University of Oxford

Combinatorial theory seminar

Graphs on surfaces and virtual knots.

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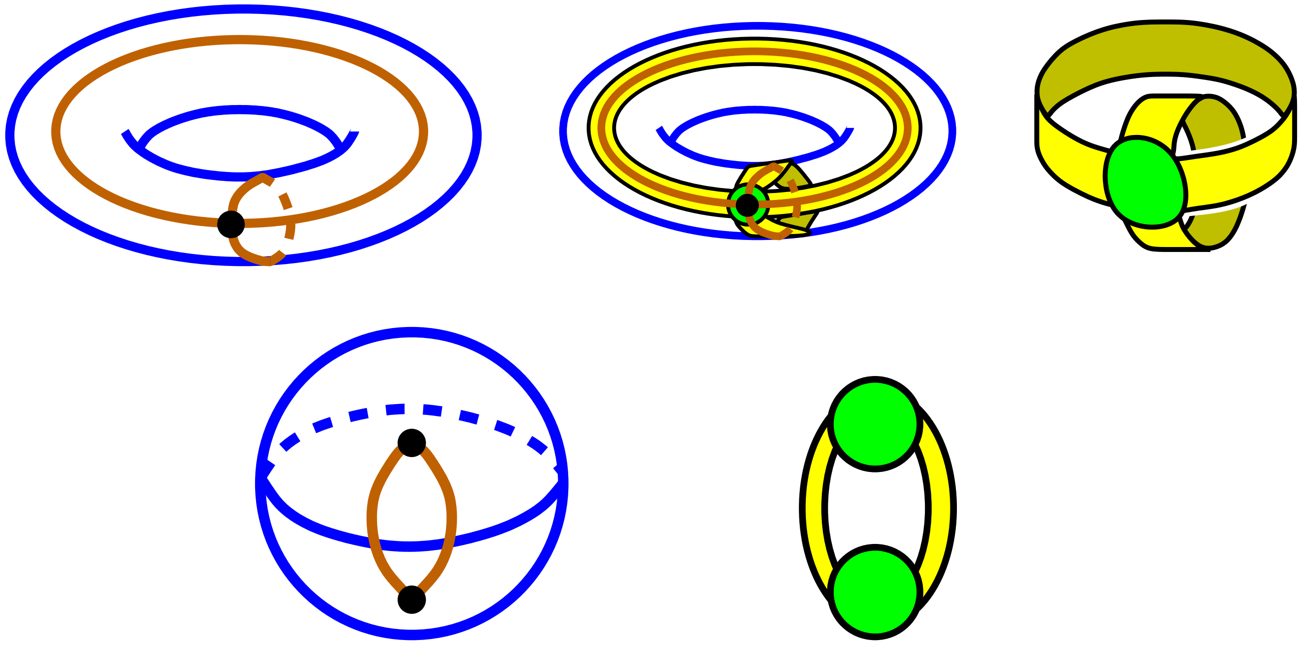
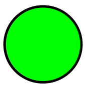
Plan

- Graphs on surfaces as ribbon graphs.
- The Tutte and Bollobás-Riordan polynomials.
- Generalized duality of ribbon graphs.
- Duality theorem for the Bollobás-Riordan polynomial.
- Contraction of loops and the Bollobás-Riordan polynomials.
- Application to knot theory (Thistlethwaite's theorem).
- Virtual links.
- The Jones polynomial as a specialization of the Bollobás-Riordan polynomial.
- Further development.

