Homework #10
Due Monday, March 31.

1. [do Carmo, 3-3, #6, p.168] (A surface with $K \equiv -1$; the Pseudosphere.)

(a) Determine an equation for the place curve $C$, which is such that the segment of the tangent line between the point of tangency and some line $r$ in the plane, which does not meet the curve, is constantly equal to 1 (this curve is called the tractrix; see Fig.1-9).

(b) Rotate the tractrix $C$ about the line $r$; determine if the “surface” of revolution thus obtained (the pseudosphere; see Fig.3-22) is regular and find out a parametrization in a neighborhood of a regular point.

(c) Show that the Gaussian curvature of any regular point of the pseudosphere is $-1$.

2. [do Carmo, 3-3, #7, p.169; Modified.] (Surfaces of revolution with Constant Curvature.)

$(\varphi(v)\cos u, \varphi(v)\sin u, \psi(v))$ is given as a surface of revolution with constant Gaussian curvature $K$. To determine the functions $\varphi$ and $\psi$, choose the parameter $v$ is such a way that $(\varphi')^2 + (\psi')^2 = 1$ (geometrically, this means that $v$ is the arc length of the generating curve $(\varphi(v), \psi(v)))$. Show that

(a) $\varphi$ satisfies $\varphi'' + K\varphi = 0$ and $\psi$ is given by $\psi = \int \sqrt{a - (\varphi')^2}dv$; thus $0 < u < 2\pi$, and the domain of $v$ is such that the last integral makes sense.

(b) All surfaces of revolution with constant curvature $K = 1$ which intersect perpendicularly the plane $xOy$ are given by

$$\varphi(v) = C \cos v, \quad \psi(v) = \int_0^v \sqrt{1 - C^2 \sin^2 v}dv,$$

where $C$ is a constant ($C = \varphi(0)$).
(c) All surfaces of revolution with constant curvature \( K = -1 \) may be given by one of the following types:

1. \( \varphi(v) = C \cosh v, \quad \psi(v) = \int_0^v \sqrt{1 - C^2 \sinh^2 v} \, dv \).
2. \( \varphi(v) = C \sinh v, \quad \psi(v) = \int_0^v \sqrt{1 - C^2 \cosh^2 v} \, dv \).
3. \( \varphi(v) = e^v, \quad \psi(v) = \int_0^v \sqrt{1 - e^{2v}} \, dv \).

3. [do Carmo, 3-3, #18, p.172]
Show that Möbius strip can be parametrized by

\[
\mathbf{x}(u, v) = \left( (2 - v \sin \frac{u}{2}) \sin u, (2 - v \sin \frac{u}{2}) \cos u, v \cos \frac{u}{2} \right)
\]
and that its Gaussian curvature is

\[
K = -\frac{1}{\left\{ \frac{1}{4}v^2 + (2 - v \sin(u/2))^2 \right\}^2}.
\]

4. [do Carmo, 3-3, #19, p.172]
Obtain the asymptotic curves of the one-sheeted hyperboloid \( x^2 + y^2 - z^2 = 1 \).

5. [do Carmo, 3-3, #20, p.172]
Determine the umbilical points of the ellipsoid \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \).

6. [do Carmo, 4-3, #1, p.237]
Show that if \( \mathbf{x} \) is an orthogonal parametrization, that is, \( F = 0 \), then

\[
K = -\frac{1}{2\sqrt{EG}} \left\{ \left( \frac{E_v}{\sqrt{EG}} \right)_v + \left( \frac{G_u}{\sqrt{EG}} \right)_u \right\}.
\]

7. [do Carmo, 4-3, #2, p.237]
Show that if \( \mathbf{x} \) is an isothermal parametrization, that is, \( E = G = \lambda(u, v) \) and \( F = 0 \), then

\[
K = -\frac{1}{2\lambda} \Delta(\log \lambda),
\]
where \( \Delta \varphi \) denotes the Laplacian \( (\partial^2 \varphi/\partial u^2) + (\partial^2 \varphi/\partial v^2) \) of the function \( \varphi \). Conclude that when \( E = G = (u^2 + v^2 + c)^{-2} \) and \( F = 0 \), then \( K = \text{const.} = 4c \).
8. [do Carmo, 4-3, #3, p.237]

Verify that the surfaces
\[ \mathbf{x}(u, v) = (u \cos v, u \sin v, \log u), \]
\[ \mathbf{x}(u, v) = (u \cos v, u \sin v, v), \]
have equal Gaussian curvature at the points \( \mathbf{x}(u, v) \) and \( \mathbf{x}(u, v) \) but that the mapping \( \mathbf{x} \circ \mathbf{x}^{-1} \) is not an isometry. This shows that the “converse” of the Gauss Theorem is not true.

9. [do Carmo, 4-3, #4, p.237]

Show that no neighborhood of a point in a sphere may be isometrically mapped into a plane.

10. [do Carmo, 4-3, #5, p.237]

If the coordinate curves form a Tchebyshev net (cf. Problems 8 and 9, HW 7), then \( E = G = 1 \) and \( F = \cos \theta \). Show that in this case
\[ K = -\frac{\theta_{uv}}{\sin \theta}. \]