ADVANCES IN SOVIET MATHEMATICS
Volume **21**, 1994

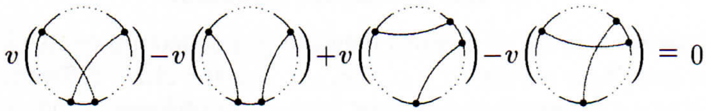
# Vassiliev Knot Invariants
# II. Intersection Graph Conjecture for Trees

S. V. CHMUTOV, S. V. DUZHIN, AND S. K. LANDO

The space of Vassiliev knot invariants has a natural filtration by order [V]. Kontsevich's theorem [K] gives a purely combinatorial description of the corresponding graded space as a space of functions $v$ ("invariants") on the set of chord diagrams satisfying certain linear equations:

one-term relations


and four-term relations




All such "invariants" form a Hopf algebra.

The intersection graph of a chord diagram is a rougher combinatorical object than the chord diagram itself, but it turns out that a lot of invariants depend only on the intersection graph. Here we prove that the one-term and four-term relations imply the coincidence of values of all invariants on two chord diagrams with the same intersection graph under the assumption that the intersection graph is a tree. A refined argument shows that this follows from the four-term relations alone.

Throughout this paper we use notations and definitions of the previous article [CDL1] which may be regarded as an introduction to the present text.

The Hopf algebra of invariants is dual to the Hopf algebra of chord diagrams (see 2.2 in [CDL1]), so the properties of either of them can be easily translated into the properties of the other. Below we shall be concerned with

Partially supported by an AMS fSU grant and a Soros Foundation grant.
Partially supported by a Soros Foundation grant.
Partially supported by an AMS fSU grant and a Soros Foundation grant.

the Hopf algebra of chord diagrams. In particular, an equality $D_1 = D_2$ will mean equality of the two diagrams in the Hopf algebra $\mathcal{N}$ (see [CDL1]), i. e. their equality modulo one- and four-term relations.

DEFINITION. The *intersection graph* $\Gamma(D)$ of a chord diagram $D$ is a graph whose vertices correspond to the chords of $D$ and two vertices are connected by an edge iff the corresponding chords intersect. (Two chords, $a$ and $b$, are said to intersect if their endpoints $a_1$, $a_2$ and $b_1$, $b_2$ appear along the circle in interchanging order $a$, $b$, $a$, $b$.)

DEFINITION. A chord diagram $D$ such that $\Gamma(D)$ is a tree will be called a *tree diagram.*

## §1. Main result

THEOREM. *If* $\Gamma(D_1) = \Gamma(D_2)$ *is a tree, then* $D_1 = D_2$.

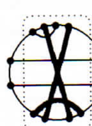
To prove the Theorem we shall describe the set of all chord diagrams with the same intersection tree graph. We shall introduce elementary transformations on chord diagrams with the property that any two diagrams with the same intersection tree graph can be connected by a sequence of these transformations. Then we shall prove that any two diagrams $D_1$ and $D_2$ differing by an elementary transformation are equal in the algebra of diagrams $\mathcal{N}$.

## §2. Key notions and statements

DEFINITION. A *share* of a chord diagram is a collection of its chords such that there exist four points $x_1$, $x_2$, $x_3$, $x_4$ on the circle different from the endpoints of all chords and satisfying the two conditions:

- no chord connects two adjacent arcs $x_1x_2$, $x_2x_3$, $x_3x_4$ or $x_4x_1$;
- no chord connects two adjacent arcs $x_1x_2$, both ends of any chord of the collection belong to the union of the two opposite arcs $x_1x_2$ and $x_3x_4$.

EXAMPLE. The collection of four chords inside the dotted oval



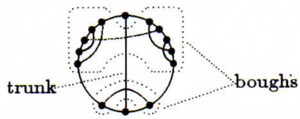
is a share. But the collection of three fat chords is not a share because there exists a fourth chord that separates their ends.

The collection formed by a single chord is always a share. The complement to a share is a share.

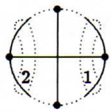
Here and below, we mark shares by dotted ovals.

DEFINITION. Pick a chord $t$ in a tree diagram $D$, and call $t$ the *trunk* of $D$. Consider the vertex $v(t)$ in $\Gamma(D)$ that corresponds to $t$. Then $\Gamma(D)\setminus v(t)$ is a forest. Each component of $\Gamma(D) \setminus v(t)$ is a tree. The collection of all chords $D$ that correspond to all vertices of one component of $\Gamma(D) \setminus v(t)$ is a share. This share will be called a *bough* of the tree diagram with respect to the chosen trunk $t$.

