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Vassiliev Knot Invariants
I. Introduction

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The present paper can be regarded both as an introduction to Vassiliev's theory of knot invariants and as an introduction to our subsequent papers [CDL2, CDL3]. Our chief goal was to explain how the initial topological problem is reduced to a purely combinatorial one. Another introduction, well suited for first reading, can be found in [CD]. For more detailed expositions we refer the reader to [BL, B-N, S, B].

The set $\mathscr{K}^*$ of all knot invariants with values in a ring $\mathbf{K}$ forms a module over $\mathbf{K}$. Since functions can be multiplied, this module carries a natural structure of an algebra. Moreover, the connected sum of knots defines, by duality, a comultiplication on $\mathscr{K}^*$ and thus supplies $\mathscr{K}^*$ with a Hopf algebra structure. Of course, this Hopf algebra is extremely complicated. Below, we describe a series of objects which can be considered as successive simplifications of the complete algebra $\mathscr{K}^*$ and whose study yields certain information about $\mathscr{K}^*$.

In a pioneering work [V], Victor Vassiliev introduced finite order knot invariants arising from the topology of discriminants in functional spaces. The submodule $\mathscr{V}$ of finite order invariants is a Hopf subalgebra in $\mathscr{K}^*$; it has a natural filtration by order. Thus $\mathscr{V}$ is a filtered Hopf algebra. Vassiliev conjectured that the "closure" of $\mathscr{V}$ coincides with the entire algebra $\mathscr{K}^*$ (i.e. Vassiliev invariants distinguish knots). The Hopf algebra $\mathscr{V}$ is also very complicated.

A further simplification is achieved by considering the corresponding graded module $\operatorname{gr}\mathscr{V}$. This module inherits the Hopf algebra structure from $\mathscr{V}$. Kontsevich's theorem [K] gives a purely combinatorial description of $\operatorname{gr}\mathscr{V}$ as the algebra $\mathscr{N}^*$ of functions on the set of *chord diagrams* satisfying certain linear equations (one- and four-term relations). The fundamental

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problem in the theory of Vassiliev knot invariants is to find an explicit description of this Hopf algebra.

Finally, a simplification of chord diagrams is given by their *intersection graphs*. These graphs encode a lot of information about chord diagrams. Two further papers by the same authors [CDL2, CDL3] are devoted to a thorough study of intersection graphs.

Since the terminology in this area is not stable yet, we have allowed ourselves to use some notions in a way slightly differing from that adopted in [A, BL, B-N, K, S].

§1. Knots and their invariants

1.1. Smooth knots. A *knot* is a smooth embedding $S^1 \to \mathbf{R}^3$ considered up to a smooth isotopy.

1.1.1. *Hopf algebra structures on the modules of knots invariants.* A *knot invariant* is a function on the set of all knots whose values lie in an associative commutative ring $\mathbf{K}$ with unity. Invariants can be added together and multiplied by elements of $\mathbf{K}$, thus the set of invariants $\mathscr{K}^*$ is a module over $\mathbf{K}$. The natural multiplication

$$\cdot : \mathscr{K}^* \otimes \mathscr{K}^* \to \mathscr{K}^*$$

is defined by the rule $(f_1 \cdot f_2)(k) = f_1(k) \cdot f_2(k)$ for any knot k.

Another operation called *comultiplication* is given by the formula

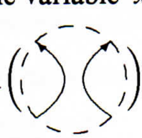
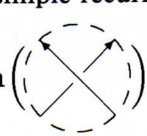
$$\delta : \mathscr{K}^* \to \mathscr{K}^* \otimes \mathscr{K}^*$$
$$\delta f(k_1 \otimes k_2) = f(k_1 \# k_2),$$

where $\#$ denotes the connected sum of knots.

These operations of multiplication and comultiplication transform $\mathscr{K}^*$ into a Hopf algebra.

Hopf algebras first appeared in [H] as algebraic structures in the cohomology rings of H-spaces. A standard reference on this subject is [MM]. We also recommend [Sp, Chapter 5].

