Wed + Fri = Mon. (mod 7)

Topics in Topology

Nov. 3, 2008

MPIM

Combinatorics of the HOMFLY polynomial

joint with M. Polyak.

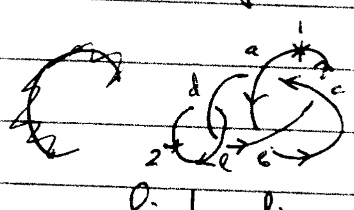
1. Gauss diagrams.
2. State model for the HOMFLY pol.
3. (~~Polyak-Viro~~ Gauss diagram) formulas for Vassiliev invariants.
4. Formulas for HOMFLY

① Gauss diagrams

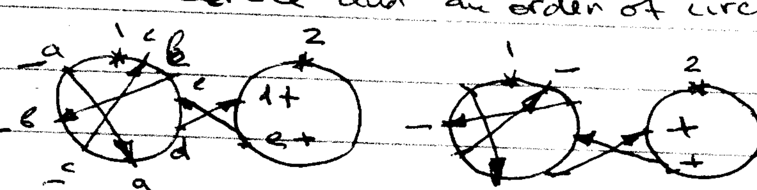
A Gauss diagram is a collection of oriented circles with ~~set~~ a number of ordered pairs of distinct points

- ordered pairs (arrows)
- sign of a pair

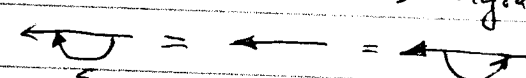
(- base point on each circle and an order of circles)

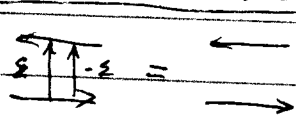
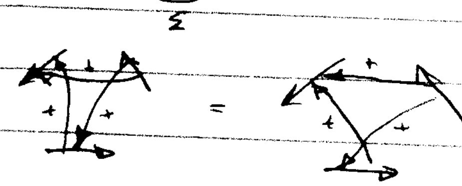


link diagram



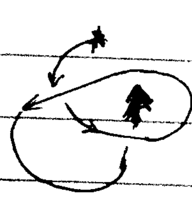
Gauss diagram.

Reidemeister moves: Ω_1: 

Ω_2:  ; Ω_3: 

A ~~k~~ (virtual) link

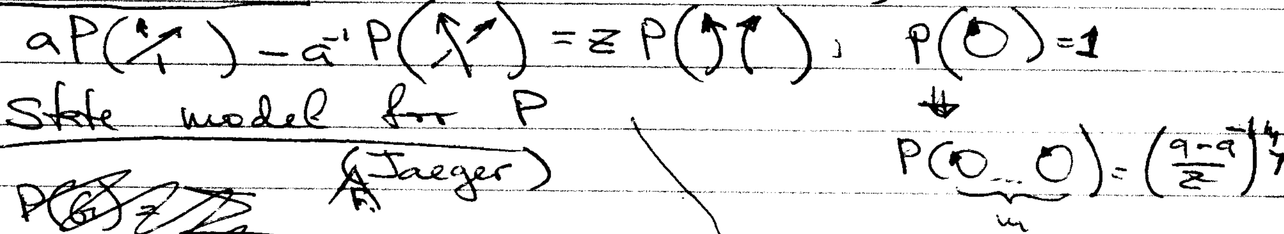
is a Gauss diagram modulo Reidemeister moves.



virtual

② HOMFLY Polynomial (Laurent polynomial in a, z)

Skein-relation:



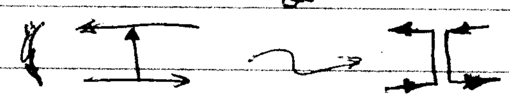
State model for P (Jaeger)

$\not{P(G)}$

A state S of G is a subset of arrows of G

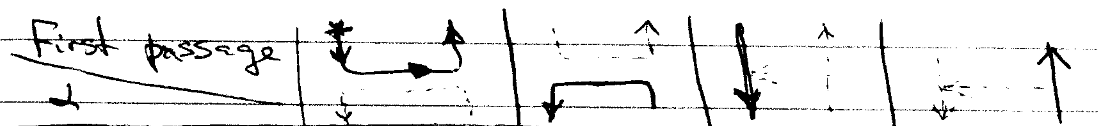
$$P(G) = \sum_S \prod_{\alpha \in G} \langle \alpha | G | S \rangle \cdot \left(\frac{a-a^{-1}}{z}\right)^{c(S)-1}$$

$G(S)$.



