Gauss diagram formulas for
Vassiliev invariants

Dec. 8, 2008

1. Vassiliev invariants
2. Conway polynomial & HOMFLYPT polynomial
3. Gauss diagrams and Polyak–Viro formulas
4. Polyak–Viro formulas for HOMFLYPT.

1. Vassiliev invariants

Def. $v : \{K\} \to \mathbb{Z}$ is called a Vassiliev invariant of order $\leq n$ if $v(K_{n+1}) = 0$ for a set of knots with $n + 1$ double points.

$v(K) := v(K) - v(K')$

2. Examples

- **Conway polynomial** $\Delta(K) = \sum \varepsilon_n(K) z^n$

  $\Delta(K) - \Delta(K') = z \Delta(K')$, $\Delta(\emptyset) = 1$

  $\varepsilon_n(K)$ is a Vassiliev invariant of order $n$.

- **HOMFLYPT polynomial** $P(L) = \sum_k \varepsilon_k(L) t^k z^l$

  $\alpha P(K) - \alpha^{-1} P(K') = z P(K')$, $P(\emptyset, 0) = 1$, $P(0, 0) = (\frac{a}{z})^{|\mu|}$

  $\alpha = e^{\frac{\pi}{\sqrt{2}}}$

3. **Lemma (Goussarov)** $\varepsilon_k(L)$ is a Vassiliev invariant of order $\leq k + l$

$a = 1$, $k = 0$, $P_0, e = e_2$

$p_{ke}(L) = \left[ A_{ke}, G_L \right]$

???
Gauss diagram (ordered)

A Gauss diagram is a collection of (based) oriented circles with
- ordered pairs of distinct points (arrows)
- sign of each pair

\[ \otimes \]

link diagram \[ \text{Gauss diagram } G_d \]

\( \otimes \) not realizable Gauss diagram

**Details**
- Product \( S = \) the space spanned by \( GD \):

\[ (A, B) = \sum_{a \in A, b \in B} a \otimes b \in S \]

\[ I(G) = \sum_{A \in G} A =: \sum \langle A, G \rangle A, I : S \times S \rightarrow S \]

\[ \langle \otimes, \otimes \rangle_G = \# \text{ of } + \text{ arrows from the second circle of } G \]

\[ \otimes \text{ to the first one.} \]

\[ \langle \otimes, \otimes \rangle_G = \# \text{ of } - \text{ arrows} \]

\[ \langle \otimes, \otimes \rangle_G = \otimes_G \]

Shorter notation:

\[ \otimes \]

\[ \langle \otimes, \otimes \rangle_G = \rho_{0,2}(K) = \chi_2(K) \]

Gaussarov theorem

Any Vassiliev knot invariant can be represented by a Freedman-Viro formulas

\[ \otimes \text{ from page 1} \]
State model for a Gauss diagram $A$

$A$ state $\mathcal{S} = \text{a subset of arrows of } A$

\[
W(A) = \sum_{S} \sum_{\text{double every arrow of } S} \left< \alpha | A | \alpha \right> \cdot \left( \frac{e^{1-2} - 1}{2} \right) = \sum_{k e} W_{k e}(A) \cdot e^{2}
\]

Tracing of $c(S)$: $\alpha$ components of the Gauss diagram obtained

For given $k$ and $l$

\[
A_{k e} = \sum_{A} W_{k e}(A) \cdot A
\]

Example $k = 0$, Conway polynomial (with Khoury, Rossi)

$S = A$; double every arrow of $A$ (arrows)

- pick those $A$ that $c(A) = 1$

Contribution of an arrow to $W_{0 e}(A)$

\[
= \left< \chi(A) \right> = 2 \epsilon
\]

Orientation of arrows is such that the first approaching is to the head of the arrow.

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