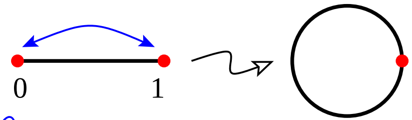
<u>Quotient Topology</u>

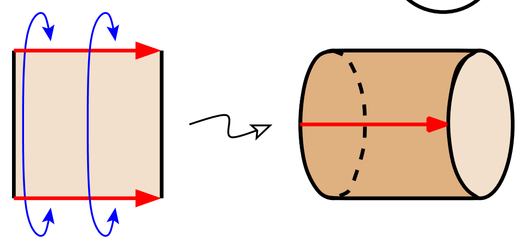
Definition. Let X be a topological space with an equivalence relation $\sim$ which is *reflexive* ($x \sim x$), *symmetric* ($x \sim y$ implies $y \sim x$), and *transitive* ($x \sim y$ and $y \sim z$ imply $x \sim z$). The set of equivalence classes $Y := X/\sim$ can be endowed with *quotient topology* whose open sets are the subsets of Y which preimages under the canonical projection $X \to X/\sim$ are open sets in X.

Examples.

(1) Let $X = I := [0, 1]$ be the unit interval with the equivalence relation, $0 \sim 1$, gluing the end-points. The quotient space $X/\sim$ is a circle S^1.



(2) Let $X = I^2 := [0, 1] \times [0, 1]$ be the unit square with the equivalence relation, $(t, 0) \sim (t, 1)$ for all $0 \leqslant t \leqslant 1$, gluing the opposite sides. The quotient space $X/\sim$ is a cylinder.



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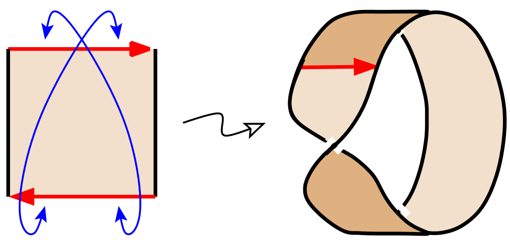
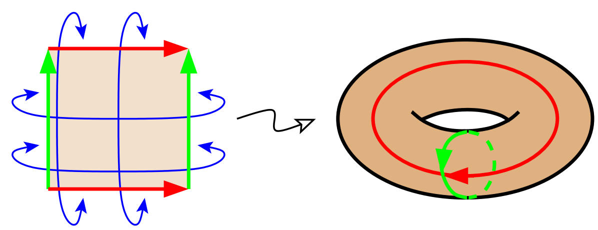
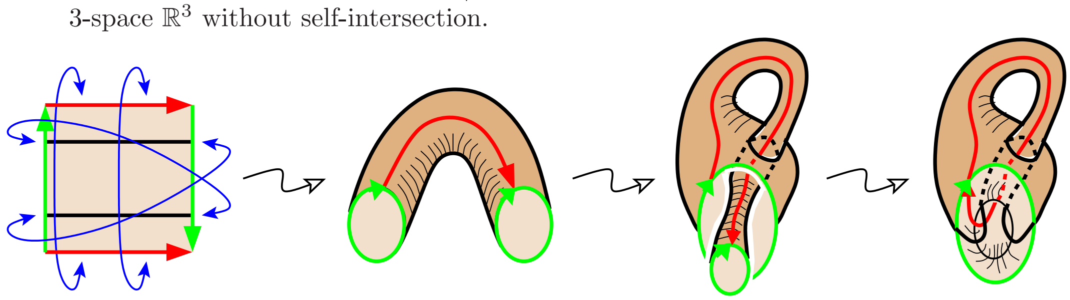


Let $X = I^2$ be the unit square with the equivalence relation, $(t, 0) \sim (1 - t, 1)$ for all $0 \leqslant t \leqslant 1$, gluing the opposite sides reversing the orientation. The quotient space $X/\sim$ is the *Möbius band*.

