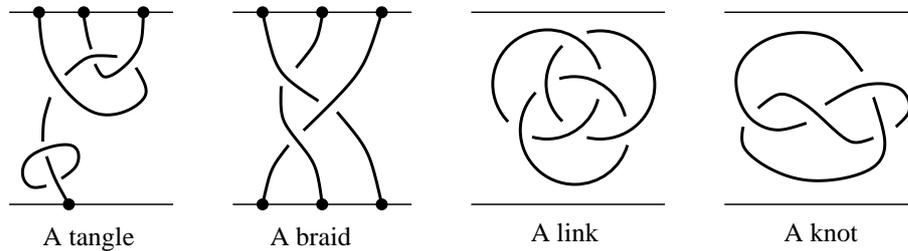# THE KONTSEVICH INTEGRAL

S. CHMUTOV, S. DUZHIN

1. INTRODUCTION

The Kontsevich integral was invented by M. Kontsevich [11] as a tool to prove the fundamental theorem of the theory of finite type (Vassiliev) invariants (see [1, 3]). It provides an invariant exactly as strong as the totality of all Vassiliev knot invariants.

The Kontsevich integral is defined for oriented tangles (either framed or unframed) in $\mathbb{R}^3$, therefore it is also defined in the particular cases of knots, links and braids.

| A tangle | A braid | A link | A knot |

As a starter, we give two examples where simple versions of the Kontsevich integral have a straightforward geometrical meaning. In these examples, as well as in the general construction of the Kontsevich integral, we represent 3-space $\mathbb{R}^3$ as the product of a real line $\mathbb{R}$ with coordinate t and a complex plane $\mathbb{C}$ with complex coordinate z.

Example 1. The number of twists in a braid with two strings $z_1(t)$ and $z_2(t)$ placed in the slice $0 \leq t \leq 1$ is equal to

$$\frac{1}{2\pi i} \int_0^1 \frac{dz_1 - dz_2}{z_1 - z_2}.$$
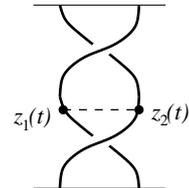
Example 2. The linking number of two spatial curves K and K' can be computed as

$$lk(K, K') = \frac{1}{2\pi i} \int_{m<t<M} \sum_j \varepsilon_j \frac{d(z_j(t) - z_j'(t))}{z_j(t) - z_j'(t)},$$

where m and M are the minimum and the maximum values of t on the link $K \cup K'$, j is the index that enumerates all possible choices of a pair of strands of the link as functions $z_j(t)$, $z_j'(t)$ corresponding to K and K', respectively, and $\varepsilon_j = \pm 1$ according to the parity of the number of chosen strands that are oriented downwards.
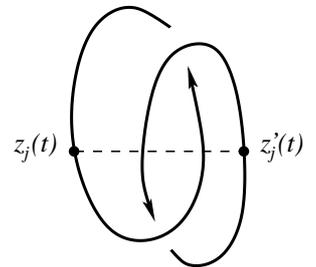
The Kontsevich integral can be regarded as a far-going generalization of these formulas. It aims at encoding all information about how the horizontal chords on the knot (or tangle) rotate when moved in the vertical direction. From a more general viewpoint, the Kontsevich integral represents the monodromy of the Knizhnik–Zamolodchikov connection in the complement to the union of diagonals in $\mathbb{C}^n$ (see [1, 17]).

2. Chord diagrams and weight systems

2.1. **Algebras $\mathcal{A}(p)$.** The Kontsevich integral of a tangle T takes values in the space of chord diagrams supported on T.

Let X be an oriented one-dimensional manifold, that is, a collection of p numbered oriented lines and q numbered oriented circles. A chord diagram of order n supported on X is a collection of n pairs of unordered points in X, considered up to an orientation- and component-preserving diffeomorphism. In the vector space formally generated by all chord diagrams of order n we distinguish the subspace spanned by all *four-term relations*

where thin lines designate chords, while thick lines are pieces of the manifold X. Apart from the shown fragments, all the four diagrams are identical. The quotient space over all such combinations will be denoted by $\mathcal{A}_n(X) = \mathcal{A}_n(p, q)$. Let $\mathcal{A}(p, q) = \oplus_{n=0}^{\infty} \mathcal{A}_n(p, q)$ and let $\hat{\mathcal{A}}(p, q)$ be the graded completion of $\mathcal{A}(p, q)$ (i.e. the space of formal infinite series $\sum_{i=0}^{\infty} a_i$ with $a_i \in \mathcal{A}_i(p, q)$. If, moreover, we divide $\mathcal{A}(p, q)$ by all *framing independence* relations (any diagram with an isolated chord, i.e. a chord joining two adjacent points of the same connected component of X, is set to 0), then the resulting space is denoted by $\mathcal{A}'(p, q)$, and its graded completion by $\hat{\mathcal{A}}'(p, q)$.

