Homework #4

Due Monday, February 3.

1. Find curve $\alpha : I \to \mathbb{R}^3$ for which the curvature and the torsion are constants, $k(s) = a$ and $\tau(s) = b$.

2. Suppose that the binormal vector $b(s)$ of a curve $\alpha(s)$ in $\mathbb{R}^3$ is constant. Prove that $\alpha$ is a plane curve.

3. Let $\beta(s)$ be a projection of a curve $\alpha(s)$ in $\mathbb{R}^3$ on its osculating plane $\langle t(s_0), n(s_0) \rangle$ at a point $s_0$. Show that $k_{\alpha}(s_0) = k_{\beta}(s_0)$.

4. Let $\alpha : I \to \mathbb{R}^3$ be an arc length parametrized curve. Then its $n$-th derivative $\alpha^{(n)}(s)$ can be expressed in the Frenet frame as

$$\alpha^{(n)}(s) = a_n(s)t(s) + b_n(s)n(s) + c_n(s)b(s).$$

Prove the following recurrent formulas for the coefficients

$$a_{n+1}(s) = a'_n(s) - k(s)b_n(s)$$

$$b_{n+1}(s) = b'_n(s) + k(s)a_n(s) + \tau(s)c_n(s)$$

$$c_{n+1}(s) = c'_n(s) - \tau(s)b_n(s).$$

5. [do Carmo, 1-5, #13, p.25]
   Assume that $\tau(s) \neq 0$ and $k'(s) \neq 0$ for all $s \in I$. Show that a necessary and sufficient condition for $\alpha(I)$ to lie on a sphere is that

$$R^2 + (R')^2T^2 = \text{const.},$$

where $R = 1/k$, $T = 1/\tau$, and $R'$ is the derivative of $R$ relative to $s$.

6. [do Carmo, 1-5, #17abc, p.26]
   A curve $\alpha$ is called helix if the tangent lines of $\alpha$ make a constant angle with a fixed direction. Assume that $\tau(s) \neq 0$, $s \in I$, and prove that:
(a) \( \alpha \) is a helix if and only if \( k/\tau = \text{const.} \).

(b) \( \alpha \) is a helix if and only if the lines containing \( n(s) \) and passing through \( \alpha(s) \) are parallel to a fixed plane.

(c) \( \alpha \) is a helix if and only if the lines containing \( b(s) \) and passing through \( \alpha(s) \) make a constant angle with a fixed direction.

7. Show that the Frenet equations for a curve in \( \mathbb{R}^3 \) can be presented in a form

\[
t' = \omega \times t, \quad n' = \omega \times n, \quad b' = \omega \times b,
\]

for some special vector \( \omega(s) \). It is called the Darboux vector.

8. Find the curvatures \( k_1, k_2, k_3 \) of the curve \( \alpha(t) = (t, t^2, t^3, t^4) \) in \( \mathbb{R}^4 \) at the point \( t = 0 \).

9. Consider a curve in \( \mathbb{R}^4 \) given by \( \alpha_{p,q}(s) = (\cos(ps), \sin(ps), \cos(qs), \sin(qs)) \), where \( p \) and \( q \) are two numbers subject to relation \( p^2 + q^2 = 1 \). Find the curvatures \( k_1, k_2, k_3 \) in terms of \( p \) and \( q \).

10. Let \( e_1, \ldots, e_n \) be the Frenet frame for an arc length parametrized curve \( \alpha : I \to \mathbb{R}^n \) with curvatures \( k_1, k_2, \ldots, k_{n-1} \). Prove that

(a) \( \alpha^{(j)}(s) = k_1(s)k_2(s)\ldots k_{j-1}(s)e_j(s) + (\text{linear combination of } e_1, \ldots, e_{j-1}) \), for \( j = 1, \ldots, n \).

(b) \( \det(\alpha', \alpha'', \ldots, \alpha^{(n)}) = \prod_{j=1}^{n-1} k_j^{n-j} \).