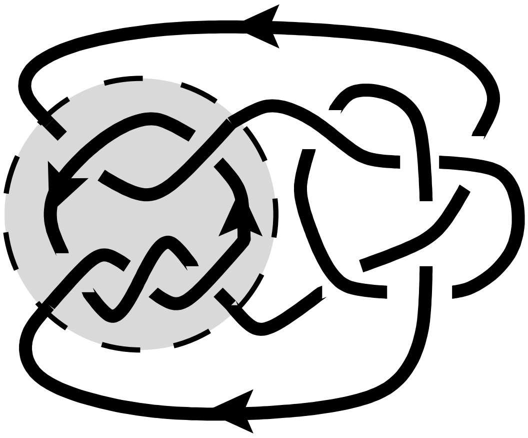
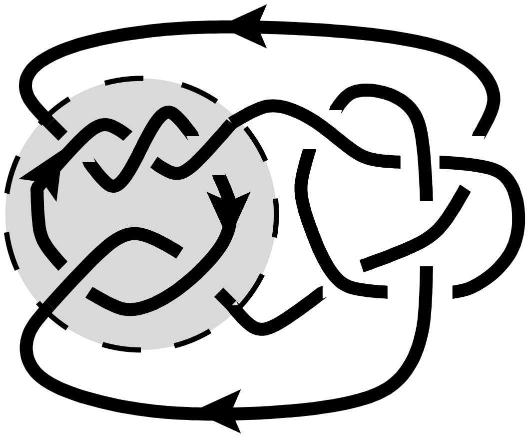
Mutant knots.



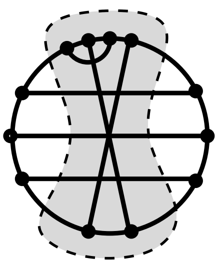
Conway
(11n34, genus $=3$)



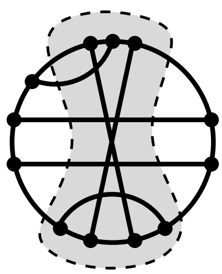
Kinoshita--Terasaka
(11n42, genus $=2$)

Mutant chord diagrams.

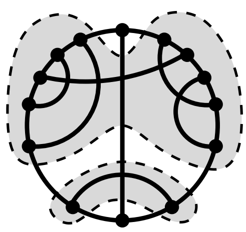
A *share* in a chord diagram is a union of two arcs of the outer circle and chords ending on them such that every chord one of whose ends belongs to these arcs has both ends on these arcs.



