Edge-reconstruction of the genus of one-vertex graphs on surfaces.

OSU Combinatoric seminar. Thursday, Nov. 15, 2007, 11:30

1) Reconstruction problems

2) Graphs on surfaces ⊃ ribbon graphs ⊃ rotation systems
   One-vertex ribbon graphs = chord diagrams on a circle

3) edge-reconstruction of graphs on surfaces

4) edge-reconstruction of the genus of one-vertex graph

5) Proof

\( G \leftarrow \text{vertex-reconstruction} \)

\[ V(G) = \{ v_1, \ldots, v_n \} \quad \text{for all } i \]

\[ E(G) = \{ e_1, \ldots, e_m \} \]

**Vertex-reconstruction conjecture:** If \( n \geq 3 \) then \( G \) is reconstructible.

Exception \( n = 2 \): \( G = \Delta_0 \), \( G' = \Delta \)

**Edge-reconstruction conjecture:** If \( m \geq 4 \) then \( G \) is edge-reconstructible.

\[ G \leftarrow \text{edge-reconstruction} \{ G - e_i \} \quad \text{for } i = 1, \ldots, m \]

Exception \( m = 3 \): \( G = \Delta_0, G' = \Gamma_0 \) and \( G = \Delta, G' = \Delta_0 \)

2) Graphs on surfaces. Orientable cell-edge embedding \( \Gamma \to \Sigma \)

- ribbon graph \( \mathcal{G} \)
- a surface \( \Sigma = (V(G), E(G)) \) and \( \mathcal{E}(G) \)
- the vertices and edges intersect by disjoint line segments.
- each such segment lies on exactly one vertex and exactly one edge.
- every edge contains exactly two such segments.
ribbon graph $\rightarrow$ rotation system $\rightarrow$ chord diagrams

= cyclic order of half-edges at each vertex

3. Edge-reconstruction of a graph on surface = means a reconstruction of rotation system from edge-deleted rotation systems

4. In. The genus of an oriented one-vertex ribbon graph is edge reconstructible. (if $n \geq 3$)

If $e'$ and $e''$ belongs to the same boundary compound (face) of $G'$, then $t$ is decreased by 1.

$$t(G) = t(G-e) - 1$$

If $e'$ and $e''$ lies in different faces $e'$ and $e''$, then $t$ is increased by 1.

$$t(G) = t(G-e) + 1$$

1. If $t(G-e_1) \neq t(G-e_2)$ or $t(G-e_1) = t(G-e_2)$, then $t(G) = \frac{t(G-e_1) + t(G-e_2)}{2}$.

2. $t(G-e_1) = t(G-e_2)$ for all $i,j$. If $t(G-e_i) \neq 2$, then either $t(G-e_1) = 1$ or $t(G) = 2$. 

If \( t(G) \geq 3 \Rightarrow t(G) = t(G-e) + 1 \)

Indeed \( t(G) = t(G-e) - 1 \), then for any edge \( e \in E(G) \) the segments \( e^1 \) and \( e^2 \) belong to the same component \( \Gamma \)

\[ \Rightarrow t(G) = 1 \]

\[ \Rightarrow t(G-e) = 2. \]

3. \( t(G-e) = 2 \) for all \( e \in E(G) \)
   either \( t(G) = 1 \) or \( t(G) = 3 \)

Intersection matrix of \( \Sigma \)

is non-degenerate.

\[ \Sigma = \begin{cases} \text{handles} \\ g \end{cases} \]

Intersection form = bilinear form on \( H_4(\Sigma) \)

\[ \langle \cdot , \cdot \rangle : H_4(E) \otimes H_4(\Sigma) \to R_2 \]

\[ \text{rank}(\langle \cdot , \cdot \rangle) = 2g \]

If \( \Sigma = \Sigma_G \), then \( H_4(\Sigma_G, F_2) \) has a basis determined by edges

\[ \dim H_4(\Sigma_G) = m = \text{edges}. \]

Intersection matrix = adjacency matrix \( A(E) \) of the intersection graph \( I_0 \)

\[ V(I_0) = \{ v_1, \ldots, v_m \} \]

\[ E(I_0) = \{ v_i; v_j \} \]

\[ G \rightarrow I_0 \]

Tutte:

\[ G-e_i \rightarrow I_0 - v_e_i \]

\[ \text{rank}(A(E)) \text{ is vertex reconstructible}. \]