Heidelberg University

WORKSHOP “The Mathematics of Knots: Theory and Application”

Combinatorics of Gauss diagrams and the HOMFLYPT polynomial.

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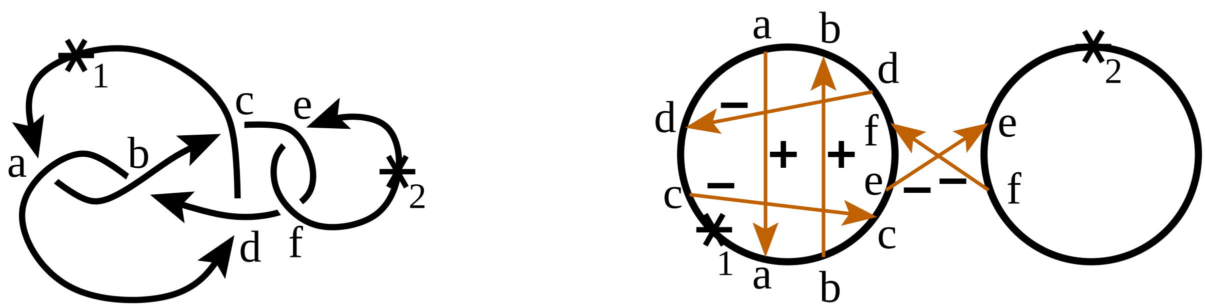
Plan

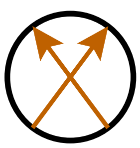
- Gauss diagrams and virtual links.
- HOMFLYPT polynomial and Jaeger's state model for it.
- Two HOMFLYPTs for virtual links.
- Gauss diagram formulas for Vassiliev invariants.

Gauss diagrams

A *Gauss diagram* is a collection of oriented circles with a distinguished set of ordered pairs of distinct points. Each pair carries a sign ± 1.

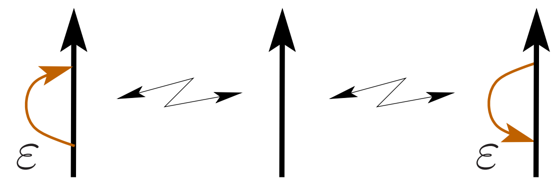


Ordered Gauss diagram is an ordered collection of circles with a base point $\ast_1, \ast_2, \ldots, \ast_m$ on each.

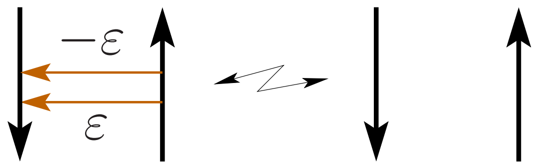
 is a not realizable Gauss diagram.

Reidemeister moves

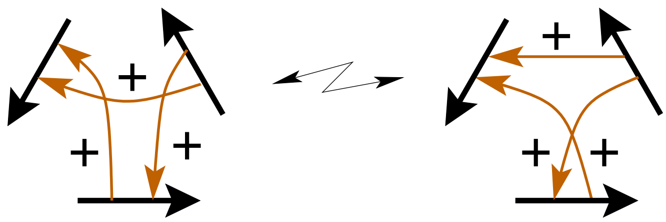
Ω_1 :



Ω_2 :

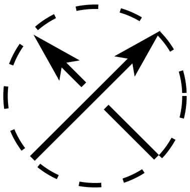
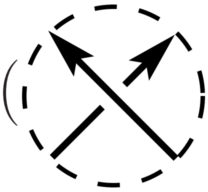
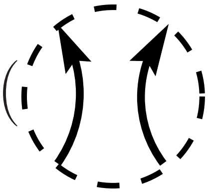


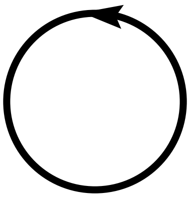
Ω_3 :



A *virtual link* is a Gauss diagram up to the Reidemeister moves.

The HOMFLYPT polynomial

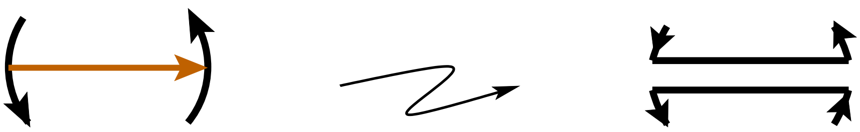
$aP($  $) - a^{-1}P($  $) = zP($  $)$;

$P($  $) \quad = \quad 1\,.$

State models on Gauss diagram

A *state* S on a Gauss diagram G is a subset of its arrows.