A share

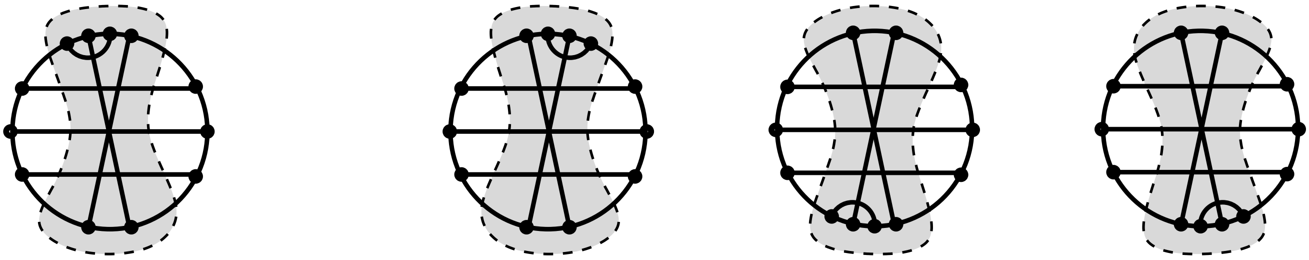


Not a share

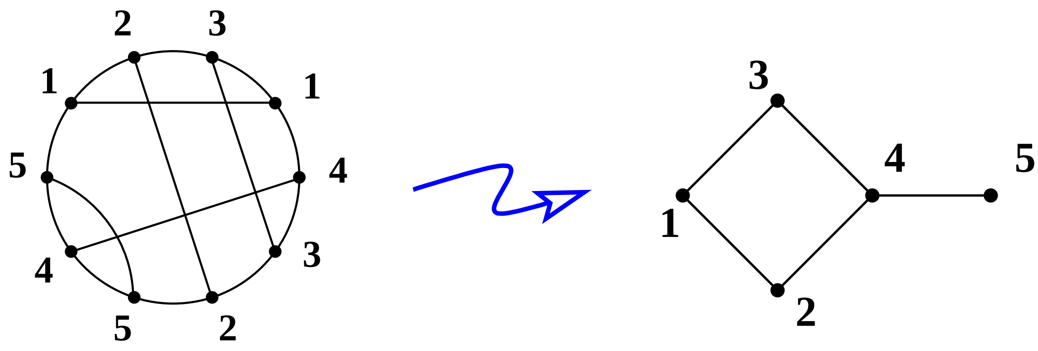


Two shares

Definition. A *mutation of a chord diagram* is another chord diagram obtained by a rotation (reflection) of a share.



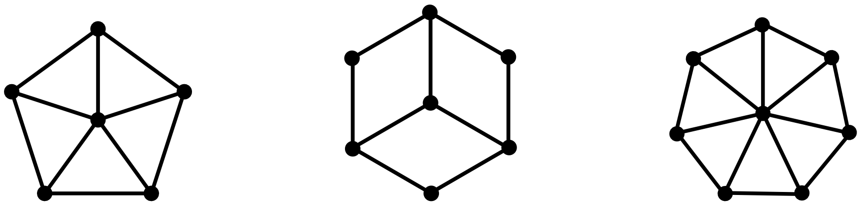
Intersection graph of a chord diagram



chord $\longrightarrow$ vertex

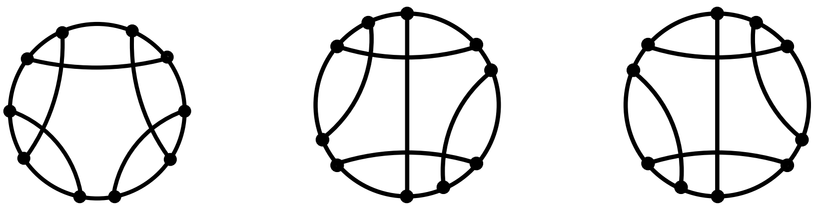
intersection $\longrightarrow$ edge

Not every graph is the intersection graph of a chord diagram:



Three diagrams with the same intersection graph

 :



Theorem. *Two chord diagrams have the same intersection graph if and only if they are related by a sequence of mutations.*

A *split* of a (simple) graph Γ is a disjoint bipartition $\{V_1, V_2\}$ of its set of vertices $V(\Gamma)$ such that each part contains at least 2 vertices, and there are subsets $W_1 \subseteq V_1$, $W_2 \subseteq V_2$ possessing the following property: all the edges of Γ connecting V_1 with V_2 form the complete bipartite graph $K(W_1, W_2)$ with the parts W_1 and W_2.

A *prime* graph is a graph with at least three vertices admitting no splits.

A. Bouchet [Bu, Statement 4.4], and C. P. Gabor, K. J. Supowit, W.-L. Hsu [GSH, Section 6]: *There is a unique way to realize a prime graph intersection graph by a chord diagram.*

Consider two graphs Γ_1 and Γ_2 each having a distinguished vertex $v_1 \in V(\Gamma_1)$ and $v_2 \in V(\Gamma_2)$, respectively, called *markers*. Construct the new graph $\Gamma = \Gamma_1 \boxtimes_{(v_1,v_2)} \Gamma_2$ whose set of vertices is

$$V(\Gamma) = \{V(\Gamma_1) - v_1\} \cup \{V(\Gamma_2) - v_2\}$$

and whose set of edges is

$$\begin{aligned} E(\Gamma) = {} & \{(v_1', v_1'') \in E(\Gamma_1) : v_1' \neq v_1 \neq v_1''\} \cup \\ & \{(v_2', v_2'') \in E(\Gamma_2) : v_2' \neq v_2 \neq v_2''\} \cup \\ & \{(v_1', v_2') : (v_1', v_1) \in E(\Gamma_1) \text{ and } (v_2, v_2') \in E(\Gamma_2)\} . \end{aligned}$$

Representation of Γ as $\Gamma_1 \boxtimes_{(v_1,v_2)} \Gamma_2$ is called a *decomposition* of Γ, Γ_1 and Γ_2 are called the *components* of the decomposition. The partition $\{V(\Gamma_1) - v_1, V(\Gamma_2) - v_2\}$ is a split of Γ.