The spaces $\mathcal{A}(p, 0) = \mathcal{A}(p)$ have the structure of an algebra (the product of chord diagrams is defined by concatenation of underlying manifolds in agreement with the orientation). Closing a line component into a circle, we get a linear map $\mathcal{A}(p, q) \to \mathcal{A}(p - 1, q + 1)$ which is an isomorphism when $p = 1$. In particular, $\mathcal{A}(S^1) \cong \mathcal{A}(\mathbb{R}^1)$ has the structure of an algebra; this algebra is denoted simply by $\mathcal{A}$; the Kontsevich integral of knots takes its values in its graded completion $\hat{\mathcal{A}}$. Another algebra of special importance is $\hat{\mathcal{A}}(3) = \hat{\mathcal{A}}(3, 0)$, because it is where the Drinfeld associators live.

2.2. **Hopf algebra structure.** The algebra $\mathcal{A}(p)$ has a natural structure of a Hopf algebra with the coproduct δ defined by all ways to split the set of chords into two disjoint parts. To give a convenient description of its primitive space, one can use generalized chord diagrams. We now allow trivalent vertices not belonging to the supporting manifold and use STU relations

to express the generalized diagrams as linear combinations of conventional chord diagrams, e.g.

Then the primitive space coincides with the subspace of $\mathcal{A}(p)$ spanned by all connected generalized chord diagrams (*connected* means that they remain connected when the supporting manifold X is disregarded).

2.3. Weight systems.

A *weight system* of degree n is a linear function on the space $\mathcal{A}_n$. Every Vassiliev invariant v of degree n defines a weight system $\mathrm{symb}(v)$ of the same degree called its *symbol*.

2.4. Algebras $\mathcal{B}(p)$.

Apart from the spaces of chord diagrams modulo four-term relations, there are closely related spaces of Jacobi diagrams. A *Jacobi diagram* is defined as a uni-trivalent graph, possibly disconnected, having at least one vertex of valency 1 in each connected component and supplied with two additional structures: a cyclic order of edges in each trivalent vertex and a labelling of univalent vertices taking values in the set $\{1, 2, \ldots, p\}$. The space $\mathcal{B}(p)$ is defined as the quotient of the vector space formally generated by all p-coloured Jacobi diagrams modulo the two types of relations:

The disjoint union of Jacobi diagrams makes the space $\mathcal{B}(p)$ into an algebra.

The symmetrization map $\chi_p : \mathcal{B}(p) \to \mathcal{A}(p)$, defined as the average over all ways to attach the legs of colour i to i-th connected component of the underlying manifold:

is an isomorphism of vector spaces (the formal PBW isomorphism [1, 15]) which is not compatible with the multiplication. The relation between $A(p)$ and $\mathcal{B}(p)$ very much resembles the relation between the universal enveloping algebra and the symmetric algebra of a Lie algebra. The algebra $\mathcal{B} = \mathcal{B}(1)$ is used to write out the explicit formula for the Kontsevich integral of the unknot (see [4] and below).

3. THE CONSTRUCTION

3.1. Kontsevich's formula.

We will explain the construction of the Kontsevich integral in the classical case of (closed) oriented knots; for an arbitrary tangle T the formula is the same, only the result is interpreted as an element of $\hat{\mathcal{A}}(T)$. As above, represent three-dimensional space $\mathbb{R}^3$ as a direct product of a complex line $\mathbb{C}$ with coordinate z and a real line $\mathbb{R}$ with coordinate t. The integral is defined for Morse knots, i. e. knots K embedded in $\mathbb{R}^3 = \mathbb{C}_z \times \mathbb{R}_t$ in such a way that the coordinate t restricted to K has only nondegenerate (quadratic) critical points. (In fact, this condition can be weakened, but the class of Morse knots is broad enough and convenient to work with.)

