be homeomorphic to the surface obtained by the simple cut along the arc. If on the resulting surface there is a substantial arc we continue the process of cutting until we will get a surface without substantial arcs. The compactness of the surface guarantees the finiteness of this process. The surface without substantial arcs at the end is homeomorphic to a disc. We think about this disc as the one whose boundary is the circle of the chord diagram we would like to construct. Now we will run the cutting process backward gluing in the narrow bands to the planar disc. At each step we should glue in a band which is topologically homeomorphic to a rectangle to the boundary of the previous step surface along the two opposite sides of the rectangle. If at some step we need to attach a new band to the boundary of the one previously glued band then we slide the new one down to the boundary of our disc which obviously gives a homeomorphic surface:

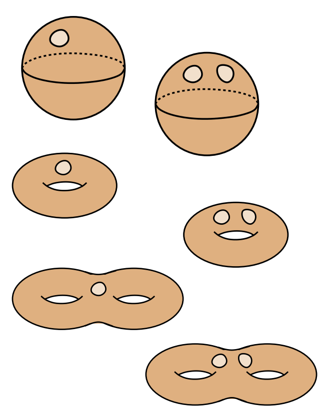


Sliding of the new band

At the end of this backward process we will get the surface homeomorphic to our original surface because we glues back exactly the same bands which we had cut out. Also this surface is represented by a disc with several bands attached to its boundary. this is clearly a surface obtained from a chord diagram. □

Exercises.

In problems (2)—(6) choose a substantial arc cutting along which reduces the surface to the one from previous exercise. Then using the chord diagram presentation from the previous problem glue in a band corresponding to that substantial arc back to get the required presentation of the given surface. This demonstrates the proof of the theorem.



(1) Represent a sphere with a hole by a chord diagram.

(2) Represent a sphere with two holes by a chord diagram.

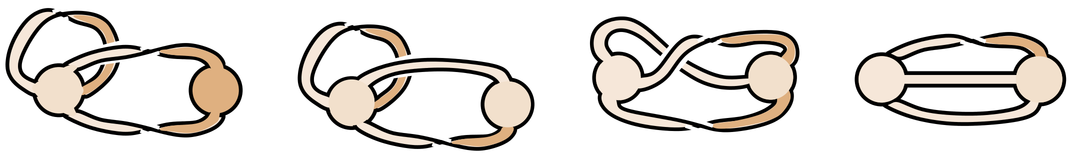
(3) Represent a torus with a hole by a chord diagram.

(4) Represent a torus with two holes by a chord diagram.

(5) Represent a double torus with one hole by a chord diagram.

(6) Represent a double torus with two holes by a chord diagram.

(7) Represent the following non-orientable surfaces by twisted chord diagrams.



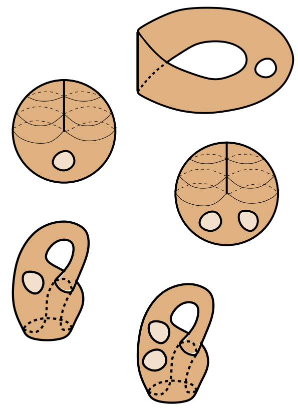
(8) Represent a Möbius band with one hole by a (twisted) chord diagram.

(9) Represent a projective plane with a hole by a (twisted) chord diagram.

(10) Represent a projective plane with two holes by a (twisted) chord diagram.

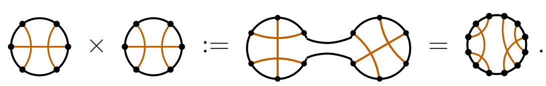
(11) Represent a Klein bottle with a hole by a (twisted) chord diagram.

(12) Represent a Klein bottle with two holes by a (twisted) chord diagram.

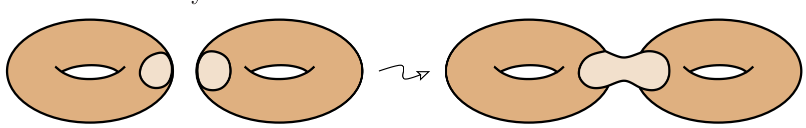


Product of chord diagrams.

Definition. The *product* of two chord diagrams D_1 and D_2 is defined to be their connected sum:



Of course, the resulting chord diagram depends on the places on D_1 and D_2 where they were joined together. However, we will see that the surfaces represented by the products $D_1 \times D_2$ for any two possible choices are homeomorphic. One can easily see that the product of two chord diagrams represent a boundary arc connected sum of the surfaces of factors:



2T relation.

We impose a relation on chord diagrams, 2T relation, when a chord end-point can be slided along the next chord to the other side:

Main Lemma. *The surfaces represented by two chord diagrams in the 2T relation are homeomorphic.*

Exercise. Prove that modulo 2T relation the product of chord diagrams is well-defined.

References

[SC] J. Scott Carter, *How surfaces intersect in space: an introduction to topology*, World Scientific, 1995.