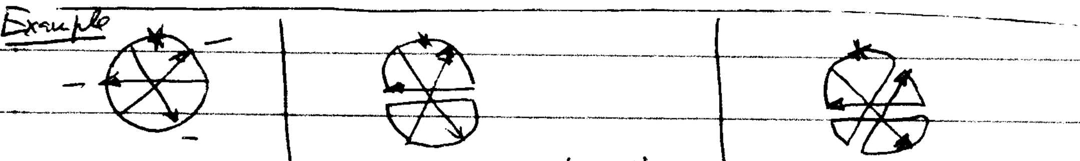
$c(S) :=$ # of circles of $G(S)$

$\langle \alpha | G | S \rangle$ depend on the first passage of endpoints of α in the tracing of $G(S)$



First passage / α				
ε	$\varepsilon a^{-\varepsilon} \cdot z$	0	$a^{-2\varepsilon}$	1

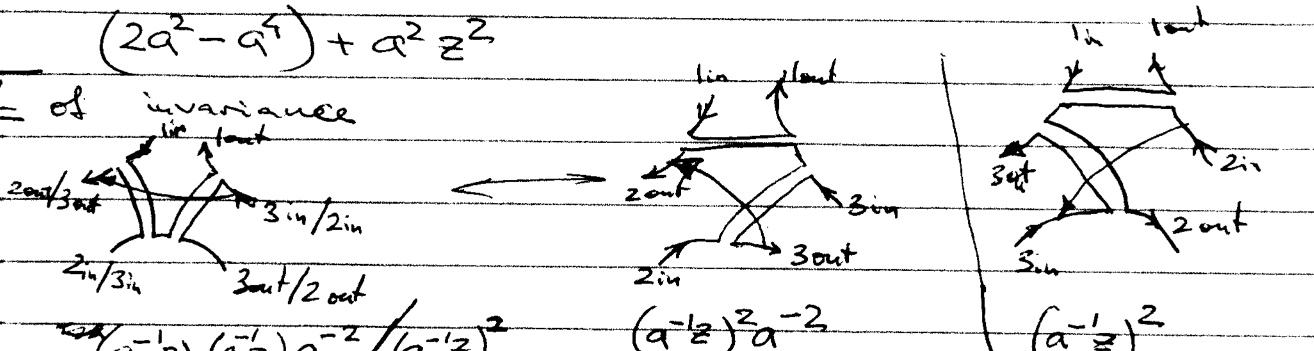
Example



$1(a^2) \cdot 1$ | $1 \cdot (-az) \cdot a^2 \cdot \left(\frac{a-a^{-1}}{\cancel{z}}\right)$ | $1 \cdot (-az) \cdot (-az)$

$$(2a^2 - a^4) + a^2 z^2$$

Proof of invariance



$(a^{-1}z)\cdot(a^{-1}z)a^{-2} / (a^{-1}z)^2$ | $(a^{-1}z)^2 a^{-2}$ | $(a^{-1}z)^2$

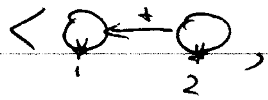
(Polyak-Viro)

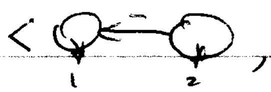
③ ~~Gauss~~ diagram formulas

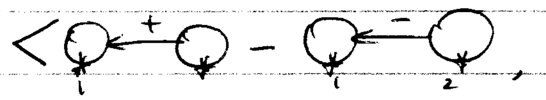
Let G be a Gauss diagram, for example with 2 circles

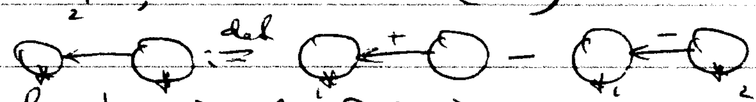
A be another Gauss diagram.

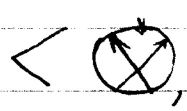
$\langle A, G\rangle :=$ # of A-subdiagrams of G

 $G\rangle =$ # of + arrows of G from the second circle to the first one.

 $G\rangle =$ # of $-$ arrows —"—

 $G\rangle = lk(G)$



 $G\rangle =$ coef. at z^2 of $P(1,z)$ the Conway polynomial.