The Kontsevich integral $Z(K)$ of the knot K is the following element of the completed algebra $\hat{\mathcal{A}}'$:

$$Z(K) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int_{\substack{t_{\min}<t_m<\cdots<t_1<t_{\max} \\ t_j \text{ are noncritical}}} \sum_{P=\{(z_j, z_j')\}} (-1)^{\downarrow_P} D_P \bigwedge_{j=1}^{m} \frac{dz_j - dz_j'}{z_j - z_j'} \ .$$

3.2. **Explanation of the constituents.** The real numbers $t_{\min}$ and $t_{\max}$ are the minimum and the maximum of the function t on K.

The integration domain is the m-dimensional simplex $t_{\min} < t_m < \cdots < t_1 < t_{\max}$ divided by the critical values into a certain number of *connected components*. For example, the following picture shows an embedding of the unknot where, for $m = 2$, the integration domain has six connected components:
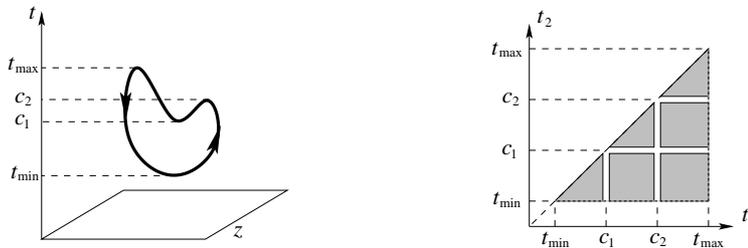
The number of summands in the integrand is constant in each connected component of the integration domain, but can be different for different components. In each plane $\{t = t_j\} \subset \mathbb{R}^3$ choose an unordered pair of distinct points (z_j, t_j) and (z_j', t_j) on K, so that $z_j(t_j)$ and $z_j'(t_j)$ are continuous branches of the knot. We denote by $P = \{(z_j, z_j')\}$ the collection of such pairs for $j = 1, \ldots, m$. The integrand is the sum over all choices of the pairing P. In the example above for the component $\{t_{\min} < t_1 < c_1, c_2 < t_2 < t_{\max}\}$ we have only one possible pair of points on the levels $\{t = t_1\}$ and $\{t = t_2\}$. Therefore, the sum over P for this component consists of only one summand. Unlike this, in the component $\{t_{\min} < t_1 < c_1, c_1 < t_2 < c_2\}$ we still have only one possibility for the level $\{t = t_1\}$, but the plane $\{t = t_2\}$ intersects our knot K in four points. So we have $\binom{4}{2} = 6$ possible pairs (z_2, z_2') and the total number of summands is six (see the picture below).

For a pairing P the symbol '$\downarrow_P$' denotes the number of points (z_j, t_j) or (z_j', t_j) in P where the coordinate t decreases along the orientation of K.

Fix a pairing P. Consider the knot K as an oriented circle and connect the points (z_j, t_j) and (z_j', t_j) by a chord. Up to a diffeomorphism, this chord does not depend on the value of t_j within a connected component. We obtain a chord diagram with m chords. The corresponding element of the algebra $\mathcal{A}'$ is denoted by D_P. In the picture below, for each connected component in our example, we show one of the possible pairings, the corresponding chord diagram with the sign $(-1)_P^{\downarrow}$ and the number of summands of the integrand (some of which are equal to zero in $\mathcal{A}'$ due to the framing independence relation).

Over each connected component, z_j and z_j' are smooth functions of t_j.

By $\bigwedge\limits_{j=1}^{m} \dfrac{dz_j - dz_j'}{z_j - z_j'}$ we mean the pullback of this form to the integration domain of variables $t_1, \ldots, t_m$. The integration domain is considered with the orientation of the space $\mathbb{R}^m$ defined by the natural order of the coordinates $t_1, ..., t_m$.

By convention, the term in the Kontsevich integral corresponding to $m = 0$ is the (only) chord diagram of order 0 with coefficient one. It represents the unit of the algebra $\mathcal{A}'$.