Let $G(S)$ be the Gauss diagram obtained by doubling every arrow in S:

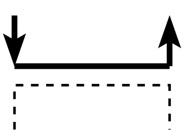
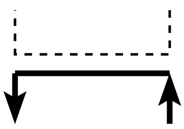
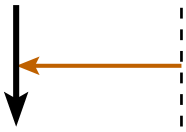
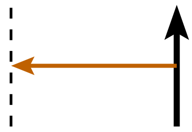
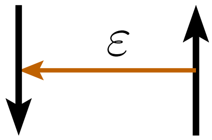
 ,

$c(S) :=$ # of circles of $G(S)$.

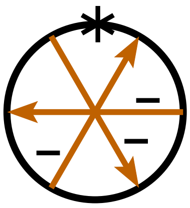
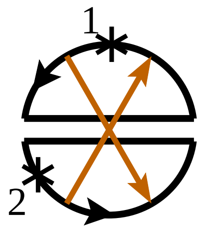
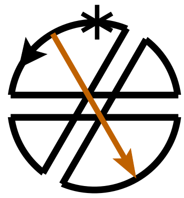
Theorem (F.Jaeger'90).

$$P(G) = \sum_{S} \prod_{\alpha \in G} \langle \alpha | G | S \rangle \cdot \left(\frac{a - a^{-1}}{z} \right)^{c(S)-1}$$

Table of local weights $\langle \alpha | G | S \rangle$:

First passage:				
	$\varepsilon a^{-\varepsilon} z$	0	$a^{-2\varepsilon}$	1

Example. For the Gauss diagram of the trefoil the states with non-zero weights are:

		
$1 \cdot a^2 \cdot 1$	$1 \cdot (-az)a^2(\frac{a-a^{-1}}{z})$	$1 \cdot (-az)(-az)$

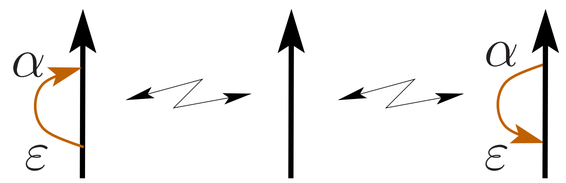
$$P(G) = (2a^2 - a^4) + z^2 a^2$$

Invariance under the Reidemeister moves

Theorem. *$P(G)$ is invariant under Reidemeister moves of ordered Gauss diagrams and thus defines an invariant of ordered virtual links.*

Proof.

Ω_1 :

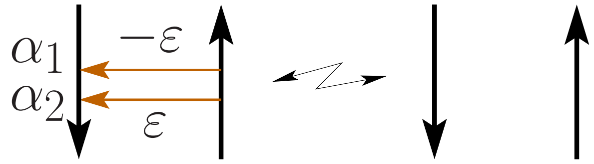


$$\left.\begin{matrix} S \\ S\cup\alpha \end{matrix}\right\} \xleftarrow{2:1} S \xrightarrow{1:2} \left\{\begin{matrix} S \\ S\cup\alpha \end{matrix}\right.$$

$$\left.\begin{matrix} \langle G|S\rangle \\ 0 \end{matrix}\right\} \qquad \langle G|S\rangle \qquad \left\{\begin{matrix} a^{-2\varepsilon}\langle G|S\rangle \\ \varepsilon a^{-\varepsilon} z \frac{a-a^{-1}}{z}\langle G|S\rangle \end{matrix}\right.$$

$$a^{-2\varepsilon} + a^{-\varepsilon}(a - a^{-1}) \equiv 1$$

Ω_2 :



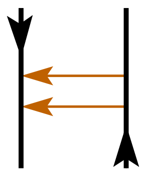
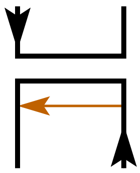
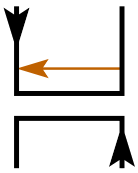
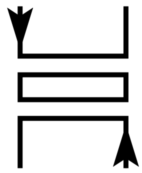
$$S,\ S \cup \alpha_1,\ S \cup \alpha_2,\ S \cup \alpha_1 \cup \alpha_2 \xleftarrow{4:1} S$$