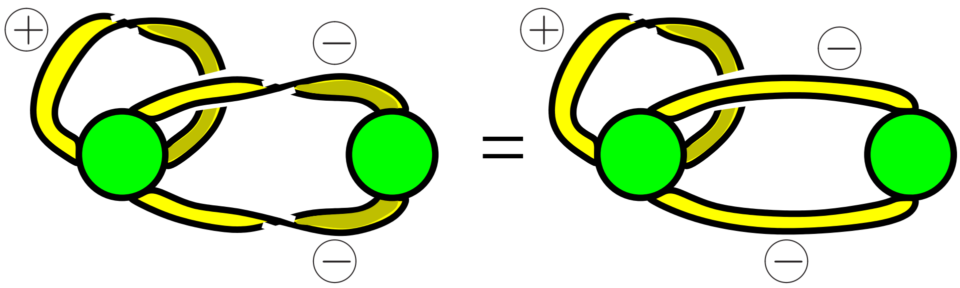
Graphs on surfaces



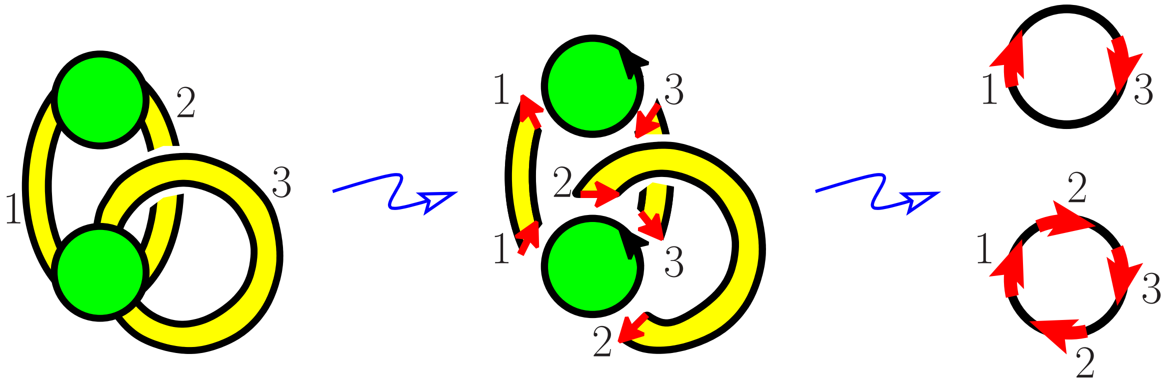
Ribbon graphs

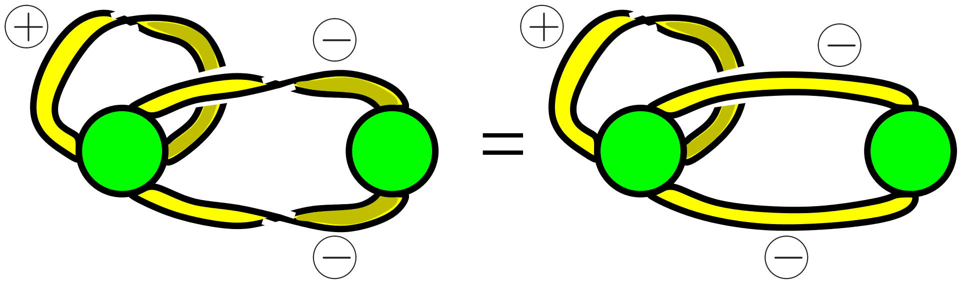
A ribbon graph G is a surface represented as a union of vertices-discs  and edges-ribbons 

- discs and ribbons intersect by disjoint line segments,
- each such line segment lies on the boundary of precisely one vertex and precisely one edge;
- every edge contains exactly two such line segments.



Arrow presentation





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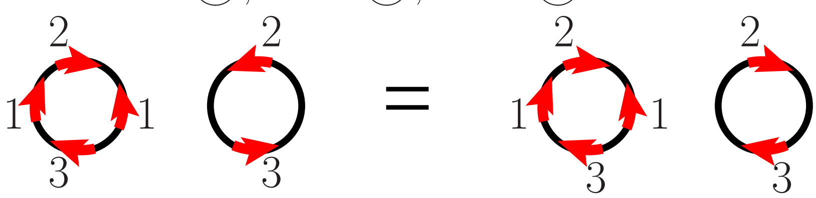
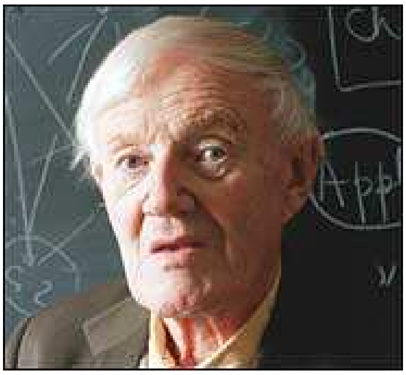


The Tutte polynomial



Let • F be a graph;

- $v(F)$ be the number of its vertices;
- $e(F)$ be the number of its edges;
- $k(F)$ be the number of components of F;
- $r(F) := v(F) - k(F)$ be the *rank* of F;
- $n(F) := e(F) - r(F)$ be the *nullity* of F;

$$T_\Gamma(x,y) \;:=\; \sum_{F\subseteq E(\Gamma)} (x-1)^{r(\Gamma)-r(F)}(y-1)^{n(F)}$$

Properties.

$T_\Gamma = T_{\Gamma-e} + T_{\Gamma/e}$	if e is neither a bridge nor a loop ;
$T_\Gamma = xT_{\Gamma/e}$	if e is a bridge ;
$T_\Gamma = yT_{\Gamma-e}$	if e is a loop ;
$T_{\Gamma_1\sqcup\Gamma_2} = T_{\Gamma_1\cdot\Gamma_2} = T_{\Gamma_1}\cdot T_{\Gamma_2}$	for a disjoint union, $G_1 \sqcup G_2$ and a one-point join, $G_1 \cdot G_2$;

$T_\bullet = 1$.

$T_\Gamma(1,1)$ is the number of spanning trees of Γ ;

$T_\Gamma(2,1)$ is the number of spanning forests of Γ ;

$T_\Gamma(1,2)$ is the number of spanning connected subgraphs of Γ ;

$T_\Gamma(2,2) = 2^{|E(\Gamma)|}$ is the number of spanning subgraphs of Γ .

The Bollobás-Riordan polynomial

Let
- F be a ribbon graph;
- $v(F)$ be the number of its vertices;
- $e(F)$ be the number of its edges;
- $k(F)$ be the number of components of F;
- $r(F) := v(F) - k(F)$ be the *rank* of F;
- $n(F) := e(F) - r(F)$ be the *nullity* of F;
- $\mathrm{bc}(F)$ be the number of boundary components of F;
- $s(F) := \dfrac{e_-(F) - e_-(\overline{F})}{2}$.

$$R_G(x, y, z) \ := \sum_F x^{r(G)-r(F)+s(F)} y^{n(F)-s(F)} z^{k(F)-\mathrm{bc}(F)+n(F)}$$

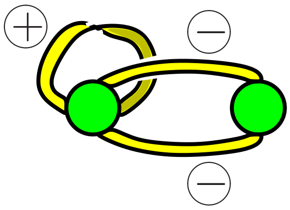
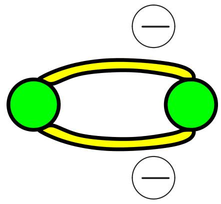
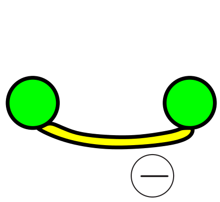
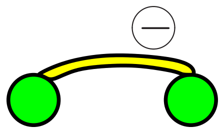
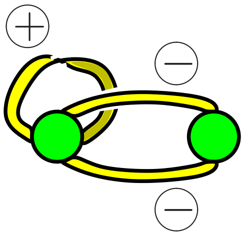
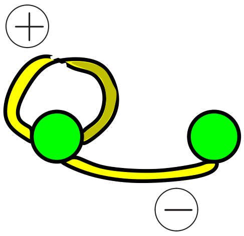
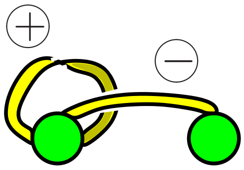
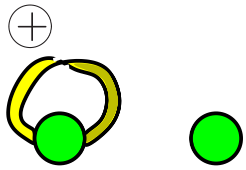
Relations to the Tutte polynomial. (All edges are positive.)