Example.



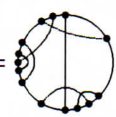

It is easy to see that each bough has the form of a "dumbbell" with two shares 1 and 2.




Definition. *An elementary transformation* of the tree diagram is a permutation of boughs with respect to some trunk.

Proposition 1. *If $D_1$, $D_2$ are tree diagrams with $\Gamma(D_1) = \Gamma(D_2)$, then $D_2$ can be obtained from $D_1$ by a sequence of elementary transformations.*

Example. For the diagrams

$D_1 =$  and $D_2 =$ 

$\Gamma(D_1) = \Gamma(D_2) =$ 

So $D_2$ can be obtained from $D_1$ by a sequence of elementary transformations (the corresponding trunks are shown in thick lines):

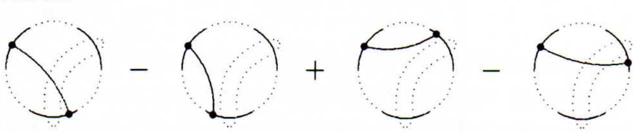
$D_1 =$  $= D_2$

Proposition 2. *If $D_1$ and $D_2$ are two tree diagrams that differ by a permutation of boughs, then $D_1 = D_2$.*

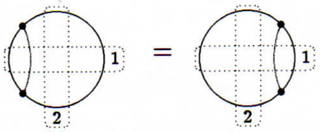
The main theorem (§1) is a consequence of Propositions 1 and 2.

In the proof of Proposition 2 we shall use the following statement and its corollary.

Statement (Generalized four-term relation). *For any share the following relation holds:*

 $= 0$

COROLLARY.




*Note that in the last figure there are no dotted arcs, i.e. each diagram consists of the two marked shares and one more chord.*

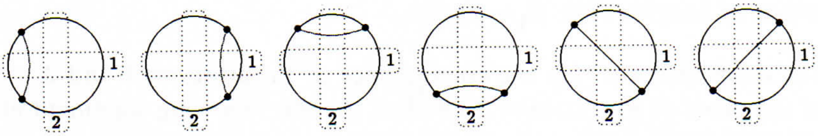
## §3. Proof of the Main Theorem

To prove the main theorem, it is sufficient to prove Propositions 1 and 2.

**3.1. Proof of Proposition 1.** We can choose trunks $t_1 \in D_1$ and $t_2 \in D_2$ that correspond to one and the same vertex $v = v(t_1) = v(t_2)$ in the tree $\Gamma = \Gamma(D_1) = \Gamma(D_2)$. There is a one-to-one correspondence between the sets of boughs of $D_1$ and $D_2$. Rearranging the boughs of $D_1$, we can dispose them on the trunk in the order of the corresponding boughs in the diagram $D_2$. Now, each bough is a tree diagram with a naturally distinguished trunk — the "handle" of the "dumbbell". So we can make a permutation of the new boughs with respect to a new trunk. Proceeding in this way, we obtain a sequence of elementary transformations that convert $D_1$ into $D_2$.

**3.2. Proof of the generalized four-term relation.** An ordinary four-term relation is determined by the choice of a chord and the choice of a point on the circle different from the endpoints of all chords. Given a share in a chord diagram and a fixed point $p$ outside of it, take all the chords of the share one by one and write the usual four-term relations for these chords and the point $p$. In the sum of all such equations, only four terms (those that give the generalized four-term relation) will survive, because every diagram with the variable endpoint of the additional chord inside the share occurs twice with opposite signs and thus cancels.

**3.3. Proof of the Corollary.** Of the six chords diagrams




$x$ $y$ $z$ $u$ $v$ $w$

four different generalized four-term relations can be composed. They can be written as

$$\begin{cases} x - v + z = 0, \\ y - w + z = 0, \\ y - v + u = 0, \\ x - w + u = 0, \end{cases}$$

where in each equation a one-term relation is taken into account. These equations imply $x = y$.

**3.4. Proof of Proposition 2.** The proof proceeds by induction with respect to a specially chosen parameter that will be called *complexity*.