Three cases:

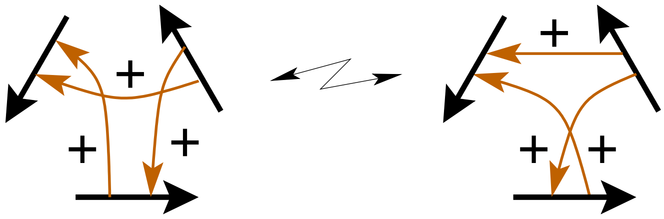
(1) the first entrance to this fragment in S is on the right string;

(2a) the first entrance is on the left string and both strings belong to the same circle of $G(S)$;

(2b) the first entrance is on the left string and the strings belong to two different circles of $G(S)$.

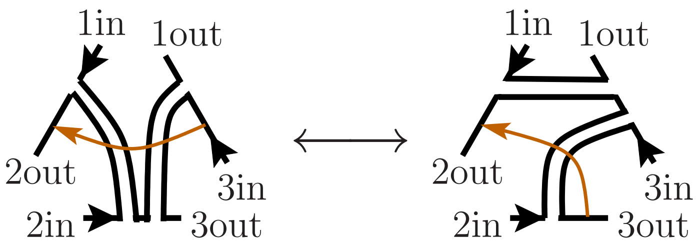
				
(1)	1	0	0	0
(2a)	1	$-\varepsilon a^{-\varepsilon} z$	$\varepsilon a^{\varepsilon} z$	$(a - a^{-1})z$
(2b)	1	$-\varepsilon a^{\varepsilon} z$	$\varepsilon a^{\varepsilon} z$	0

Ω_3 :

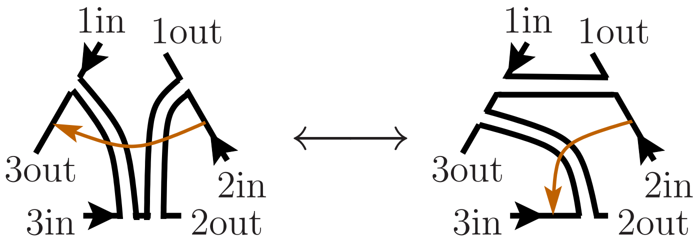


Two of the 14 cases:

$a^{-1}z \cdot a^{-1}z \cdot a^{-2}$



$a^{-1}z \cdot a^{-1}z \cdot 1$



□

Corollary.

1. *HOMFLYPT extends to an invariant of ordered virtual links.*

2. *Interchanging "head" and "tail" of the arrows in the table of local weight of the Jaeger model gives another extension of the HOMFLYPT to virtual links.*

3. *These two extensions coincide on classical links.*

Gauss diagram formulas

Let $\mathcal{S}$ be the space generated by all Gauss diagrams. A map $I : \mathcal{S} \to \mathcal{S}$ is defined as

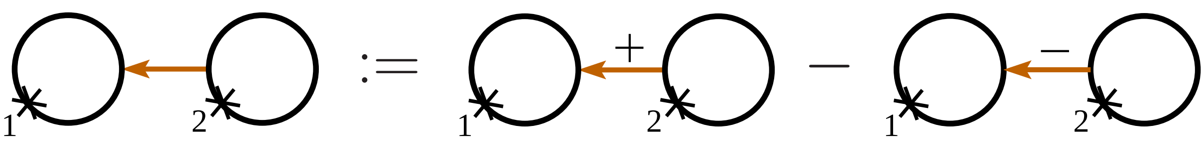
$$I(G) = \sum_{A \subseteq G} A =: \sum \langle A, G \rangle A$$

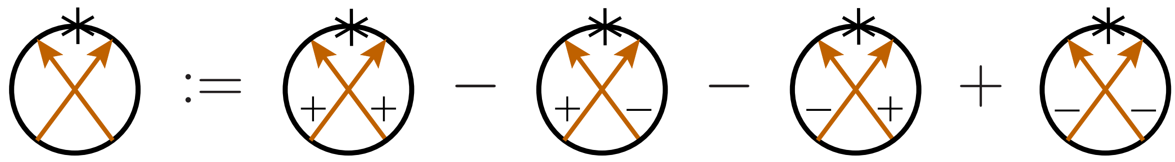
The pairing $\langle A, G \rangle$ extends to a bilinear pairing $\langle \cdot, \cdot \rangle : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$.