3.3. **Framed version of the Kontsevich integral.** Let K be a framed oriented Morse knot with writhe number $w(K)$. Denote the corresponding knot without framing by $\bar{K}$. The framed version of the Kontsevich integral can be defined by the formula

$$Z^{\mathrm{fr}}(K) = e^{\frac{w(K)}{2}\Theta} \cdot Z(\bar{K}) \in \hat{\mathcal{A}},$$

where Θ is the chord diagram with one chord and the integral $Z(\bar{K}) \in \hat{\mathcal{A}}'$ is understood as an element of the completed algebra $\hat{\mathcal{A}}$ (without 1-term relations) by virtue of a natural inclusion $\mathcal{A}' \to \mathcal{A}$ defined as identity on the primitive subspace of $\mathcal{A}'$.

See [8, 15] for other approaches.

4. BASIC PROPERTIES

4.1. **Constructing the universal Vassiliev invariant.** The Kontsevich integral $Z(K)$

(1) converges for any Morse knot K,
(2) is invariant under deformations of the knot in the class of Morse knots,
(3) behaves in a predictable way under the deformation that adds a pair of new critical points to a Morse knot:

$$Z\left(\begin{array}{c}\end{array}\right) = Z(H) \cdot Z\left(\begin{array}{c}\end{array}\right).$$

Here the first and the third pictures depict two embeddings of an arbitrary knot, differing only in the shown fragment, $H = \bigcup$ is the *hump* (unknot embedded in $\mathbb{R}^3$ in the specified way), and the product is the product in the completed algebra $\hat{\mathcal{A}}'$ of

chord diagrams. The last equality allows one to define a genuine knot invariant by
the formula

$$I(K) = Z(K)/Z(H)^{c/2},$$

where c denotes the number of critical points of K and the ratio means the division
in the algebra $\hat{\mathcal{A}}'$ according to the rule $(1 + a)^{-1} = 1 - a + a^2 - a^3 + \dots$

The expression $I(K)$ is sometimes referred to as the 'final' Kontsevich integral
as opposed to the 'preliminary' Kontsevich integral $Z(K)$. It represents a universal
Vassiliev invariant in the following sense: *Let w be a weight system, i.e. a linear
functional on the algebra $\hat{\mathcal{A}}'$. Then the composition $w(I(K))$ is a numerical Vassiliev
invariant, and any Vassiliev invariant can be obtained in this way.*

The final Kontsevich integral for framed knots is defined in the same way, using
the hump H with zero writhe number.

4.2. Is universal Vassiliev invariant universal? At present, it is not known
whether the Kontsevich integral separates knots, or even if it can tell the orientation
of a knot. However, the corresponding problem is solved, in the affirmative, in the
case of braids and string links (theorem of Kohno–Bar-Natan ([2, 10]).

4.3. Omitting long chords. We will state a technical lemma which is highly im-
portant in the study of the Kontsevich integral. It is used in the proof of the multi-
plicativity, in the combinatorial construction etc.

Suppose we have a Morse knot K with a distinguished tangle T.
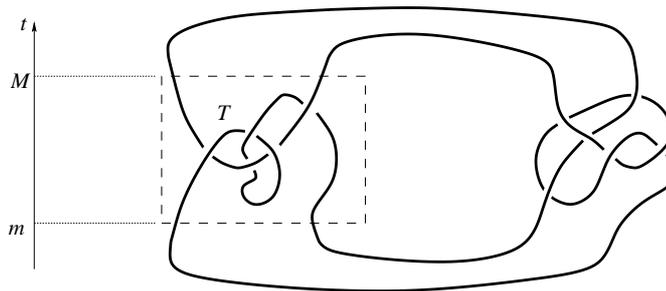
Let m and M be the maximal and minimal values of t on the tangle T. In the
horizontal planes between the levels m and M we can distinguish two kinds of chord:
'short' chords that lie either inside T or inside $K \setminus T$, and 'long' chords that connect
a point in T with a point in $K \setminus T$. Denote by $Z_T(K)$ the expression defined by the
same formula as the Kontsevich integral $Z(K)$ where only short chords are taken into
consideration. More exactly, if C is a connected component of the integration domain
(see Section 3) whose projection on the coordinate axis t_j is entirely contained in the
segment $[m, M]$, then in the sum over the pairings P we include only those pairings
that include short chords.

Lemma. 'Long' chords can be omitted when computing the Kontsevich integral:
$Z_T(K) = Z(K)$.

4.4. Kontsevich's integral and operations on knots. The Kontsevich integral
behaves in a nice way with respect to the natural operations on knots, such as mirror
reflection, changing the orientation of the knot, mutation of knots (see [5]), cabling
(see [20]). We give some details regarding the first two items.