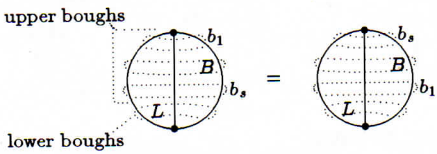
Let $\pi$ be the permutation of boughs of a tree diagram $D_1$ transforming it into $D_2$.

DEFINITION. Numerate the boughs from top to bottom along the trunk: $b_1, \dots, b_k$. Let $s$ be the minimum number such that $\pi$ acts identically on $b_{s+1}, \dots, b_k$. The *complexity* $c(D_1, \pi)$ of the pair $(D_1, \pi)$ is the total number of chords in the boughs $b_1, \dots, b_s$.

**Induction base.** If $c(D_1, \pi) = 2$, then $D_1 = D_2$

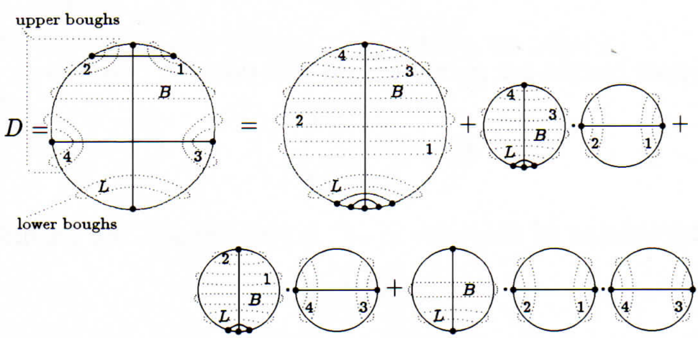
**Induction hypothesis.** Suppose that for $c(D_1, \pi) < m$ the statement of Proposition 2 holds.

**Induction step.** Let us prove the statement for $c(D_1, \pi) = m$. Since the permutation group is generated by transpositions of the first element with other elements of the underlying finite set, it is sufficient to prove the following fact:




where the total number of chords in the upper boughs is exactly $m$. Note that if both shares of the dumbbell $b_1$ are empty, then this fact follows from the Corollary and the induction hypothesis. In the general case we need the following lemma.

LEMMA 1. *Suppose that the statement of Proposition 2 holds for all pairs $(D_1, \pi)$ with $c(D_1, \pi) < m$. Then*



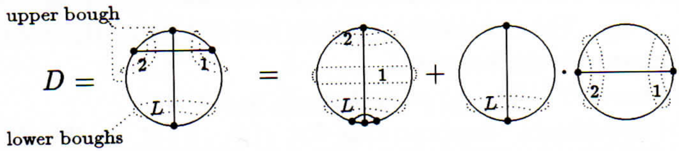

*for a tree diagram $D$ such that the total number of chords in the upper boughs is exactly $m$.*

The number of chords in the upper boughs of the big diagram in the right hand side is $m - 2$. By the induction hypothesis, the class of the diagram in $\mathcal{N}$ does not change under any permutation of its upper boughs. In particular, the right hand side stays invariant when the two pairs of shares $(1, 2)$ and

(3, 4) are swapped. Proposition 2 thus follows from Lemma 1.

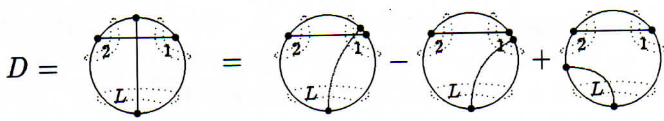
To prove Lemma 1, we need yet another lemma.

LEMMA 2. *Suppose that the statement of Proposition 2 holds for all pairs* $(D_1, \pi)$ *such that* $c(D_1, \pi) < m$. *Then*

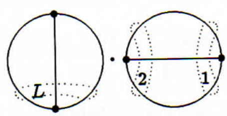



*for any tree diagram* $D$ *with the number of chords in the upper boughs less than* $m$.

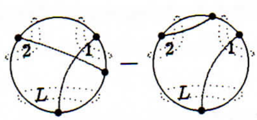
**3.5. Proof of Lemma 2.** By the generalized four-term relation (see §2)



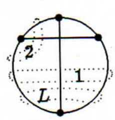
The last diagram is the product



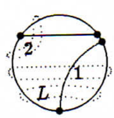
The difference of the first two diagrams on the right, by the generalized four-term relation, is equal to



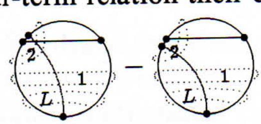
After its upper boughs are permuted, the first diagram becomes