A Gauss diagram formula for a link invariant v is a linear combination $\sum \lambda_i A_i$ presenting v in a form

$$v(L) = \langle \sum \lambda_i A_i, G_L \rangle$$

Shorter notation.





Theorem of Goussarov.
Any Vassiliev knot invariant can be represented by a Gauss diagram formula.

Coefficients of the HOMFLYPT polynomial

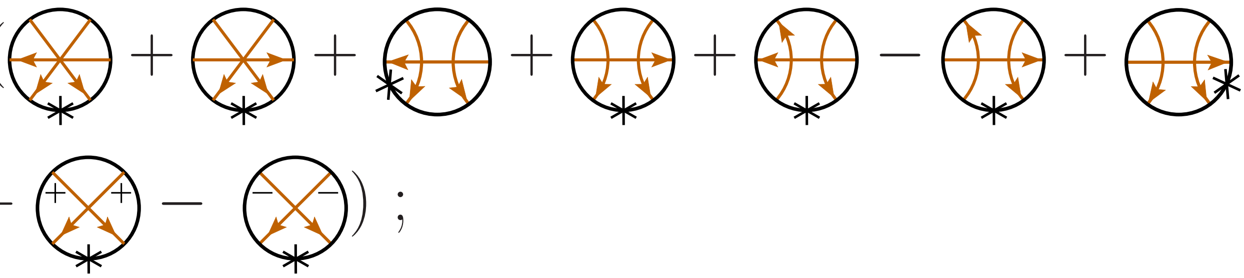
$$P(L)|_{a=e^h} =: \sum p_{k,l}(L) h^k z^l$$

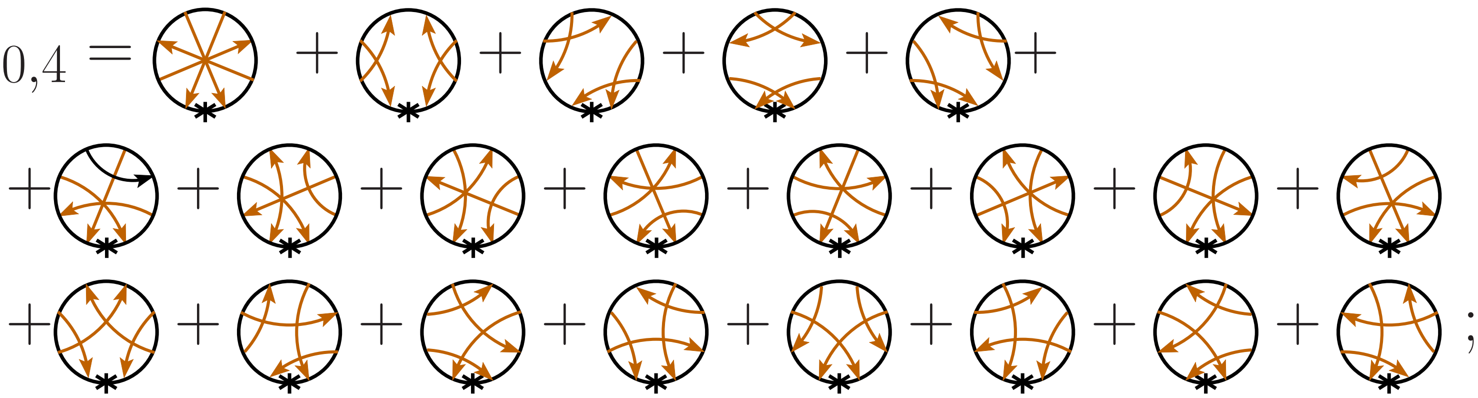
Goussarov's Lemma.
The coefficient $p_{k,l}$ is a Vassiliev invariant of order $\leqslant k + l$.

$$p_{k,l}(K) =: \langle A_{k,l}, G_K \rangle$$

$A_{0,2} =$  ; $\quad A_{2,0} = 0$;

$A_{0,3} = 0$; $\quad A_{3,0} = -4A_{1,2}$;

$A_{1,2} = -2($  $)$;

$A_{0,4} =$  ;

$A_{2,2} = 78$ terms.