Fact 1. Let R be the operation that sends a knot to its mirror image. Define the corresponding operation R on chord diagrams as multiplication by $(-1)^n$ where n is the order of the diagram. Then the Kontsevich integral commutes with the operation R: $Z(R(K)) = R(Z(K))$, where by $R(Z(K))$ we mean simultaneous application of R to all the chord diagrams participating in $Z(K)$.

Corollary. The Kontsevich integral $Z(K)$ and the universal Vassiliev invariant $I(K)$ of an amphicheiral knot K consist only of even order terms. (A knot K is called *amphicheiral*, if it is equivalent to its mirror image: $K = R(K)$.)

Fact 2. Let S be the operation on knots which inverts their orientation. The same letter will also denote the analogous operation on chord diagrams (inverting the orientation of the outer circle or, which is the same thing, axial symmetry of the diagram). Then the Kontsevich integral commutes with the operation S of inverting the orientation: $Z(S(K)) = S(Z(K))$.

Corollary. The following two assertions are equivalent:
— Vassiliev invariants do not distinguish the orientation of knots,
— all chord diagrams are symmetric: $D = S(D)$ modulo four-term relations.

The calculations of [9] show that up to order 12 all chord diagrams are symmetric. For bigger orders the problem is still open.

4.5. Multiplicative properties.
The Kontsevich integral for tangles is multiplicative:

$$Z(T_1) \cdot Z(T_2) = Z(T_1 \cdot T_2)$$

whenever the product $T_1 \cdot T_2$, defined by vertical concatenation of tangles, exists. Here the product in the left-hand side is understood as the image of the element $Z(T_1) \otimes Z(T_2)$ under the natural map $\mathcal{A}(T_1) \otimes \mathcal{A}(T_2) \to \mathcal{A}(T_1 \cdot T_2)$.

This simple fact has two important corollaries:

(1) For any knot K the Kontsevich integral $Z(K)$ is a group-like element of the Hopf algebra $\hat{\mathcal{A}}'$, i.e.

$$\delta(Z(K)) = Z(K) \otimes Z(K),$$

where δ is the comultiplication in $\mathcal{A}$ defined above.

(2) The final Kontsevich integral, taken in a different normalization

$$I'(K) = Z(H)I(K) = \frac{Z(K)}{Z(H)^{c/2-1}}.$$

is multiplicative with respect to the connected sum of knots:

$$I'(K_1 \# K_2) = I'(K_1)I'(K_2),$$

4.6. Arithmetical properties.
For any knot K the coefficients in the expansion of $Z(K)$ over an arbitrary basis consisting of chord diagrams are rational (see [11, 15] and below).

5. Combinatorial construction of the Kontsevich integral

5.1. Sliced presentation of knots.
The idea is to cut the knot into a number of standard simple tangles, compute the Kontsevich integral for each of them and then recover the integral of the whole knot from these simple pieces.

More exactly, we represent the knot by a family of plane diagrams continuously depending on a parameter $\varepsilon \in (0, \varepsilon_0)$ and cut by horizontal planes into a number of slices with the following properties.

(1) At every boundary level of a slice (dashed lines in the pictures below) the distances between various strings are asymptotically proportional to different whole powers of the parameter ε.

(2) Every slice contains exactly one special event and several strictly vertical strings which are farther away (at lower powers of ε) from any string participating in the event than its width.

(3) There are three types of special events:

min/max: $m =$ ⌣ $M =$ ⌢

braiding: $B_+ =$ ⤬ $B_- =$ ⤬

associativity: $A_+ =$ ⫿ $A_- =$ ⫿
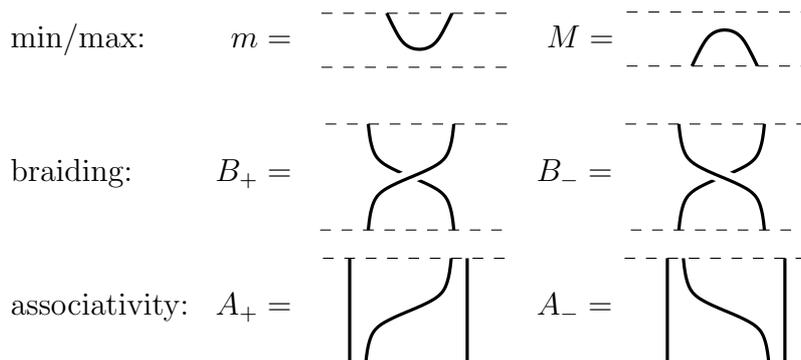
where, in the two last cases, the strings may be replaced by bunches of parallel strings which are closer to each other than the width of this event.