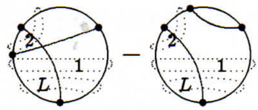
Since multiplication of diagrams in $\mathscr{N}$ is well defined, the second diagram is equal to



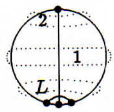
Therefore, by the usual four-term relation their difference equals



The generalized four-term relation applied to this difference yields

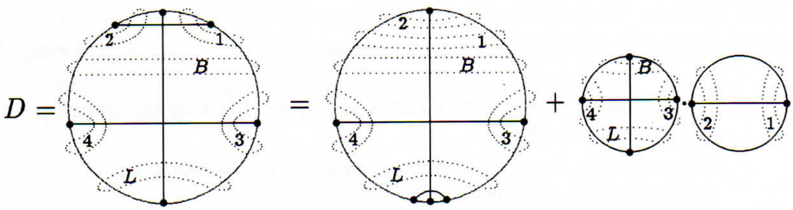


Here the last diagram vanishes by the one-term relation, while the first diagram is equal to

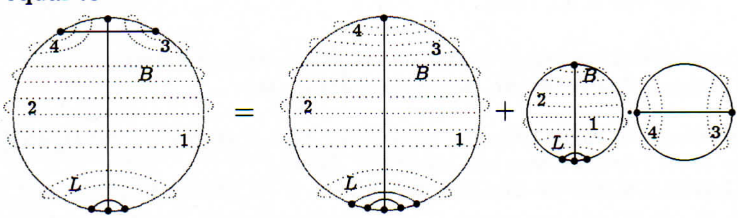


by the Corollary of the generalized four-term relation (see §2). This completes the proof of Lemma 2.

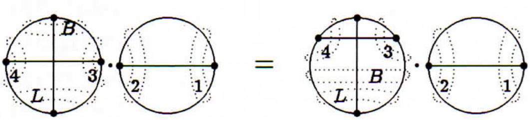
**3.6. Proof of Lemma 1.** By Lemma 2

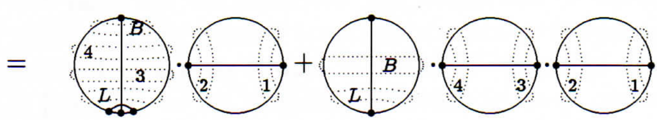


By the induction hypothesis and then by Lemma 2, the big diagram on the right is equal to



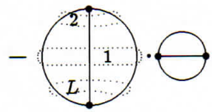
Applying the hypothesis and Lemma 2 once more, we obtain:





This completes the proof of Lemma 1, Proposition 2 and the Theorem.

REMARK. Now that Proposition 2 is proved, another look at Lemmas 1 and 2 shows that their statements hold without any assumptions on the tree diagram. Moreover, it is easy to see that a more careful argument gives the analogs of the main theorem, Proposition 2 and so on in the algebra $\mathscr{M}$ (see [CDL1]) of diagrams considered modulo only four-term relations. In this case in the right hand part of the equality of Lemma 2 the following additional term appears:





## Acknowledgments

In our first version of the proof of the main theorem we had only a topological proof of the Corollary from §2, which did not use the four-term relation. The present proof of the Corollary (sec. 3.3) is due to Don Zagier who suggested it during the Jerusalem combinatorics conference in May, 1993. We also thank A. Zvonkin, J. Birman and T. Stanford for useful discussions and interest in our work.


## References


[CDL1] S. V. Chmutov, S. V. Duzhin, and S. K. Lando, *Vassiliev knot invariants I. Introduction*, this volume.

[K] M. Kontsevich, *Feynman diagrams and Vassiliev invariants*, talk at the conference on quantum groups, Paris, July 1992.

[V] V. A. Vassiliev, *Cohomology of knot spaces*, Theory of Singularities and Its Applications (V. I. Arnold, ed.), Advances in Soviet Math., vol. 1, Amer. Math. Soc., Providence, RI, 1990.



Program Systems Institute, Pereslavl-Zalessky, 152140 Russia
*E-mail address*: chmutov@math.botik.yaroslavl.su

The University of Aizu, Tsuruga, Ikki-machi, Aizu-Wakamatsu, 965 Japan (On leave of absence from Program Systems Institute, Russia)
*E-mail address*: duzhin@u-aizu.ac.jp

Independent University of Moscow, Moscow Institute of New Technologies, Moscow, Russia
*E-mail address*: lando@math.imu.msk.su