

Wed + Fri = Mon. (Wed 7)

Topics in Topology

Nov. 3, 2008

MPIM

Combinatorics of the HOMFLY polynomial joint with M. Polyak

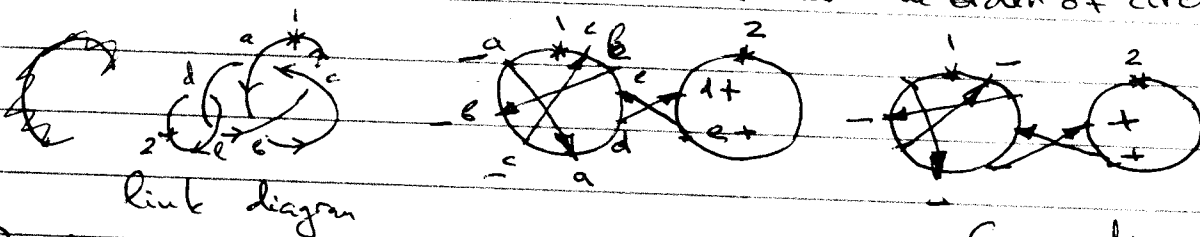
1. Gauss diagrams.
2. State model for the HOMFLY pd.
3. (~~Polyak-Viro~~ Gauss diagram) formulas for Vassiliev invariants.
4. Formulas for HOMFLY

① Gauss diagrams

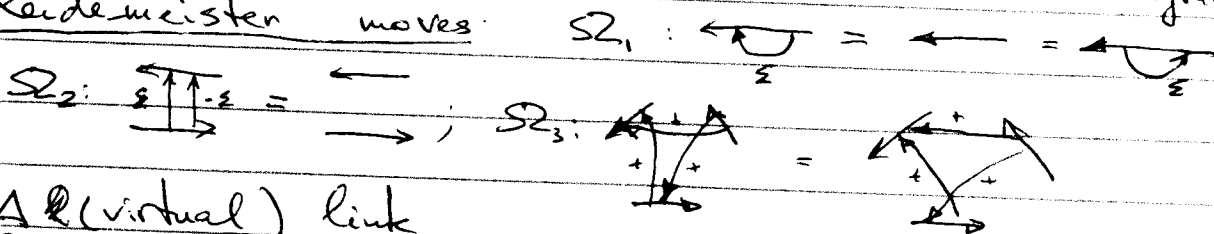
A Gauss diagram is a collection of ^{oriented} circles with ~~an~~ a number of ordered pairs of distinct points

- ordered pairs (arrows)
- sign of a pair

(- base point on each circle and an order of circles)

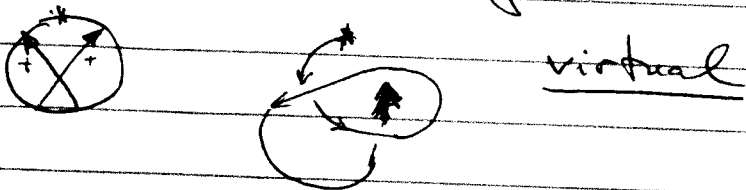


Reidemeister moves



A (virtual) link

is a Gauss diagram modulo Reidemeister moves.



② HOMFLY Polynomial (Laurent polynomial in a, z)

Skein-relation:

$$aP(\uparrow) - a^{-1}P(\downarrow) = zP(\uparrow\uparrow), \quad P(\bigcirc) = 1$$

Skein model for P

$$P(G) = \sum_{\text{states } S \text{ of } G} \langle \alpha | G | S \rangle$$

(Jaeger)




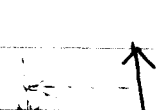
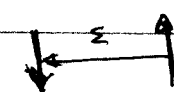
$$P(\underbrace{\bigcirc \dots \bigcirc}_n) = \left(\frac{a-a^{-1}}{z}\right)^n$$

A state S is a subset of arrows of G

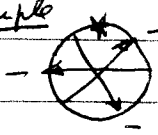
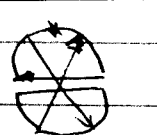
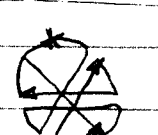
$$P(G) = \sum_S \prod_{\alpha \in G} \langle \alpha | G | S \rangle \cdot \left(\frac{a-a^{-1}}{z}\right)^{c(S)-1}$$

$G(S)$:  $c(S) := \#$ of circles of $G(S)$

$\langle \alpha | G | S \rangle$ depend on the first passage of endpoints of α in the tracing of $G(S)$