Goussarov theorem:

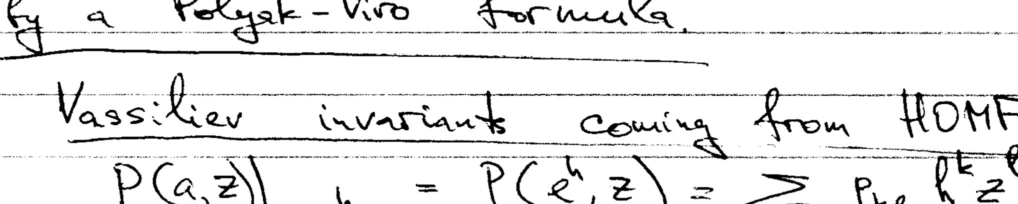
Any Vassiliev knot invariant can be expressed by a Polyak-Viro formula.

Vassiliev invariants coming from HOMFLY

$$P(a,z)\big|_{a=e^h} = P(e^h, z) = \sum_{k,\ell} p_{k\ell}\, h^k z^\ell$$

Lemma $p_{k,\ell}(L)$ is a Vassiliev invariant of order $\le k+\ell$

Power series state model



	$\varepsilon e^{-\varepsilon h} z$	0	$e^{-2\varepsilon h}-1$	0

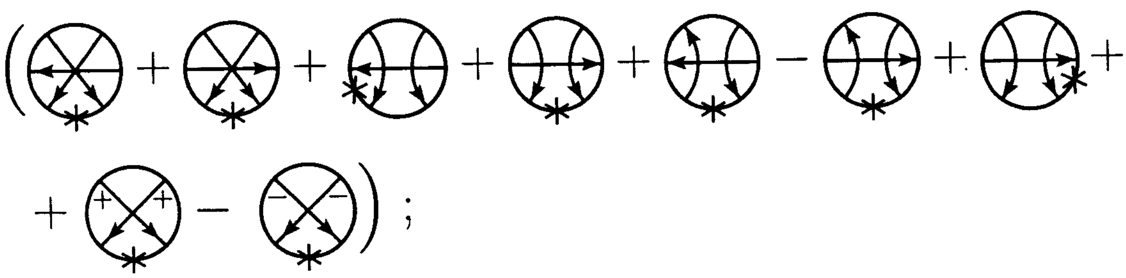
$$W(A) = \sum_S \prod_{\alpha\in A} \langle \alpha | A | S\rangle \left(\frac{e^h - e^{-h}}{z}\right)^{c(S)-1} =: \sum_{k,\ell} w_{k,\ell}(A)\, h^k z^\ell$$

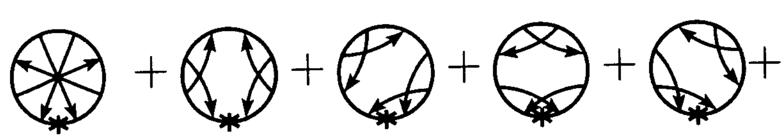
Theorem

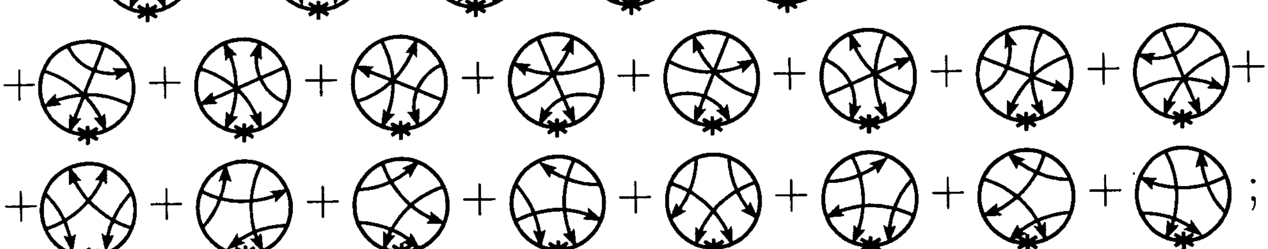
$$p_{k,\ell}(G) = \langle \Delta_{k\ell}, G \rangle = \langle \sum_{A} w_{k\ell}(A) \cdot A, G \rangle$$

$A_{0,2} =$  ; $\quad A_{2,0} = 0$;

$A_{0,3} = 0$; $\quad A_{3,0} = -4A_{1,2}$;

$A_{1,2} = -2$ 

$A_{0,4} =$ 



$A_{2,2} = 78$ terms.