Homework #1

Due Monday, January 13.

1. Prove that the curvature does not depend of the choice of parameter. In other words, if $s$ is a different parameter so that $s = f(t)$ for some function $f(\cdot)$, and primes denote derivatives relative to $s$: $x' := \frac{dx}{ds}\bigg|_{s=f(0)}$, $x'' := \frac{d^2x}{ds^2}\bigg|_{s=f(0)}$, $y' := \frac{dy}{ds}\bigg|_{s=f(0)}$, $y'' := \frac{d^2y}{ds^2}\bigg|_{s=f(0)}$, then

$$\frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{|x'y'' - x''y'|}{(x'^2 + y'^2)^{3/2}}.$$

2. [Do Carmo,1-5, #7a, p.23]

The evolute of a curve is the set of all its centers of curvature. For a parametrized curve $\alpha(t)$ with the signed curvature $k(t)$ and the unit normal vector $n(t)$, evolute $\beta(t)$ can be given by the formula

$$\beta(t) = \alpha(t) + \frac{1}{k(t)} n(t).$$

Show that the tangent line of the evolute $\beta$ at $t$ is normal to $\alpha$ at $t$.

3. A parabola is given parametrically $x(t) = t$, $y(t) = at^2$. Find its curvature and its evolute. Draw a picture of the parabola and the evolute on the same figure.

4. An ellipse is given parametrically $x(t) = a \cos t$, $y(t) = b \sin t$. Find its curvature and its evolute. Draw a picture of the ellipse and the evolute on the same figure.

5. A hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can be parametrize using hyperbolic functions $\sinh t := \frac{e^t - e^{-t}}{2}$ and $\cosh t := \frac{e^t + e^{-t}}{2}$. The parametrization is $x(t) = a \cosh t$, $y(t) = b \sinh t$. Find the (signed) curvature and evolute of the hyperbola. Draw a picture of the hyperbola and its evolute on the same figure.

6. Find a formula for the curvature of a curve given in polar parametrized form $r(t)$, $\theta(t)$. 
7. Find a formula for the curvature of a curve given as a graph of function \( y = f(x) \) in terms of function \( f(x) \).

8. Find a formula for the curvature of a curve given implicitly by an equation \( F(x, y) = 0 \) in terms of function \( F(x, y) \).

9. Find a formula for the curvature of a curve given as a solution of the differential equation \( P(x, y)dx + Q(x, y)dy = 0 \) in terms of functions \( P(x, y) \) and \( Q(x, y) \).

10. Find the curvature of the following curves.
    
    (a) \( y = \sin x \) at a vertex \( x = \pi/2 \).
    
    (b) \( y = a \cosh(x/a) \), a catenary.
    
    (c) \( r^2 = a^2 \cos(2\theta) \), a lemniscate.
    
    (d) \( r = a(1 + \cos(\theta)) \), a cardioid.
    
    (e) \( r = a\theta \), a spiral of Archimedes.