1.1.2. *Example: the Conway polynomial.* The most famous of all knot invariants is probably the Alexander polynomial [Al]. It takes values in the ring of polynomials in one variable. This invariant is defined not only for knots but also for *links*. The Conway polynomial [C] differs from the Alexander polynomial by a normalization and a change of variable. We will denote the value of this invariant on a knot (or link) k by $\mathrm{Con}(k)$ and regard it as a polynomial in the variable x. It can be defined by the simple recurrent rule

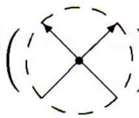
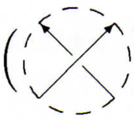
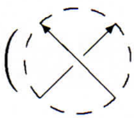
$x \cdot \mathrm{Con}\Big($  $\Big) = \mathrm{Con}\Big($  $\Big) - \mathrm{Con}\Big($  $\Big)$

and the initial value $\mathrm{Con}(\text{trivial knot}) = 1$.

By these three figures we mean three knots (links) which are identical outside a small ball and are as shown on the picture inside the ball. Arrows mark the link orientation.

1.2. Singular knots. *Singular knots* are smooth mappings $S^1 \to \mathbf{R}^3$ having only transversal double points as singularities and, like ordinary knots, are considered up to isotopy equivalence.

1.2.1. *Vassiliev invariants.* The Vassiliev's crucial idea was to study *prolongations* of the invariants to *singular knots*. To prolong a knot invariant f to the space of all singular knots, all one has to do is to successively apply the rule

$f\Big($  $\Big) = f\Big($  $\Big) - f\Big($  $\Big)$ (1)

One may check that for any knot invariant f, initially defined only on non-singular knots, this prolongation procedure always gives a consistent result.

DEFINITION. A function on singular knots that satisfies the defining condition (1) will be called a *Vassiliev knot invariant.*

DEFINITION. A Vassiliev knot invariant is *of order no greater than* n if it vanishes on any singular knot with more than n double points.

Let $\mathscr{V}_n \subset \mathscr{K}^*$ be the submodule which contains invariants of order no greater than n and let $\mathscr{V}$ denote the module of all finite order Vassiliev invariants. Then

$$\mathscr{V}_0 \subseteq \mathscr{V}_1 \subseteq \mathscr{V}_2 \subseteq \cdots \subseteq \mathscr{V}_n \subseteq \cdots \subset \mathscr{V}$$

is a filtration in $\mathscr{V}$.

Note that product of two invariants of orders n and m is an invariant of order $n+m$. It is not so difficult to see that the coproduct of a finite order invariant belongs to $\mathscr{V} \otimes \mathscr{V} \subset \mathscr{K}^* \otimes \mathscr{K}^*$. So the submodule $\mathscr{V} \subset \mathscr{K}^*$ of Vassiliev knot invariants forms a filtered Hopf subalgebra in $\mathscr{K}^*$.

Using this filtration, we can study a simpler graded module

$$\operatorname{gr}\mathscr{V} = \mathscr{V}_0 \oplus \mathscr{V}_1/\mathscr{V}_0 \oplus \mathscr{V}_2/\mathscr{V}_1 \oplus \cdots \oplus \mathscr{V}_n/\mathscr{V}_{n-1} \oplus \ldots$$

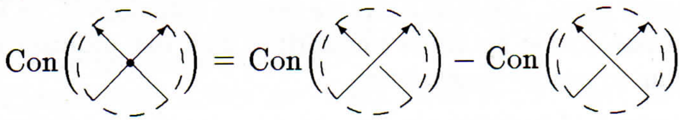
This module is also a Hopf algebra (it inherits the structure from $\mathscr{V}$).

In §2 we introduce a purely combinatorial graded module $\mathscr{A}^*$ consisting of functions on chord diagrams. Kontsevich's theorem [K] says that $\operatorname{gr}\mathscr{V} \cong \mathscr{A}^*$ when $\mathbf{K}$ is a field of characteristic zero. Over an arbitrary ring of coefficients $\operatorname{gr}\mathscr{V}$ is isomorphic to a submodule of $\mathscr{A}^*$. We think that the equality $\operatorname{gr}\mathscr{V} = \mathscr{A}^*$ is always true, but, to the best of our knowledge, this is still an open problem.