$$R_G(x-1, y-1, 1) = T_G(x, y)$$

If G is planar (genus zero):

$$R_G(x-1, y-1, z) = T_G(x, y)$$

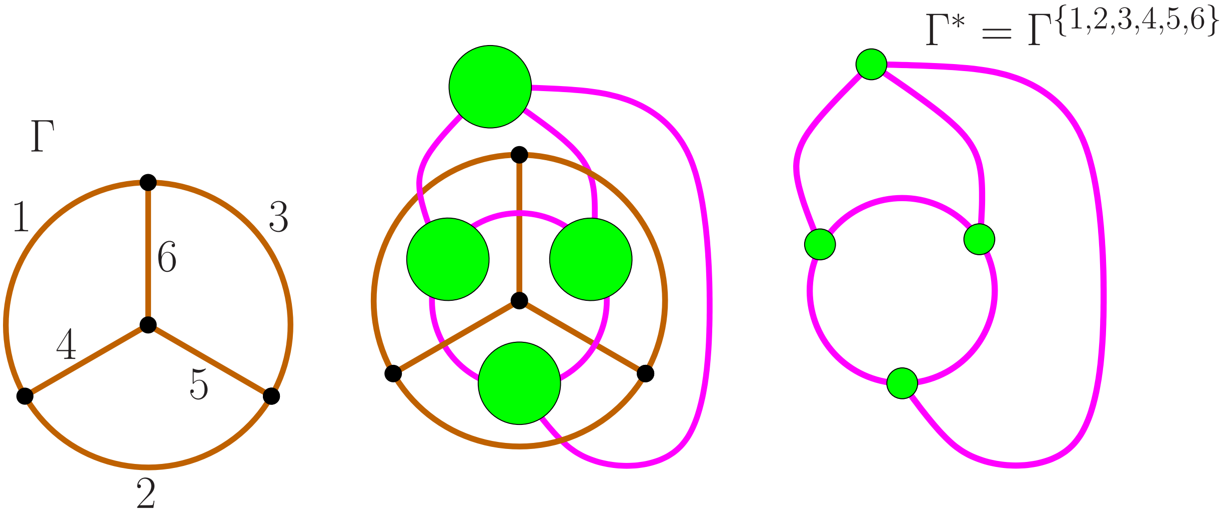
Example.

				
$(k, r, n, \mathrm{bc}, s)$	$(1, 1, 1, 2, 1)$	$(1, 1, 0, 1, 0)$	$(1, 1, 0, 1, 0)$	$(2, 0, 0, 2, -1)$
				
	$(1, 1, 2, 1, 1)$	$(1, 1, 1, 1, 0)$	$(1, 1, 1, 1, 0)$	$(2, 0, 1, 2, -1)$

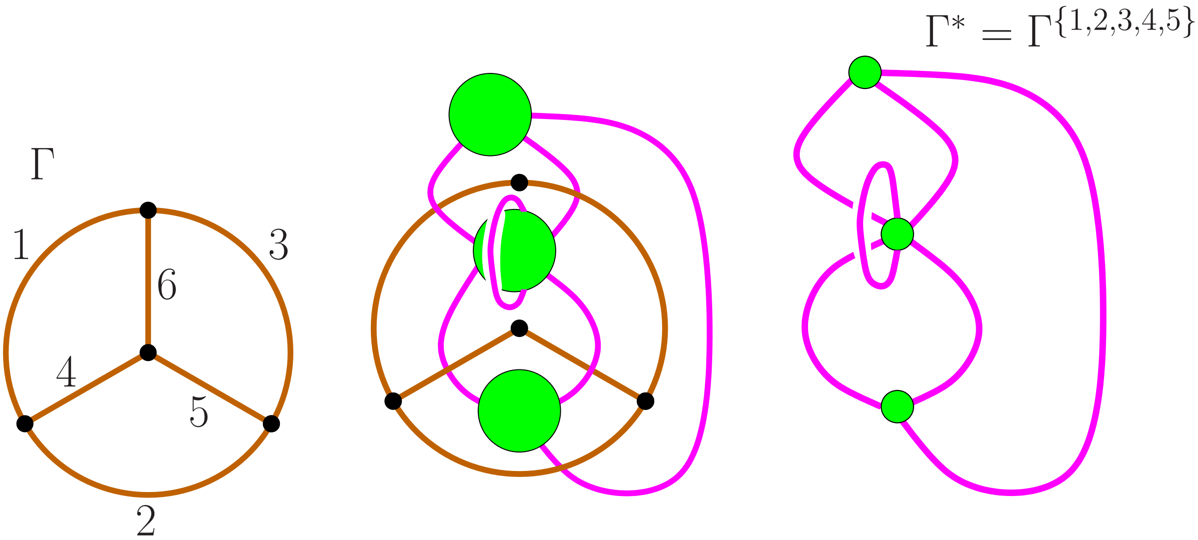
- $r(F) := v(F) - k(F)$;
- $n(F) := e(G) - r(F)$;
- $\mathrm{bc}(F)$ is the number of boundary components;
- $s(F) := \dfrac{e_-(F) - e_-(\overline{F})}{2}$.