5.2. **Recipe of computation of the Kontsevich integral.** Given such a sliced representation of a knot, the combinatorial algorithm to compute its Kontsevich integral consists in the following:

(1) Replace each special event by a series of chord diagrams supported on the corresponding tangle according to the rule

$$
\begin{aligned}
m, M &\mapsto 1, \\
B_+ &\mapsto R, \quad B_+ \mapsto R^{-1}, \\
A_+ &\mapsto \Phi, \quad A_- \mapsto \Phi^{-1},
\end{aligned}
$$

where

$$
R = \text{╱╲} \cdot \exp\left(\frac{\text{╠═╣}}{2}\right) = \text{╱╲} + \frac{1}{2}\text{╱╲} + \frac{1}{2 \cdot 2^2}\text{╱╲} + \frac{1}{3! \cdot 2^3}\text{╱╲} + \dots
$$

$$
\Phi = 1 - \frac{\zeta(2)}{(2\pi i)^2}[a, b] - \frac{\zeta(3)}{(2\pi i)^3}([a, [a, b]] + [b, [a, b]]) + \dots
$$

($\Phi \in \hat{\mathcal{A}}(3)$ is the Knizhnik-Zamolodchikov Drinfeld associator defined below; it is an infinite series in two variables $a = $ ⫿⫿↑, $b = $ ↑⫿⫿).

(2) Compute the product of all these series from top to bottom taking into account the connection of the strands of different tangles, thus obtaining an element of the algebra $\hat{\mathcal{A}}'$.

To accomplish the algorithm, we need two auxiliary operations on chord diagrams:

(1) $S_i : \mathcal{A}(p) \to \mathcal{A}(p)$ defined as multiplication by $(-1)^k$ on a chord diagram containing k endpoints of chords on the string number i. This is the correction

term in the computation of R and Φ in the case when the tangle contains some strings oriented downwards (the upwards orientation is considered as positive).

(2) $\Delta_i : \mathcal{A}(p) \to \mathcal{A}(p+1)$ acts on a chord diagram D by doubling the i-th string of D and taking the sum over all possible lifts of the endpoints of chords of D from the i-th string to one of the two new strings. The strings are counted by their bottom points from left to right. This operation can be used to express the combinatorial Kontsevich integral of a generalized associativity tangle (with strings replaced by bunches of strings) in terms of the combinatorial Kontsevich integral of a simple associativity tangle.

5.3. Example. Using the combinatorial algorithm, we compute the Kontsevich integral of the trefoil knot 3_1 to the terms of degree 2. A sliced presentation for this knot shown in the picture implies that $Z(3_1) = S_3(\Phi)R^{-3}S_3(\Phi^{-1})$ (here the product from left to right corresponds to the multiplication of tangles from top to bottom). Up to degree 2, we have: $\Phi = 1 + \frac{1}{24}[a,b] + \ldots$, $R = X(1 + \frac{1}{2}a + \frac{1}{8}a^2 + \ldots)$, where X means that the two strands in each term of the series must be crossed over at the top. The operation S_3 changes the orientation of the third strand, which means that $S_3(a) = a$ and $S_3(b) = -b$. Therefore, $S_3(\Phi) = 1 - \frac{1}{24}[a,b] + \ldots$, $S_3(\Phi^{-1}) = 1 + \frac{1}{24}[a,b] + \ldots$, $R^{-3} = X(1 - \frac{3}{2}a + \frac{9}{8}a^2 + \ldots)$ and $Z(3_1) = (1 - \frac{1}{24}[a,b] + \ldots)X(1 - \frac{3}{2}a + \frac{9}{8}a^2 + \ldots)(1 + \frac{1}{24}[a,b] + \ldots) = 1 - \frac{3}{2}Xa - \frac{1}{24}abX + \frac{1}{24}baX + \frac{1}{24}Xab - \frac{1}{24}Xba + \frac{9}{8}Xa^2 + \ldots$ Closing these diagrams into the circle, we see that in the algebra $\mathcal{A}$ we have $Xa = 0$ (by the framing independence relation), then $baX = Xab = 0$ (by the same relation, because these diagrams consist of two parallel chords) and $abX = Xba = Xa^2 = \bigotimes$. The result is: $Z(3_1) = 1 + \frac{25}{24}\bigotimes + \ldots$. The final Kontsevich integral of the trefoil (in the multiplicative normalization, see page 7) is thus equal to $I'(3_1) = Z(3_1)/Z(H) = (1 + \frac{25}{24}\bigotimes + \ldots)/(1 + \frac{1}{24}\bigotimes + \ldots) = 1 + \bigotimes + \ldots$
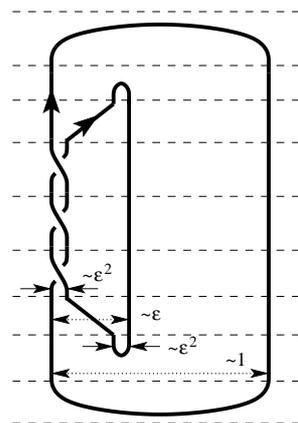
5.4. Drinfeld associator and rationality. The Drinfeld associator used as a building block in the combinatorial construction of the Kontsevich integral, can be defined as the limit