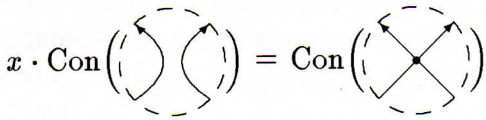
1.2.2. *Example: coefficients of the Conway polynomial as Vassiliev invariants.* Let $c_n(k)$ be the coefficient of x^n in the Conway polynomial $\operatorname{Con}(k)$.

Here we explain that c_n is a knot invariant of order no greater than n (see also [BL, B-N]).

According to sec. 1.2.1, we prolong the Conway polynomial to singular knots by the rule



So



Now it is obvious that if the knot k contains more than n double points then $\mathrm{Con}(k)$ has no term x^i with $i \leq n$.

§2. Chord diagrams

2.1. Notions.

DEFINITION. *A chord diagram of order* n is a circle with a distinguished set of n unordered pairs of points, regarded up to an orientation preserving diffeomorphism of the circle.

A chord diagram may be depicted as a circle with a set of n chords all of whose endpoints are distinct. Of course, these chords can be drawn as lines or curves whose actual shape is irrelevant; what matters is the way they bind their endpoints into pairs.

DEFINITION. *The chord diagram of a singular knot* $S^1 \to \mathbf{R}^3$ is the oriented circle S^1 with the preimages of every double point connected by a chord.

2.2. From knot invariants to functions on chord diagrams. Consider a Vassiliev knot invariant v of order no greater than n, $v \in \mathscr{V}_n$. We shall show that the value of v on a knot with n double points depends only on the chord diagram of the knots, but not on the knot itself.

Indeed, let k_1, k_2 be two singular knots with n double points that have identical chord diagrams. By an appropriate isotopy, the knot k_2 can be transformed so that

a) the corresponding double points of k_2 coincide with those of k_1;

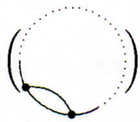
b) the corresponding arcs of k_2 coincide with those of k_1 in a small neighborhood of each double point;

c) the arcs of k_1 and k_2 have no common points outside of these small neighborhoods.

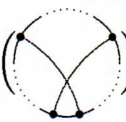
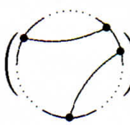
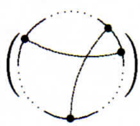
Now take two subsequent double points of k_2 and an arc of k_2 connecting these points. This arc may be transformed into the corresponding arc of k_1 by an isotopy which contains a sequence of "perestroikas", each "perestroika" being a passage through a knot with an additional double point. Due to the defining condition (1) the value of v does not change under such "perestroikas", whence the required assertion.

The values of the invariant do not thus depend on the specific behavior of the knotted curve, being completely defined by the corresponding chord diagram, i.e., by the order in which double points are encountered while moves along the curve. Moreover, if the restrictions of two invariants of order no greater than n on the set of all singular knots with precisely n double points coincide, then their difference is an invariant of order no greater than $n-1$.[1] Hence, to an element $v \in \mathscr{V}_n/\mathscr{V}_{n-1}$ we can assign a function on the set of chord diagrams with n chords. Functions on chord diagrams that appear in this way always satisfy two types of conditions (see [V, BL, B-N, CD]):

(one-term relation)

v  $= 0$ (2)

(four-term relation)

v  $-v$  $+v$  $-v$  $= 0$ (3)

for an arbitrary fixed position of $(n-2)$ chords (which are not drawn here) and the two additional chords positioned as shown in the picture. Here and below, dotted arcs suggest that there might be further chords attached to their points, while on the solid portions of the circle all the endpoints are explicitly shown.

Denote the set of all functions v (with values in $\mathbf{K}$) on the chord diagrams satisfying equations (2) and (3) by $\mathscr{A}^*$.