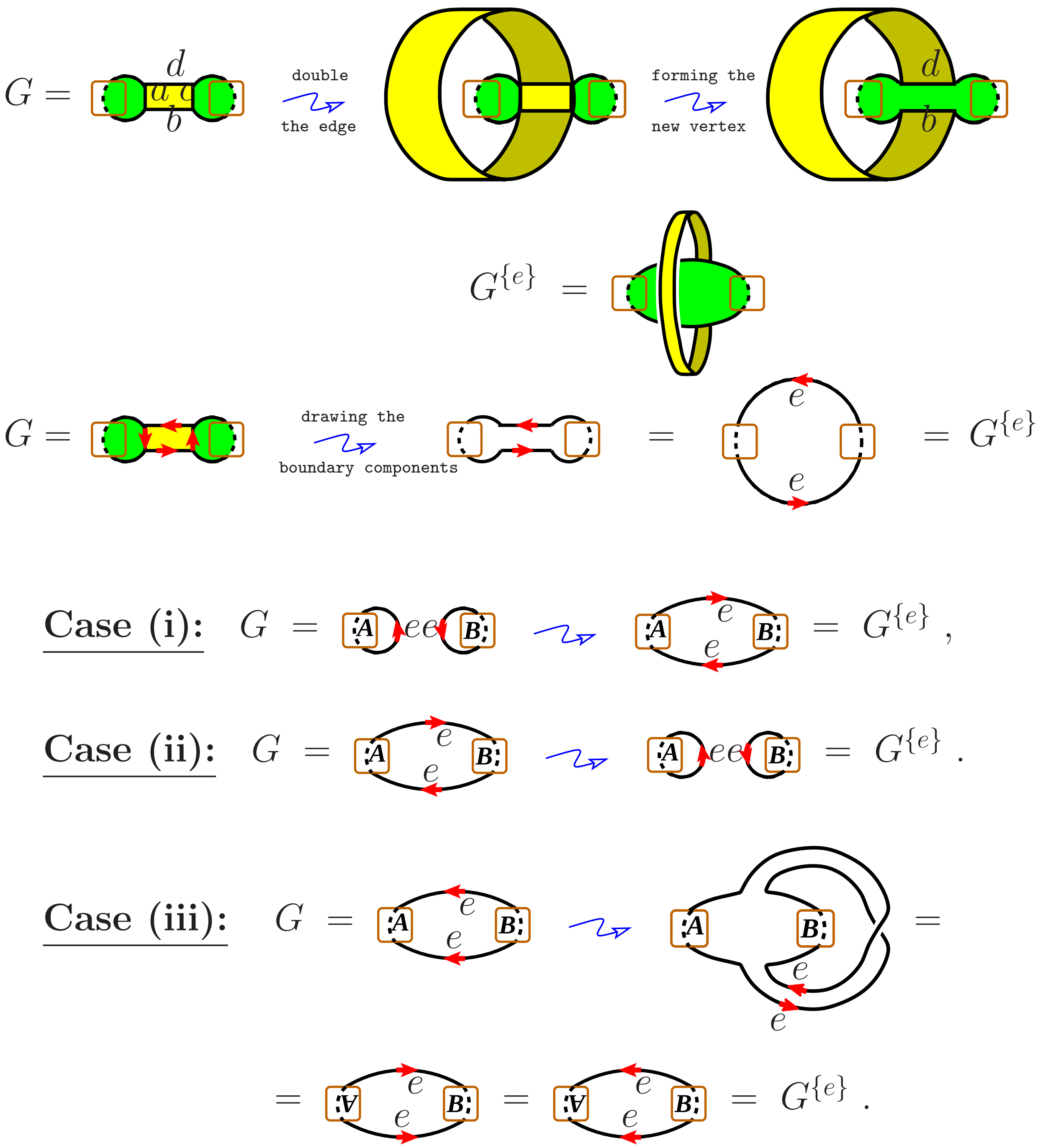
$$R_G(x, y, z) = x + 2 + y + xyz^2 + 2yz + y^2 z \ .$$

Duality

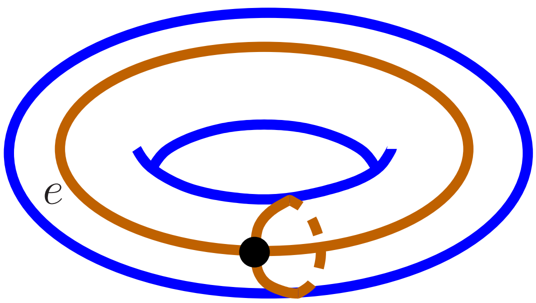


Generalized duality

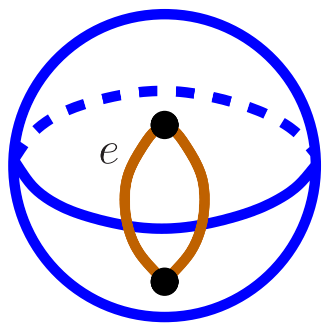



G =
d
a c
b
double
the edge
forming the
new vertex
d
b
G^{e} =
G =
drawing the
boundary components
=
e
e
= G^{e}
Case (i):
G =
A
ee
B
e
e
= G^{e} ,
Case (ii):
G =
A
e
e
B
ee
= G^{e} .
Case (iii):
G =
A
e
e
B
=
e
e
= G^{e} .

