Edge-reconstruction of the genus of one-vertex graphs on surfaces.

OSU Combinatoric seminar, Thursday, Nov. 15, 2007, 1:30 MA-240

1) Reconstruction problems
2) Graphs on surfaces ⇒ ribbon graphs → rotation systems
   One-vertex ribbon graphs = chord diagrams on a circle
3) edge-reconstruction of graphs on surfaces
4) edge-reconstruction of the genus of one vertex graph.
5) Proof.

① G, $V(G) = \{v_1, \dots, v_n\}$, $E(G) = \{e_1, \dots, e_m\}$ ⟶ $\{G - v_i\}$ for all i (vertex-reconstruction ←)

Vertex-reconstruction conjecture: If $n \geq 3$ then G is vertex reconstructible

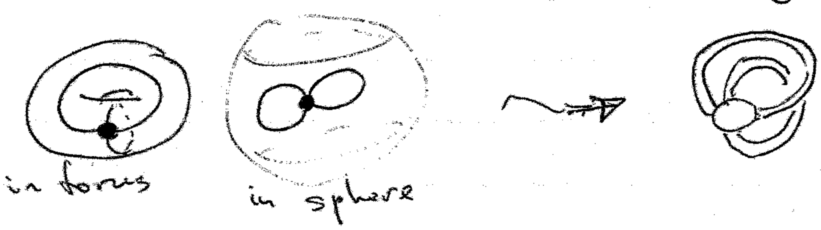
Exception $n = 2$: G = • •, G' = •—•

Edge-reconstruction conjecture: If $m \geq 4$ then G is edge-reconstructible.

G ⟵(edge-reconstruction) $\{G - e_i\}$ for $i = 1, \dots, m$

Exceptions $m = 3$: G = ⌇⌇, G' = Γ∘ and G = ⅄, G' = △∘

② Graphs on surfaces. (orientable) cellular embedding $\Gamma \hookrightarrow \Sigma$



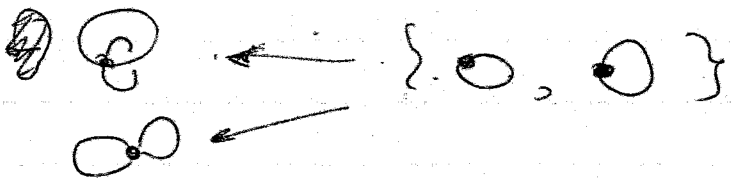
ribbon graph Σ_G is a surface $\Sigma_G = V(G) \cup E(G)$

$V(G) = \{$discs$\}$ - vertices $E(G) = \{$disc-edges$\}$ $F(G) = \{$disc-faces$\}$ $f_1, \dots, f_t$

- the vertices and edges intersect by disjoint line segments.
- each such segment lies on exactly one vertex and exactly one edge
- every edge contains exactly two such segments.

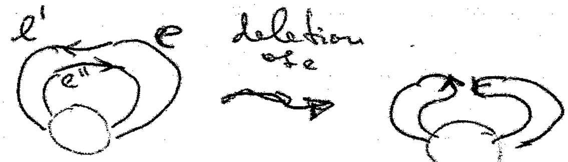
ribbon graph → rotation system → chord diagrams
= cyclic order of
half-edges ~~incident~~
at each vertex

③ Edge – reconstruction of a graph on surface ~~=~~ means a reconstruction of rotation system from edge-deleted rotation systems

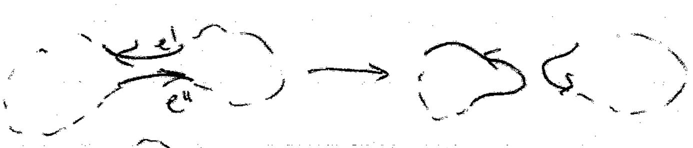


④ Th. The genus of an oriented one-vertex ribbon graph is edge reconstructible. (if $m \geq 3$)

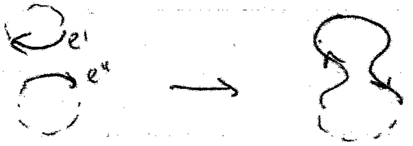
$2-2g = \chi(\Sigma) = ~~v~~ - m + f \Rightarrow$ reconstructibility of $g \Leftrightarrow$ reconstructibility of f



if e' and e'' belongs to the same boundary compound (face) f then f is increases by 1



$f(G) = f(G-e) - 1$



if e' and e'' ~~lie~~ lies in different faces f' and f'' then f is decreasing by 1

$f(G) = f(G-e) + 1$

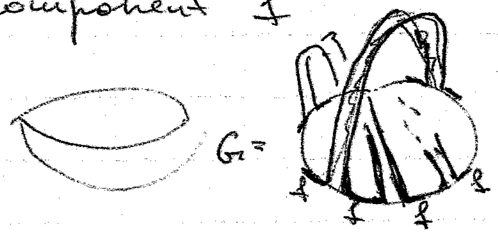
1. if $f(G-e_1) \neq f(G-e_2) \Rightarrow f(G) = \frac{f(G-e_1) + f(G-e_2)}{2}$

2. $f(G-e_i) = f(G-e_j)$ for all i, j

if ~~$f(G-e) \neq 2$ then either~~ $f(G-e) = 1 \Rightarrow f(G) = 2$.

if $t(G-e) \geq 3 \Rightarrow t(G) = t(G-e)+1$

indeed if $t(G) = t(G-e) - 1$, then for any edge $e \notin G$ the segments e' and e'' belong to the same ~~face~~ (boundary component f



$\Rightarrow t(G) = 1$

$\Downarrow$

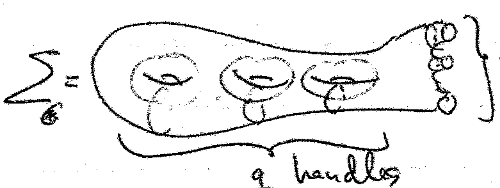
$t(G-e) = 2.$

3. $t(G-e) = 2$ for all $e \in E(G)$

either $t(G) = 1$ or $t(G) = 3$

$\Updownarrow$

Intersection matrix of Σ_G is nondegenerate.



$H_1(\Sigma, \mathbb{F}_2)$ = vector space$/\mathbb{F}_2$ of dim $= 2g+t-1$

intersection form = bilinear form on $H_1(\Sigma)$

$\langle \cdot, \cdot \rangle : H_1(\Sigma) \otimes H_1(\Sigma) \to \mathbb{F}_2$

$\mathrm{rank}(\langle \cdot, \cdot \rangle) = 2g$

if $\Sigma = \Sigma_G$, then $H_1(\Sigma_G, \mathbb{F}_2)$ has a basis determined by edges

$\dim H_1(\Sigma_G) = m = \#$ edges.

Intersection matrix = adjacency matrix $A(I_G)$ of the intersection graph $= I_G$, $V(I_G) = \{v_{e_1}, \dots v_{e_m}\}$

$E(I_G) = \{ (v_{e_i}, v_{e_j})$ if e_i interlaced, 0, otherwise $\}$

$G \rightsquigarrow I_G$

$\downarrow \qquad \downarrow$

$G - e_i \to I_G - v_{e_i}$

Tutte: $\mathrm{rank}(A(I_G))$ is vertex reconstructible.