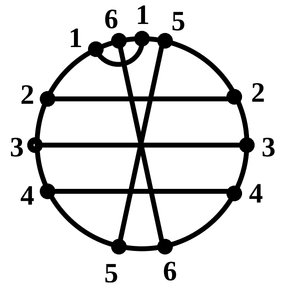
A decomposition of a graph is said to be *canonical* if the following conditions are satisfied:

(i) each component is either a prime graph, or a complete graph K_n, or a star S_n, which is the tree with a vertex, the *center*, adjacent to n other vertices;

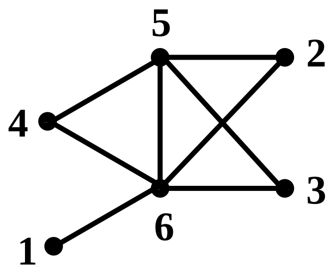
(ii) no two components that are complete graphs are neighbors, that is, their markers are not connected by a dashed edge;

(iii) the markers of two components that are star graphs connected by a dashed edge are either both centers or both not centers of their components.

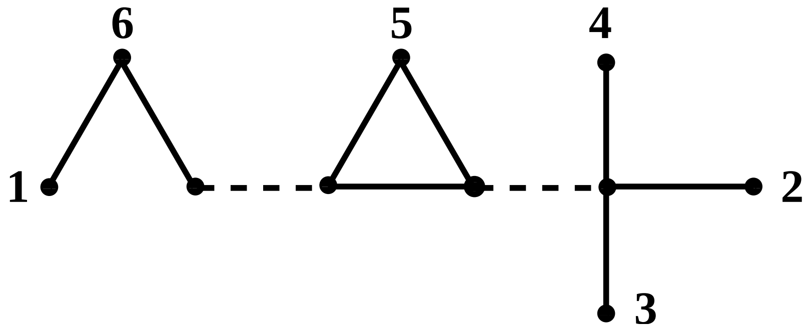
W. H. Cunningham proved [Cu, Theorem 3] that each graph with at least six vertices possesses a unique canonical decomposition.