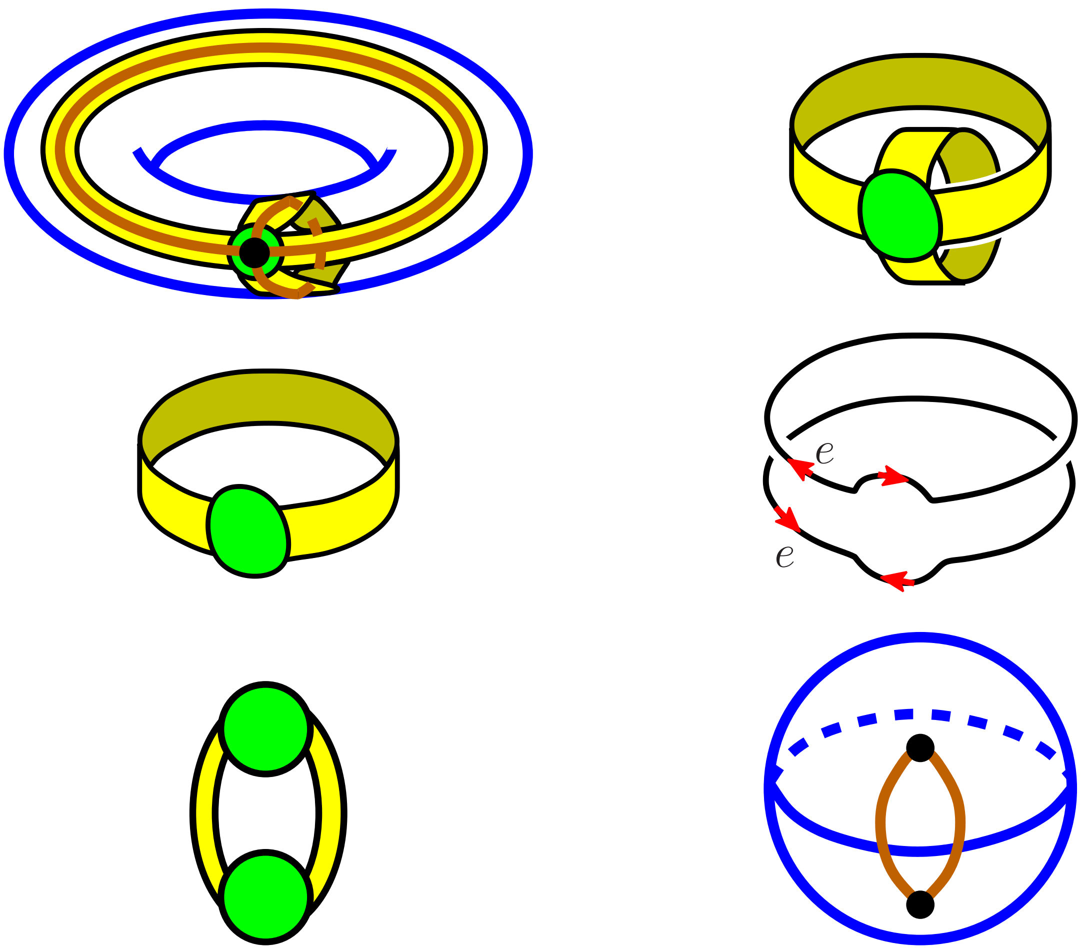
Examples

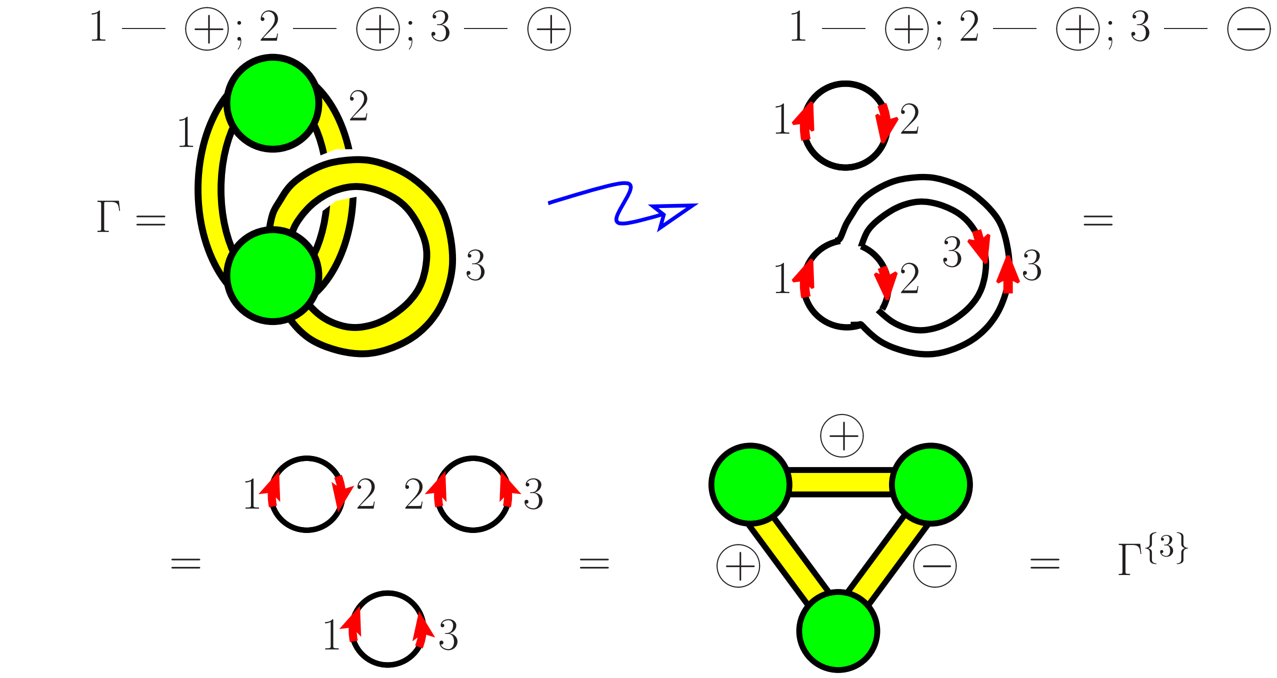


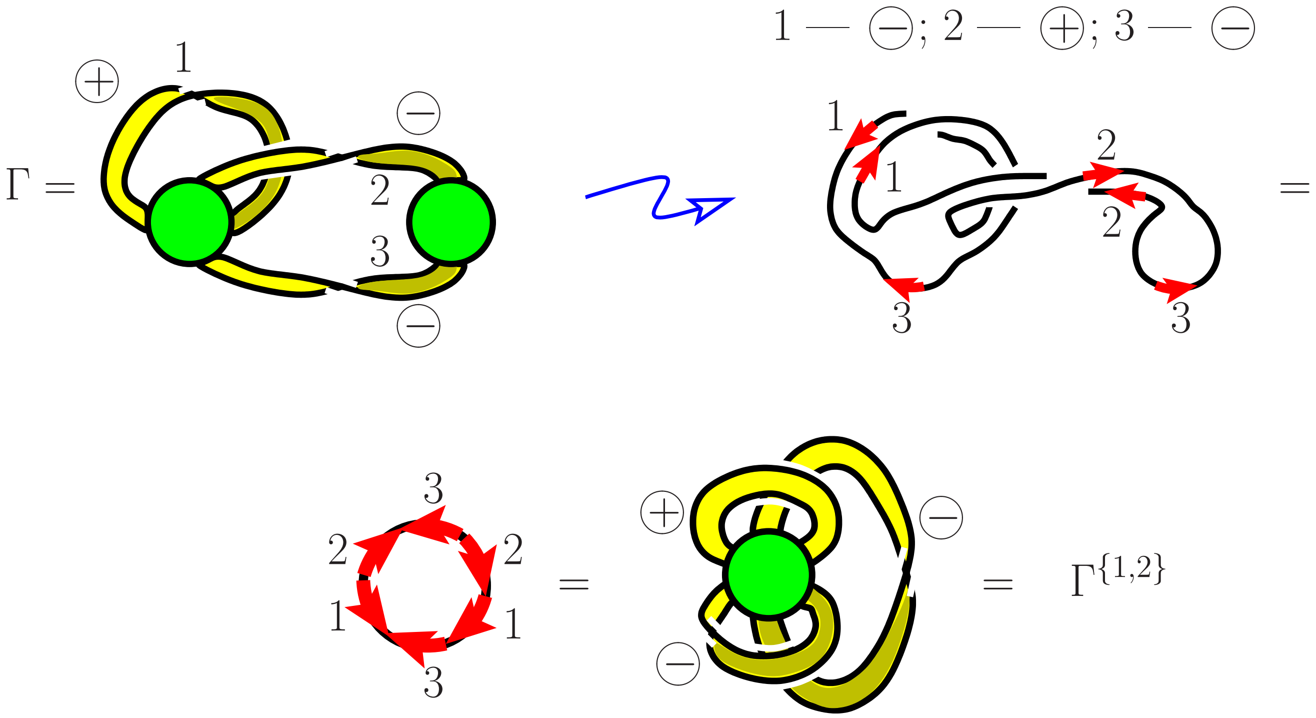
Graph Γ on a torus



Dual graph $\Gamma^{\{e\}}$ with respect to the edge e is embedded into a sphere




1 — ⊕; 2 — ⊕; 3 — ⊕
1 — ⊕; 2 — ⊕; 3 — ⊖
Γ =
=
$\Gamma^{\{3\}}$


1 — ⊖; 2 — ⊕; 3 — ⊖
Γ =
=
$\Gamma^{\{1,2\}}$

Duality theorem [Ch]

For any choice of the subset of edges E'. the restriction of the polynomial $x^{k(G)}y^{v(G)}z^{v(G)+1}R_G(x,y,z)$ to the surface $xyz^2=1$ is invariant under the generalized duality:

$$x^{k(G)}y^{v(G)}z^{v(G)+1}R_G(x,y,z)\Big|_{xyz^2=1} = x^{k(G')}y^{v(G')}z^{v(G')+1}R_{G'}(x,y,z)\Big|_{xyz^2=1}$$

where $G' := G^{E'}$.

Idea of the proof.

$$x^{k(G)}y^{v(G)}z^{v(G)+1}R_G(x,y,z) = \sum_F M_G(F)$$

One-to-one correspondence $E(G) \supseteq F \leftrightarrow F' \subseteq E(G')$:

An edge e of G' belongs to the spanning subgraph F' if and only if either $e \in E'$ and $e \notin F$, or $e \notin E'$ and $e \in F$.

$$M_G(F)\Big|_{xyz^2=1} = M_{G'}(F')\Big|_{xyz^2=1},$$

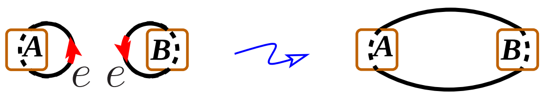
Corollary

Let G be a connected plane ribbon graph, i.e. its underlying graph Γ is embedded into the plane. Then