$$\Phi_{\mathrm{KZ}} = \lim_{\varepsilon \to 0} \varepsilon^{-b} Z(AT_\varepsilon)\varepsilon^a,$$

where $a = $ ↑↑↑, $b = $ ↑↑↑), and AT_ε is the positive associativity tangle (special event A_+ shown above) with the distance between the vertical strands constant 1 and the distance between the close endpoints equal to ε. An explicit formula for Φ_{KZ} was found by Le and Murakami [15]; it is written as a nested summation over four variable multiindices and therefore does not provide an immediate insight into the structure of the whole series; we confine ourselves by quoting the beginning of the series (note that Φ_{KZ} is a group-like element in the free associative algebra with 2 generators,

hence its logarithm belongs to the corresponding free Lie algebra):

$$\log(\Phi_{KZ}) = -\zeta(2)[x, y] - \zeta(3)([x, [x, y]] + [y, [x, y]])$$
$$- \frac{\zeta(2)^2}{10}(4[x, [x, [x, y]]] + [y, [x, [x, y]]] + 4[y, [y, [x, y]]])$$
$$- \zeta(5)([x, [x, [x, [x, y]]]] + [y, [y, [y, [x, y]]]])$$
$$+ (\zeta(2)\zeta(3) - 2\zeta(5))([y, [x, [x, [x, y]]]] + [y, [y, [x, [x, y]]]])$$
$$+ (\frac{1}{2}\zeta(2)\zeta(3) - \frac{1}{2}\zeta(5))[[x, y], [x, [x, y]]]$$
$$+ (\frac{1}{2}\zeta(2)\zeta(3) - \frac{3}{2}\zeta(5))[[x, y], [y, [x, y]]] + \dots$$

where $x = \frac{1}{2\pi i}a$ and $y = \frac{1}{2\pi i}b$. In general, Φ_{KZ} is an infinite series whose coefficients are *multiple zeta values* ([15, 22])

$$\zeta(a_1, \dots, a_n) = \sum_{0 < k_1 < k_2 < \dots < k_n} k_1^{-a_1} \dots k_n^{-a_n}.$$

There are other equivalent definitions of Φ_{KZ}, in particular one in terms of the asymptotical behaviour of solutions of the simplest Knizhnik–Zamolodchikov equation

$$\frac{dG}{dz} = \left(\frac{a}{z} + \frac{b}{z-1}\right)G,$$

where G is a function of a complex variable taking values in the algebra of series in two non-commuting variables a and b (see [7]).

It turns out (theorem of Le and Murakami [15]) that the combinatorial Kontsevich integral does not change if Φ_{KZ} is replaced by another series in $\hat{\mathcal{A}}(3)$ provided it satisfies certain axioms (among which the pentagon and hexagon relations are the most important, see [7, 15]).

Drinfeld [7] proved the existence of an associator $\Phi_{\mathbb{Q}}$ with rational coefficients. Using it instead of Φ_{KZ} in the combinatorial construction, we obtain the following

Theorem. ([15]) The coefficients of the Kontsevich integral of any knot (tangle) are rational when $Z(K)$ is expanded over an arbitrary basis consisting of chord diagrams.