REDEFINITION. Elements of $\mathscr{A}^*$ will be referred to as *invariants.*

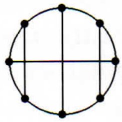
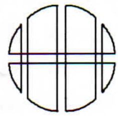
We hope that the double meaning of the word *"invariant"* (used both for functions on knots and for functions on chord diagrams) will not lead to confusion.

THEOREM [K]. *When* $\mathbf{K}$ *is a field of characteristic zero elements of* $\mathscr{V}_n/\mathscr{V}_{n-1}$ *satisfy no other relations besides those derived from one-term relations* (2) *and four-term relations* (3). *This means that the Hopf algebras* $\mathscr{A}^*$ *and* $\operatorname{gr}\mathscr{V}$ *coincide in these cases.*

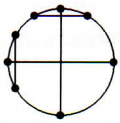
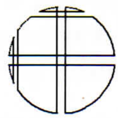
2.3. Example: coefficients of the Conway polynomial as elements of $\mathscr{A}^*$. It is easy to see from the definition in 1.1.2 that the constant term c_0 of the Conway polynomial is equal to 1 for any knot and to zero for any link with more than one component. The formulas of 1.2.2 explain the following algorithm that computes the value of c_n on a given chord diagram. Let D be a chord diagram with n chords. Replace each chord of D by a pair of

[1] In [CD], the restriction of an invariant v of order no greater than n on the set of all singular knots with exactly n double points was called the *symbol* of v.

parallel chords ("double it") and count the number of components of the curve thus obtained. If the number of components is more than one, then $c_n(D) = 0$. If there is only one component, then $c_n(D) = 1$. In particular,

for $D =$  $\longrightarrow$ 

the number of components is three, so $c_4(D) = 0$;

for $D =$  $\longrightarrow$ 

the number of components is only one, so $c_4(D) = 1$.

2.4. The Hopf algebra of chord diagrams. Let $\mathscr{M}_n$ be the **K**-module of chord diagrams of order n modulo all four-term relations

 $-$  $+$  $-$  $= \quad 0$

Consider the graded module

$$\mathscr{M} = \mathscr{M}_0 \oplus \mathscr{M}_1 \oplus \mathscr{M}_2 \oplus \dots$$

and the dual graded module

$$\mathscr{M}^* = \mathscr{M}_0^* \oplus \mathscr{M}_1^* \oplus \mathscr{M}_2^* \oplus \dots$$

where $\mathscr{M}_n^* = \oplus Hom(\mathscr{M}_n, \mathbf{K})$.

The module $\mathscr{M}^*$ inherits the Hopf algebra structure from the module $\mathscr{K}^*$, and the module $\mathscr{M}$ carries the structure of the dual Hopf algebra. Explicit description of this structure is given by the following two definitions.

DEFINITION. Let D_1, D_2 be two chord diagrams of order n_1, n_2 respectively. The *product* $[D_1] \cdot [D_2]$ of classes $[D_1] \in \mathscr{M}_{n_1}$, $[D_2] \in \mathscr{M}_{n_2}$ is a class $[D] \in \mathscr{M}_{n_1+n_2}$ defined as follows.

Break the circle of the diagram D_1 at a point x_1 different from all the ends of the chords of D_1, and break the circle of the diagram D_2 at a point x_2 different from all the ends of the chords of D_2. Gluing the two broken circles in a new circle, with their orientations taken into account, one obtains a new chord diagram D of order $n_1 + n_2$. We set $[D_1] \cdot [D_2] = [D]$.

DEFINITION. Let D be a chord diagram of order n. Any subset J of the chords of D determines two diagrams D_J and $D_{J'}$, where D_J contains only the chords belonging to the set J, and $D_{J'}$ consists of the chords belonging to the complement of J. We set

$$\delta([D]) = \sum_J [D_J] \otimes [D_{J'}],$$

where the sum is taken over all subsets J.