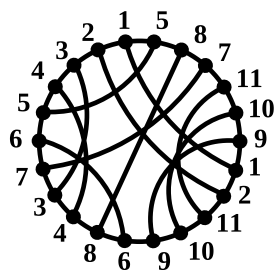
A chord diagram



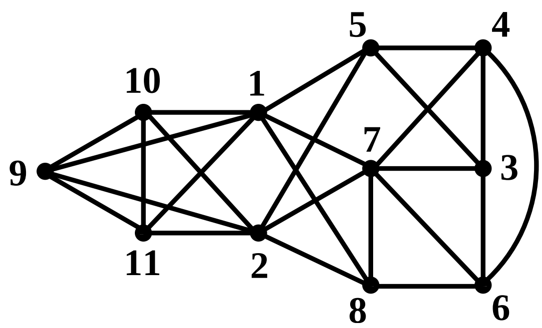
The intersection graph



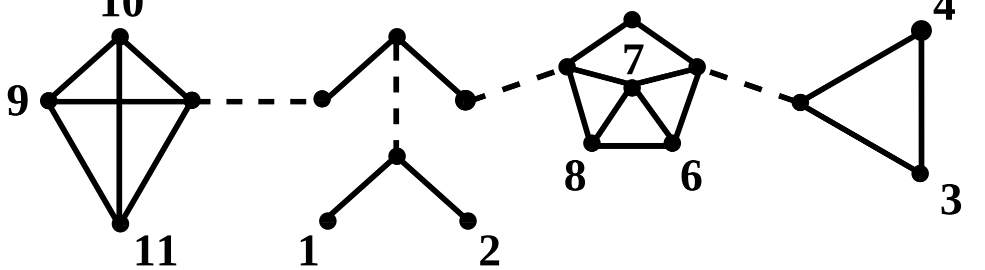
The canonical decomposition



Chord diagram

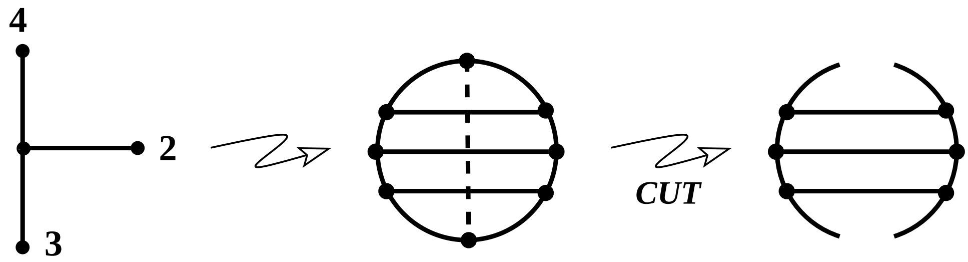


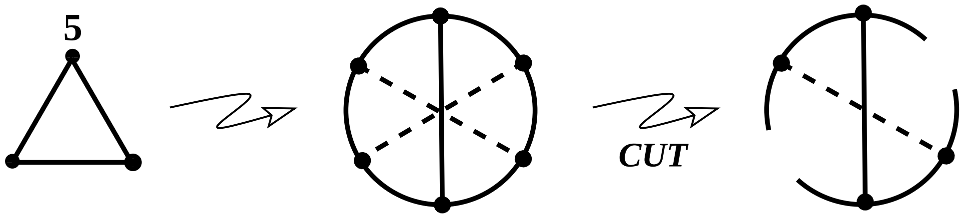
Intersection graph

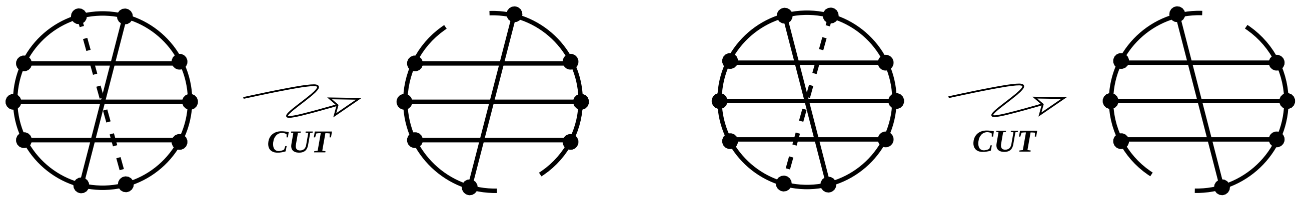


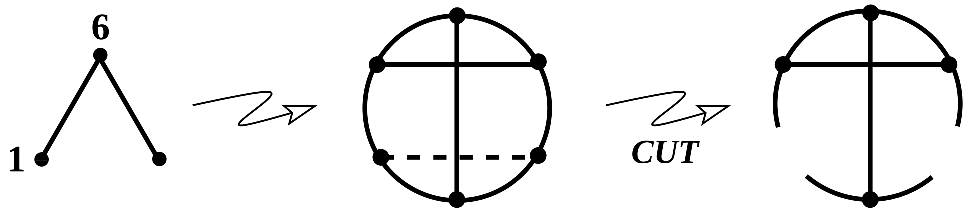
Canonical decomposition

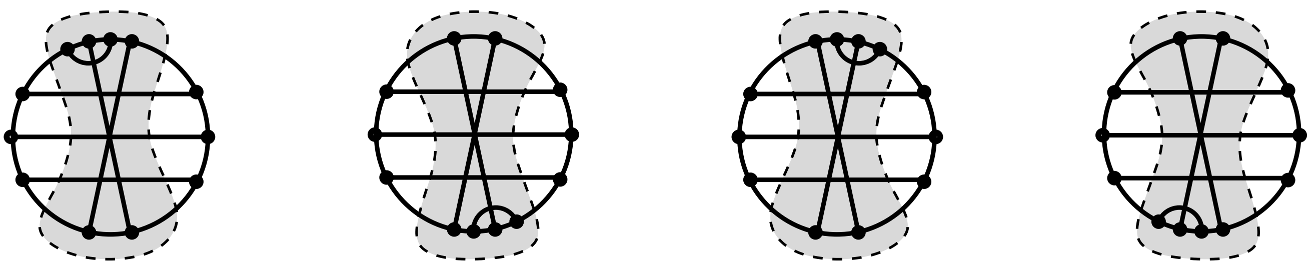
Idea of the proof.



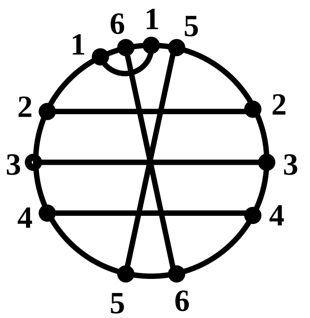


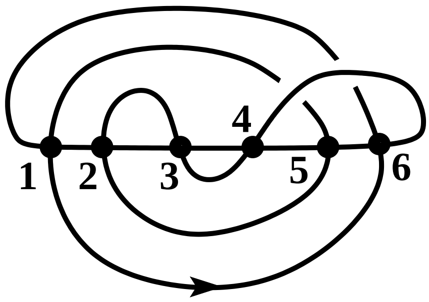


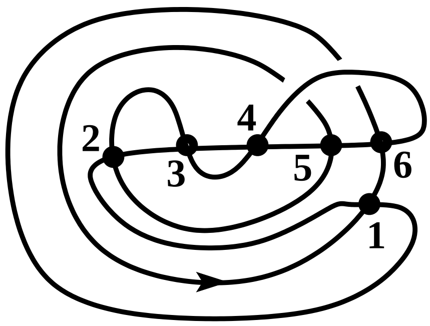




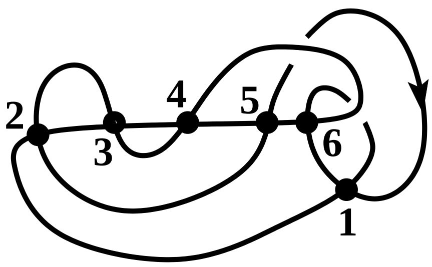
THEOREM. *If a Vassiliev invariant knot invariant does not distinguish mutant knots, then the corresponding weight system depends only on the intersection graphs of chord diagrams.*