6. Explicit formulas for the Kontsevich integral

6.1. **The wheels formula.** Let O be the unknot; the expression $I(O) = Z(H)^{-1}$ is referred to as the *Kontsevich integral of the unknot*. A closed form formula for $I(O)$ was proved in [4]:

Theorem.

$$I(O) = \exp \sum_{n=1}^{\infty} b_{2n} w_{2n} = 1 + (\sum_{n=1}^{\infty} b_{2n} w_{2n}) + \frac{1}{2}(\sum_{n=1}^{\infty} b_{2n} w_{2n})^2 + \dots$$

Here b_{2n} are modified Bernoulli numbers, i.e. the coefficients of the Taylor series

$$\sum_{n=1}^{\infty} b_{2n} x^{2n} = \frac{1}{2} \ln \frac{e^{x/2} - e^{-x/2}}{x},$$

($b_2 = 1/48$, $b_4 = -1/5760$, $b_6 = 1/362880, \dots$), and w_{2n} are the *wheels*, i. e. Jacobi diagrams of the form

$$w_2 = \quad , \quad w_4 = \quad , \quad w_6 = \quad , \quad \dots$$

The sums and products are understood as operations in the algebra of Jacobi diagrams $\mathcal{B}$, and the result is then carried over to the algebra of chord diagrams $\mathcal{A}$ along the isomorphism χ (see Section 2).

6.2. **Generalizations.** There are several generalizations of the wheels formula:

1. Rozansky's *rationality conjecture* [18] proved by A.Kricker [12] affirms that the Kontsevich integral of any (framed) knot can be written in a form resembling the wheels formula. Let us call the *skeleton* of a Jacobi diagram the regular 3-valent graph obtained by 'shaving off' all univalent vertices. Then the wheels formula says that all diagrams in the expansion of $I(O)$ have one and the same skeleton (circle), and the generating function for the coefficients of diagrams with n legs is a certain analytic function, more or less rational in e^x. In the same way, the theorem of Rozansky and Kricker states that the terms in $I(K) \in \hat{\mathcal{B}}$, when arranged by their skeleta, have the generating functions of the form $p(e^x)/A_K(e^x)$, where A_K is the Alexander polynomial of K and p is some polynomial function. Although this theorem does not give an explicit formula for $I(K)$, it provides a lot of information about the structure of this series.

2. J.Marché [16] gives a closed form formula for the Kontsevich integral of torus knots $T(p, q)$.

The formula of Marché, although explicit, is rather intricate, and here, by way of example, we only write out the first several terms of the final Kontsevich integral I' for the trefoil (torus knot of type (2,3)), following [21]:

$$I'(\quad) = \bigcirc - \quad + \quad - \frac{31}{24} \quad + \frac{5}{24} \quad + \frac{1}{2} \quad + \dots$$

6.3. **First terms of the Kontsevich integral.** A Vassiliev invariant v of degree n is called *canonical* if it can be recovered from the Kontsevich integral by applying a homogeneous weight system, i.e. if $v = \mathrm{symb}(v) \circ I$. Canonical invariants define a grading in the filtered space of Vassiliev invariants which is consistent with the filtration. If the Kontsevich integral is expanded over a fixed basis in the space of chord diagrams $\hat{\mathcal{A}}'$, then the coefficient of every diagram is a canonical invariant. According to [19, 21], the expansion of the final Kontsevich integral up to degree 4 can be written as follows:

$$I'(K) = \bigcirc - c_2(K) \quad - \frac{1}{6} j_3(K) \quad$$
$$+ \frac{1}{48} \left(4j_4(K) + 36c_4(K) - 36c_2^2(K) + 3c_2(K) \right) \quad$$
$$+ \frac{1}{24} \left(-12c_4(K) + 6c_2^2(K) - c_2(K) \right) \quad$$
$$+ \frac{1}{2} c_2^2(K) \quad + \dots$$

where c_n are coefficients of the Conway polynomial $\nabla_K(t) = \sum c_n(K)t^n$ and j_n are modified coefficients of the Jones polynomial $J_K(e^t) = \sum j_n(K)t^n$. Therefore, up to degree 4, the basic canonical Vassiliev invariants of unframed knots are c_2, j_3, j_4, $c_4 + \frac{1}{6}c_2$ and c_2^2.

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