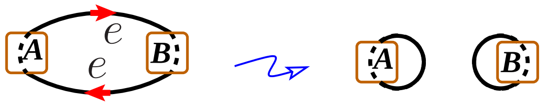
$$T_\Gamma(x,y) = T_{\Gamma^*}(y,x)$$

Duality and contraction-deletion

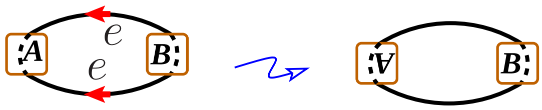
Definition: $G/e \ := \ G^{\{e\}} - e$

Case (i). $G =$ 

$= G/e$.

Case (ii). $G =$ 

$= G/e$.

Case (iii). $G =$ 

$= G/e$.

Lemma. *Let $e \notin E' \subset E(G)$. Then*

$$(G/e)^{E'} = G^{E'\cup e} - e \qquad \textit{and} \qquad (G-e)^{E'} = G^{E'\cup e}/e \ .$$

Proposition. *Let e be positive edge of G. Then*

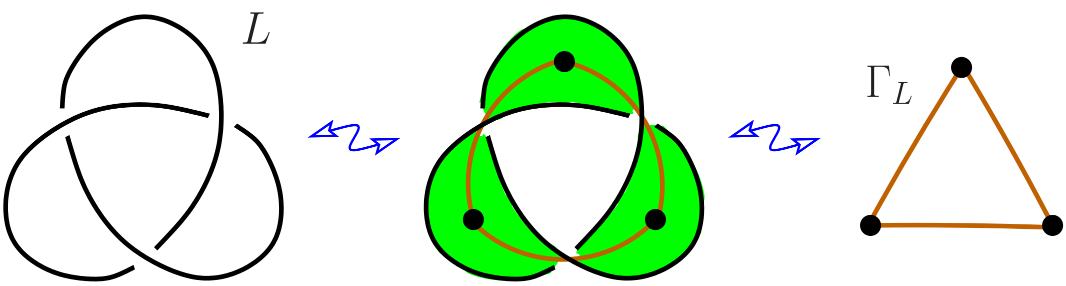
- $R_G = R_{G/e} + R_{G-e}$ *if e is ordinary;*
- $R_G = (x+1)R_{G/e}$ *if e is a bridge;*
- $R_G = (y+1)R_{G-e}$ *if e is a trivial orientable loop;*
- $R_G = (yz+1)R_{G-e}$ *if e is a trivial non-orientable loop;*

For nontrivial loops we have:

- $R_G = R_{G-e} + yzR_{G/e}$ *if e is a non-orientable loop;*
- $R_G|_{xyz^2=1} = R_{G-e}|_{xyz^2=1} + (y/x)R_{G/e}|_{xyz^2=1}$ *if e is a non-trivial orientable loop.*

M. B. Thistlethwaite'87 [Th],
L. Kauffman, K.Murasugi, F.Jaeger

Up to a sign and a power of t the Jones polynomial $V_L(t)$ of an alternating link L is equal to the Tutte polynomial $T_{\Gamma_L}(-t, -t^{-1})$.