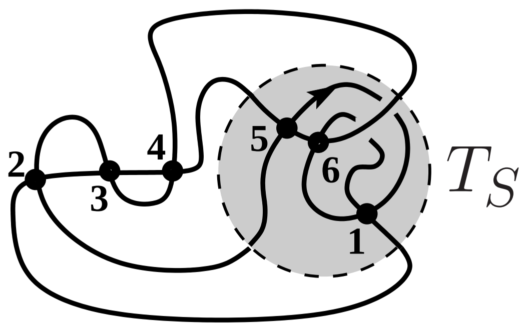




Sliding the double point 1



Shrinking the arcs



Forming the tangle T_S

References

[Bu] A. Bouchet *Reducing prime graphs and recognizing circle graphs*, Combinatorica, **7**, no. 3, 243–254 (1987)

[Cu] W. H. Cunningham, *Decomposition of directed graphs*, SIAM J. Algor. Discrete Math.,**3**, no. 2, 214–228 (1982)

[GSH] C. P. Gabor, K. J. Supowit, W.-L. Hsu, *Recognizing circle graphs in polynomial time*, Journal of the ACM (JACM) (3), **36**, no. 3, 435–473 (1989)

[CL] S. V. Chmutov, S. K. Lando, *Mutant knots and intersection graphs*, Preprint `arXiv:math.GT/0704.1013`. To appear in *Algebraic and Geometric Topology.*