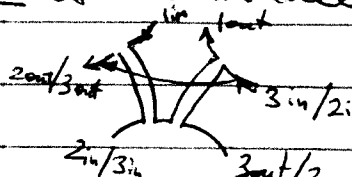
First passage				
	za^{-z}	z	a^{-2z}	1

Example

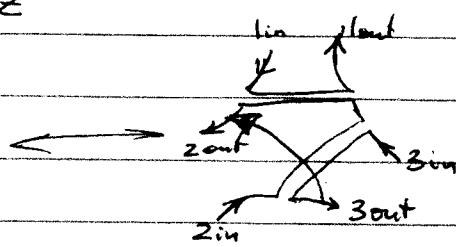
		
$1(a^z) \cdot 1$	$1(-az) \cdot a \cdot \left(\frac{a-a^{-1}}{z}\right)$	$1(-az) \cdot (-az)$

$$(2a^2 - a^4) + a^2 z^2$$

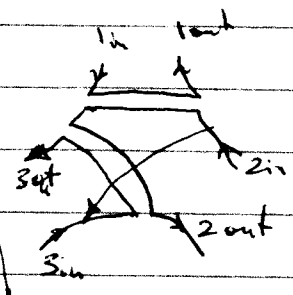
Proof of invariance



$$(a^{-1}z)(a^{-1}z)a^{-2} / (a^{-1}z)^2$$



$$(a^{-1}z)^2 a^{-2}$$



$$(a^{-1}z)^2$$

③ (Polyak-Viro)
Gauss diagram formulas

Let G be a Gauss diagram, for example with 2 circles

A is another Gauss diagram.

$\langle A, G \rangle := \#$ of A -subdiagrams of G

$\langle \begin{matrix} \text{---}^+ \text{---} \\ \text{---}^- \text{---} \end{matrix}, G \rangle = \#$ of $\begin{matrix} \text{---}^+ \text{---} \\ \text{---}^- \text{---} \end{matrix}$ arrows of G from the second circle to the first one.

$\langle \begin{matrix} \text{---}^- \text{---} \\ \text{---}^+ \text{---} \end{matrix}, G \rangle = \#$ of $\begin{matrix} \text{---}^- \text{---} \\ \text{---}^+ \text{---} \end{matrix}$ arrows $\text{---}^+ \text{---}$

$\langle \begin{matrix} \text{---}^+ \text{---} \\ \text{---}^- \text{---} \end{matrix} - \begin{matrix} \text{---}^- \text{---} \\ \text{---}^+ \text{---} \end{matrix}, G \rangle = \text{lk}(G)$

$\langle \begin{matrix} \text{---}^+ \text{---} \\ \text{---}^- \text{---} \end{matrix} \otimes \begin{matrix} \text{---}^- \text{---} \\ \text{---}^+ \text{---} \end{matrix}, G \rangle = \text{coef. at } z^2 \text{ of } P(1, z)$
the Conway polynomial.

Goussarov theorem:

Any Vassiliev knot invariant can be expressed by a Polyak-Viro formula.

Vassiliev invariants coming from HOMFLY.

$$P(a, z)|_{a=eh} = P(e^h, z) = \sum_{k, l} P_{k, l} h^k z^l$$

Lemma $P_{k, l}(L)$ is a Vassiliev invariant of order $\leq k+l$

Power series state model

First passage α				
$\downarrow z \uparrow$	$z e^{-2h} z$	0	$e^{-2h} - 1$	0

$$W(A) = \sum_S \prod_{\alpha \in A} \langle \alpha | A | S \rangle \left(\frac{e^h - e^{-h}}{z} \right)^{\text{sgn}(S) - 1} =: \sum_{k, l} w_{k, l}(A) h^k z^l$$

Theorem

$$P_{k,e}(G) = \langle A_{ke}, G \rangle = \left\langle \sum_A w_{ke}(A) \cdot A, G \right\rangle$$

COEFFICIENTS OF THE HOMFLYPT POLYNOMIAL

$$A_{0,2} = \text{Diagram} ; \quad A_{2,0} = 0 ;$$

$$A_{0,3} = 0 ; \quad A_{3,0} = -4A_{1,2} ;$$

$$A_{1,2} = -2 \left(\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5} - \text{Diagram 6} + \text{Diagram 7} + \right. \\ \left. + \text{Diagram 8} - \text{Diagram 9} \right) ;$$

$$A_{0,4} = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5} + \\ + \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10} + \text{Diagram 11} + \text{Diagram 12} + \text{Diagram 13} + \text{Diagram 14} + \\ + \text{Diagram 15} + \text{Diagram 16} + \text{Diagram 17} + \text{Diagram 18} + \text{Diagram 19} + \text{Diagram 20} + \text{Diagram 21} + \text{Diagram 22} ;$$

$$A_{2,2} = 78 \text{ terms.}$$