Gauss diagram formulas for Vassiliev invariants

Univ. of Liverpool ①
Singularities seminar.

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joint with M. Polyak

1. Vassiliev invariants
2. Conway polynomial & HOMFLYPT polynomial
3. Gauss diagrams and Polyak–Viro formulas
4. ~~Polyak–Viro formulas for HOMFLYPT.~~

① Vassiliev invariants

Def. $v: \underset{\text{set of knots}}{\mathcal{K}} \to \mathbb{C}$ is called a Vassiliev invariant of order $\leq n$ if $v|_{\mathcal{K}_{n+1}} \equiv 0$

$\mathcal{K}_{n+1}$: set of knots with $n+1$ double points.

$$v(\times) := v(\nearrow\!\!\!\!\nwarrow) - v(\nwarrow\!\!\!\!\nearrow)$$

② Examples

linking #

The Conway polynomial $\nabla(K) = \sum_n c_n(K) \cdot z^k$

$$\nabla(\nearrow\!\!\!\!\nwarrow) - \nabla(\nwarrow\!\!\!\!\nearrow) = z\,\nabla(\uparrow\uparrow), \quad \nabla(\bigcirc) = 1$$

$c_n(K)$ is a Vassiliev invariant of order n.

The HOMFLYPT polynomial $P(L) = \sum_{k,l} p_{k,l}(L)\, h^k z^l$

$$a P(\nearrow\!\!\!\!\nwarrow) - a^{-1} P(\nwarrow\!\!\!\!\nearrow) = z P(\uparrow\uparrow), \quad P(\bigcirc) = 1, \quad P(\underbrace{\bigcirc \dots \bigcirc}_{m}) = \left(\frac{a - a^{-1}}{z}\right)^{m-1}$$

$$\boxed{a = e^h}$$

Lemma (Goussarov) $p_{k,l}(L)$ is a Vassiliev invariant of order $\leq k+l$

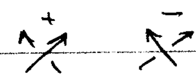
$a = 1$, $h = 0$, $\Rightarrow$ $\boxed{p_{0,l} = c_l}$

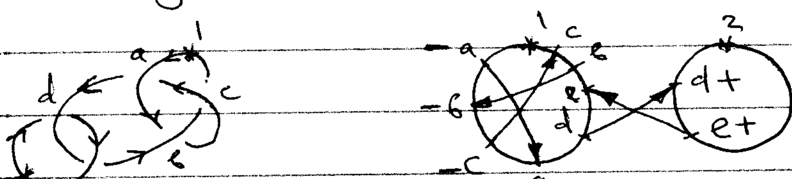
$$p_{k,l}(L) = \langle \underbrace{A_{k,l}}_{???}, G_L \rangle$$

on page?

Gauss diagrams

A Gauss diagram is a (ordered) collection of (based) oriented circles with

- ordered pairs of distinct points (arrows)
- sign of each pair. 



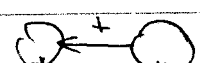
link diagram D Gauss diagram G_D

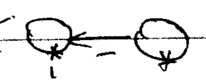
 not realizable Gauss diagram

~~Scalar product~~ S = the space spanned by GD: ~~$(A,G)=\begin{cases}1 & \text{if } A\subseteq G\\ 0, & \text{otherwise}\end{cases}$~~ $I: S\to S$

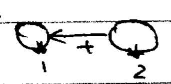
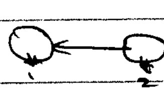
$$I(G)=\sum_{A\subseteq G} A =: \sum \langle A,G\rangle A$$

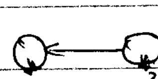
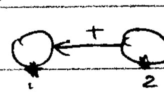
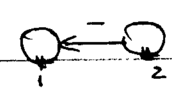
$\langle\cdot,\cdot\rangle: S\times S\to\mathbb{C}$ bilinear pairing.

$\langle$  $, G\rangle$ = # of +-arrows from the second circle of G to the first one.

$\langle$  $, G\rangle$ = # of −-arrows — " —

$\langle \sum \lambda_i A_i, G_K\rangle$ ← Polyak-Viro formula

$\langle$  $-$  $, G\rangle = lk(G)$

Shorter notation:  :=  − 

$\langle$  $, G_K\rangle = p_{0,2}(K) = c_2(K)$

Goussarov theorem

Any Vassiliev ~~invar~~ knot invariant can be represented by a Polyak-Viro formulas

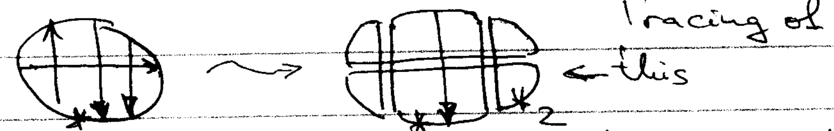
* from page 1

State model for a Gauss diagram A

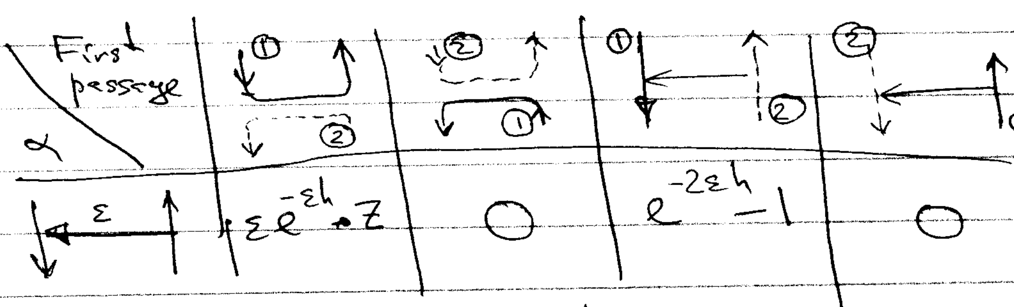
A **state** S = a subset of arrows of A

$$W(A) = \sum_{\substack{S \\ \text{state}}} \prod_{\substack{\alpha\in A \\ \text{arrow}}} \langle \alpha | A | S\rangle \cdot \left(\frac{e^{h}-e^{-h}}{z}\right)^{c(S)-1} =: \sum_{k,\ell} w_{k\ell}(A)\, h^k z^\ell$$

double every arrow of S:



$c(S) :=$ # components of the Gauss diagram obtained.



Th. For given k and l

$$A_{k\ell} = \sum_{A} w_{k\ell}(A)\cdot A$$

Example $k=0$, Conway polynomial
(with Khoury, Rossi)

$A_{0\ell}$,
$S=A$: • double every arrow of A
(ℓ arrows) • pick those A that $c(A)=1$
contribution of an arrow to $w_{0,\ell}(A)$
$= \langle \alpha | A | A\rangle = \varepsilon\xi$
• orientation of an arrow is such that ~~the first approaching is to the head~~ of the arrow.