The unity in the Hopf algebra $\mathcal{M}$ is represented by the chord diagram with an empty set of chords. The unity in the dual algebra $\mathcal{M}^*$ is the function equal to 1 on the unit diagram and equal to zero on any other diagram.

Factorizing modules $\mathcal{M}_n$ over all 1-term relations

$$= \quad 0$$

one obtains a **K**-module $\mathcal{N}_n$. The Hopf algebra $\mathcal{N} = \mathcal{N}_0 \oplus \mathcal{N}_1 \oplus \mathcal{N}_2 \oplus \dots$ is a quotient algebra of $\mathcal{M}$. The corresponding dual Hopf algebra $\mathcal{N}^*$ is naturally realized as a subalgebra $\mathcal{N}^* \subset \mathcal{M}^*$ consisting of functions that satisfy the 1-term relations (2).

Definition. $\mathcal{M}$ will be referred as to *the algebra of chord diagrams*. Elements of $\mathcal{M}^*$ will be called *pre-invariants*.

2.5. Renormalization.

Although the algebra $\mathcal{N}^*$ (functions satisfying the one- and four-term relations) is more closely related to $\mathcal{V}$ than the wider algebra $\mathcal{M}^*$ (functions satisfying only four-term relations), the latter is easier to handle in certain circumstances.

In fact, there is no real difference between studying the one or the other, because there exists a projection of $\mathcal{M}^*$ onto $\mathcal{N}^*$ which splits the natural inclusion $\mathcal{N}^* \to \mathcal{M}^*$. This projection kills the primitive element of $\mathcal{M}^*$ in dimension 1 while preserving all the remaining primitive generators intact. We shall give an explicit construction of this projection for $\mathcal{M}^*$ although everything can be done for an arbitrary Hopf algebra.

Let $X_n : \mathcal{M}_n \to \mathcal{M}_n$ be the *chord separating operator in dimension n* defined by the formula

$$X_n : D \mapsto u \cdot \sum_{i=1}^{n} D_i .$$

Here D is an n-chord diagram, D_i is the $(n-1)$-chord diagram obtained from D by deleting its i-th chord; u is the (only) diagram with 1 chord.

We set

$$\mathrm{Rn}_n = \prod_{i=1}^{n} \left(1 - \frac{X_n}{i}\right)$$

and call the operator $\mathrm{Rn} : \mathcal{M} \to \mathcal{M}$ the *renormalization operator*.

Proposition.

1. *The operator* X *is a differentiation of the algebra* $\mathcal{M}$.
2. *The operator* Rn *is a homomorphism of Hopf algebras.*
3. *The operator* $\mathrm{Rn}^* : \mathcal{M}^* \to \mathcal{M}^*$ *is a projection onto* $\mathcal{N}^* \subset \mathcal{M}^*$.

Due to the last statement of the proposition, any pre-invariant $f \in \mathcal{M}^*$ can be easily converted into an invariant $\mathrm{Rn}^*(f) \in \mathcal{N}^*$.

For practical usage, the following formula for renormalization is more convenient.

Proposition [B-N]. *The value of the renormalization operator on a chord diagram D is given by the formula*

$$\mathrm{Rn}(D) = \sum_J (-u)^{|J'|} \cdot D_J,$$

where J runs over all subsets of chords of the diagram D, the diagram D_J consists of all chords from J, and $|J'|$ is the number of chords in the complement of J.

§3. Intersection graph of a chord diagram

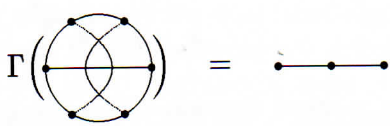
The notion of intersection graph of a chord diagram proves to be an effective tool both in obtaining estimates for the number of Vassiliev invariants and in constructing families of invariants. It turns out that many of the invariants previously constructed elsewhere depend only on the intersection graph of a diagram and not on the diagram itself.

Below, we only give the basic definitions and mention the main results. In two separate papers, [CDL2, CDL3], we present a detailed study of intersection graphs and of the related notions and constructions.