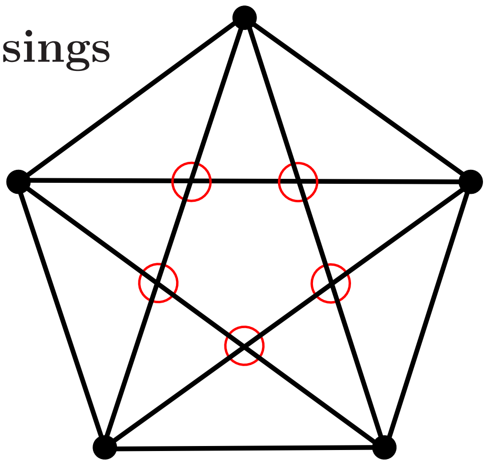
$$V_L(t) = t + t^3 - t^4 = -t^2(-t^{-1} - t + t^2)$$

$$T_{\Gamma_L}(x, y) = y + x + x^2$$

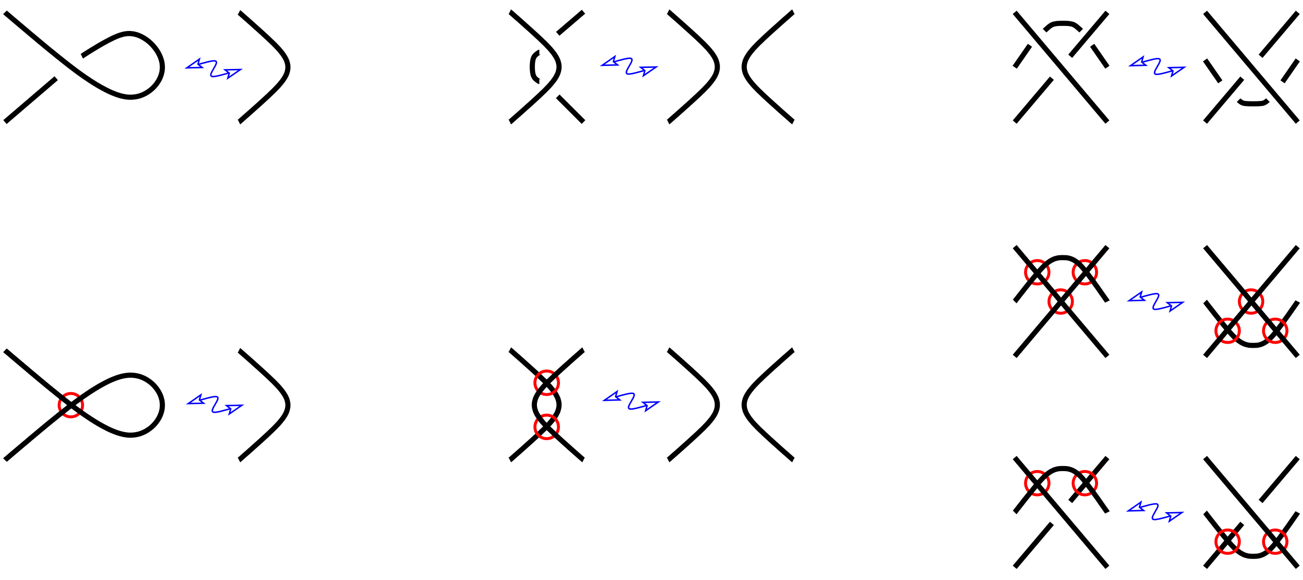
$$T_{\Gamma_L}(-t, -t^{-1}) = -t^{-1} - t + t^2$$

Virtual links

Virtual crossings

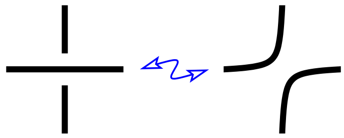


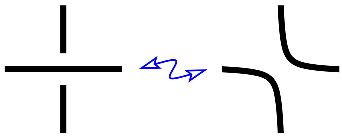
Reidemeister moves



The Kauffman bracket

Let L be a virtual link diagram.

A-splitting: 

B-splitting: 

A *state* S is a choice of either A- or B-splitting at every classical crossing.

$\alpha(S) = \#(\text{of } A\text{-splittings in } S)$

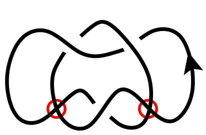
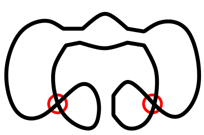
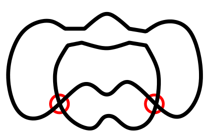
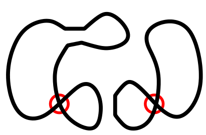
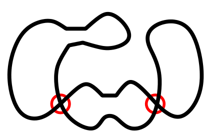
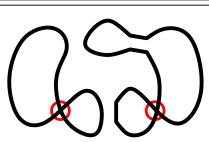
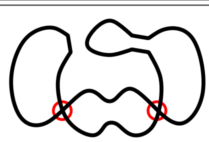
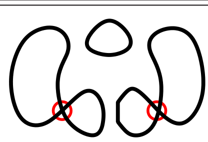
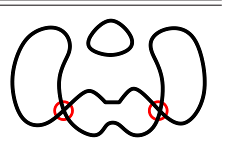
$\beta(S) = \#(\text{of } B\text{-splittings in } S)$