(4) Let $X = I^2$ be the unit square with the equivalence relation, $(t, 0) \sim (t, 1)$ and $(0, t) \sim (1, t)$ for all $0 \leqslant t \leqslant 1$, gluing the opposite sides in pairs. The quotient space $X/\sim$ is the *torus*.

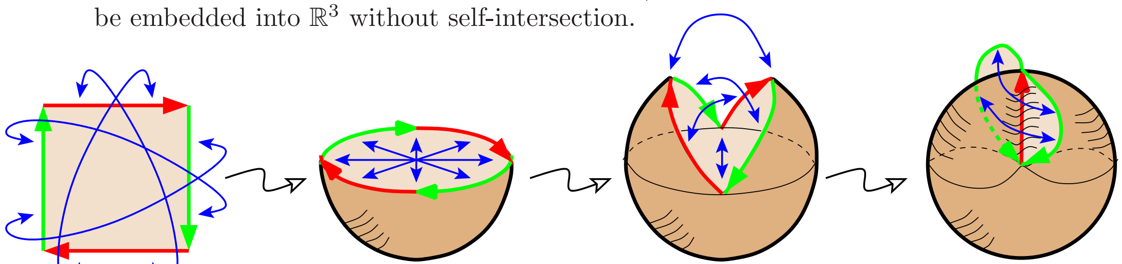


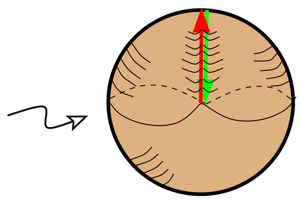
(5) Let $X = I^2$ be the unit square with the equivalence relation, $(t, 0) \sim (t, 1)$ and $(0, t) \sim (1, 1 - t)$ for all $0 \leqslant t \leqslant 1$, gluing the opposite sides in pairs and reversing the orientation of one pair. The quotient space $X/\sim$ is the *Klein bottle* which cannot be embedded into 3-space $\mathbb{R}^3$ without self-intersection.

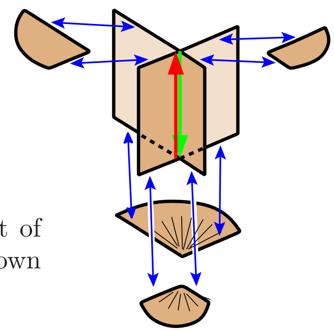


On the standard picture of the Klein bottle the set of self-intersection is represented by a circle where the tube hits itself and goes through to the interior of itself. If we cut the Klein bottle along the black contour we will get two Möbius bands. So one may think about the Klein bottle as two Möbius bands glued together along their boundary circles.

(6) Let $X = I^2$ be the unit square with the equivalence relation, $(t, 0) \sim (1-t, 1)$ and $(0, t) \sim (1, 1-t)$ for all $0 \leqslant t \leqslant 1$, gluing the opposite sides in pairs and reversing the orientation of both pairs. The quotient space $X/\sim$ is the *projective plane*. It also cannot be embedded into $\mathbb{R}^3$ without self-intersection.







The standard picture is called the *cross cap*. The set of self-intersections is an interval. Its neighborhood is shown on the right picture.

Exercises.

(1) Prove that the open interval $(0, 1)$ is homeomorphic to the whole line $\mathbb{R}$.
(2) Prove that the open interval $(0, 1)$ is not homeomorphic to the closed interval $[0, 1]$.
(3) Prove that the circle in Example (1) obtained by gluing the end-points of the interval is homeomorphic to the circle $S^1 := \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ with the topology induced from $\mathbb{R}^2$.
(4) Prove that the circle S^1 is not homeomorphic to the figure 8, also known as the *wedge* of two circles $S^1 \vee S^1$.
(5) Prove that the cylinder and the annulus $1 \leqslant x^2 + y^2 \leqslant 2$ are homeomorphic.
(6) Prove that the cylinder is not homeomorphic to the Möbius band.
(7) Prove that the spheres of different radii are homeomorphic.
(8) Prove that a sphere is not homeomorphic to a torus.