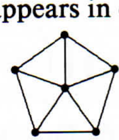
3.1. Definition and examples.

Definition. *The intersection graph* $\Gamma(D)$ of a chord diagram D is a graph whose vertices correspond to the chords of D and two vertices are connected by an edge iff the corresponding chords intersect. (Two chords, a and b, are said to intersect if their endpoints a_1, a_2 and b_1, b_2 appear in interchanging order a_1, b_1, a_2, b_2 along the circle.)

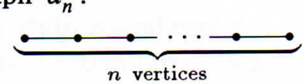
For example,



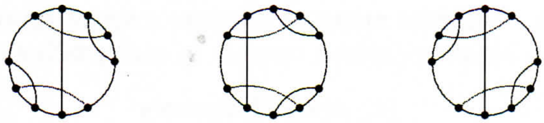
Note that not every graph can be obtained as the intersection graph of a diagram. The simplest example appears in degree 6 and is given by the graph



Different diagrams may have one and the same intersection graph. For example, consider the graph a_n:



The number of diagrams with this intersection graph is 2^{n-4} for even n, and $2^{n-4}+2^{\frac{n-5}{2}}$ for odd n. In particular, for $n=5$ there are three diagrams having a_n as intersection graph:



3.2. Intersection Graph Conjecture.

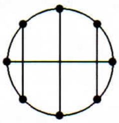
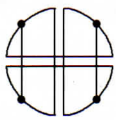
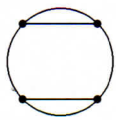
3.2.1. *Conjecture If D_1 and D_2 are two chord diagrams whose intersection graphs are the same, $\Gamma(D_1) = \Gamma(D_2)$, then $v([D_1]) = v([D_2])$ for any $v \in \mathcal{N}^*$.*

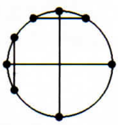
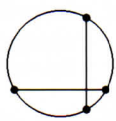
This conjecture has been verified in the following situations:

- for any $v \in \mathcal{N}_n^*$ up to $n \leq 8$ (we have checked this by a computer calculation);
- for any invariant v coming from the defining representations of Lie algebras $\mathfrak{gl}(N)$ or $\mathfrak{so}(N)$ according to D. Bar-Natan's construction [B-N];
- when $\Gamma(D_1) = \Gamma(D_2)$ is a tree (see [CDL2]) or, more generally, D_1, D_2 belong to the forest subalgebra (see [CDL3]).

REMARK. If the intersection graph conjecture is true, then Vassiliev knot invariants do not distinguish knots with opposite orientation (see [B-N]).

3.2.2. *Example. Coefficients of Conway polynomial as functions on the intersection graphs.* Here we prove that the value of c_n on a chord diagram depends only on the intersection graph of the diagram. We prove this by induction over the number of chords n. For $n = 1, 2$ the fact is obvious. For arbitrary n we have: $c_n(D) = c_{n-2}(D')$, where D' is obtained from D by doubling any two intersecting chords. In particular,

for $D =$  $D' =$  $=$ 

for $D =$  $D' =$  $=$ 

Now it is easy to see that $\Gamma(D')$ depends only on $\Gamma(D)$.

3.3. What are the subsequent papers about. In [CDL2] we investigate chord diagrams whose intersection graph is a forest. These diagrams generate a Hopf subalgebra in $\mathcal{M}$ called the *forest algebra.* We prove the Intersection Graph Conjecture for pairs of diagrams belonging to the forest subalgebra.

In [CDL3] we study the forest subalgebra and a certain quotient algebra of $\mathcal{M}$ (Hopf algebra of *weighted graphs*). These two algebras turn out to be isomorphic and their structure is rather simple: they have precisely one primitive element in every dimension. The restriction of the homomorphism from $\mathcal{M}$ onto the algebra of weighted graphs on the forest subalgebra is an isomorphism. The dual mapping provides a set of invariants for chord diagrams containing one primitive element in each grading of $\mathscr{M}^*$.

§4. Acknowledgments

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