$\delta(S) = \#(\text{of circles in } S)$

$$[L](A, B, d) \ := \ \sum_S A^{\alpha(S)} B^{\beta(S)} d^{\delta(S)-1}$$

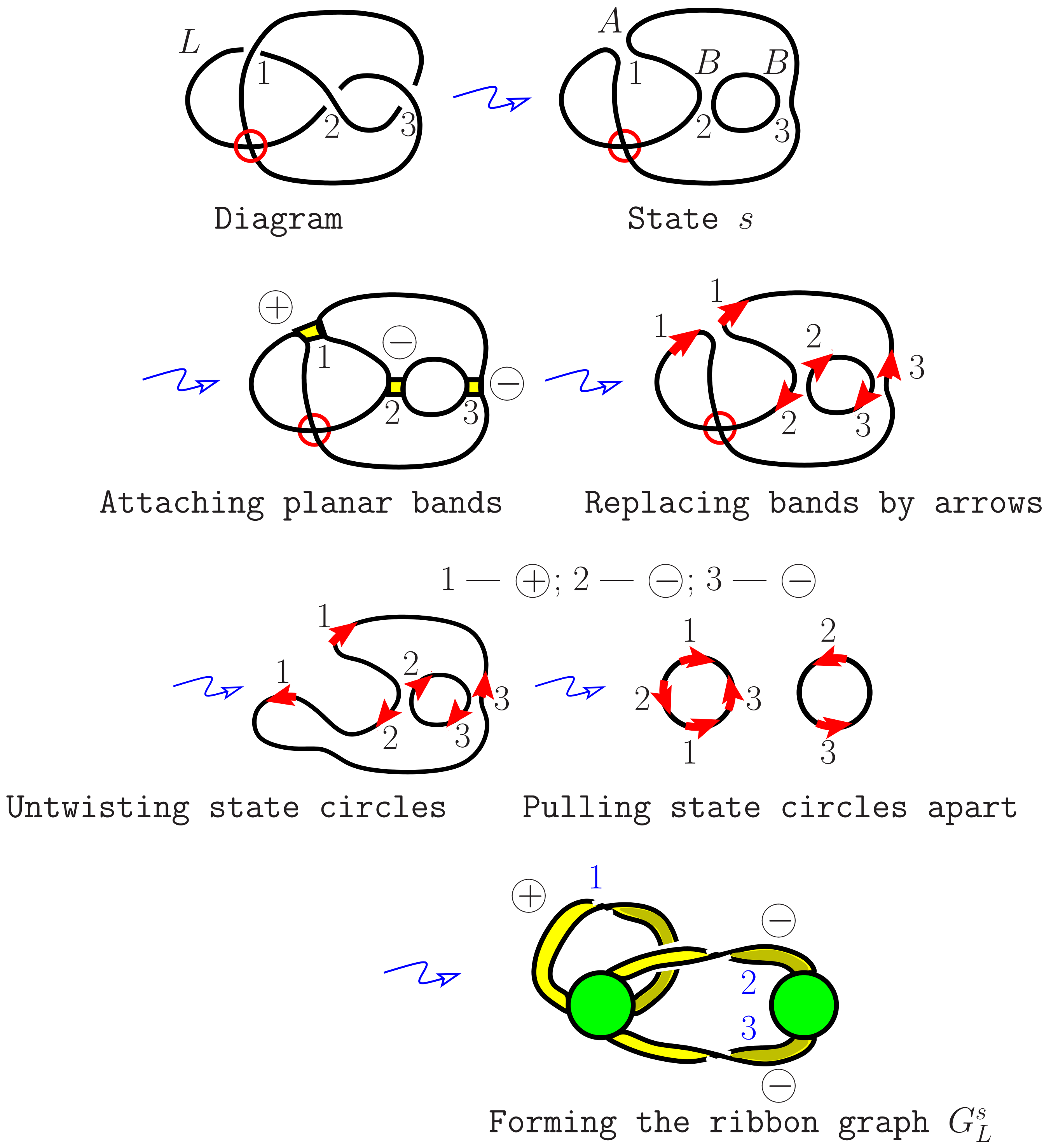
$$J_L(t) \ := (-1)^{w(L)} t^{3w(L)/4} [L](t^{-1/4}, t^{1/4}, -t^{1/2} - t^{-1/2})$$

Example

				
(α, β, δ)	$(3, 0, 1)$	$(2, 1, 2)$	$(2, 1, 2)$	$(1, 2, 1)$
				
	$(2, 1, 2)$	$(1, 2, 1)$	$(1, 2, 3)$	$(0, 3, 2)$

$[L] = A^3 + 3A^2Bd + 2AB^2 + AB^2d^2 + B^3d$; $J_L(t) = 1$

Construction of a ribbon graph from a virtual link diagram



Theorem [Ch]

Let L be a virtual link diagram with e classical crossings, G_L^s be the signed ribbon graph corresponding to a state s, and $v := v(G_L^s)$, $k := k(G_L^s)$. Then $e = e(G_L^s)$ and

$$[L](A,B,d) = A^e \left(x^k y^v z^{v+1} R_{G_L^s}(x,y,z) \Big|_{x=\frac{Ad}{B},\ y=\frac{Bd}{A},\ z=\frac{1}{d}} \right) .$$

Idea of the proof.

One-to-one correspondence between states s' of L and spanning subgraphs F' of G_L^s:

> *An edge e of G_L^s belongs to the spanning subgraph F' if and only if the corresponding crossing was split in s' differently comparably with s.*

Theorem of [CP]: The state s comes from a checkerboard coloring of the diagram L.

Theorem of [CV]: The state s is the Seifert state, i.e. all splittings preserve the orientation of L.

Theorem of [DFKLS]: The state $s = s_A$, i.e. all splittings are A-splittings.

Fabien Vignes-Tourneret (Vienna).

The multivariate signed Bollobás-Riordan polynomial, preprint `arXiv:math.CO/0811.1584`.

The combinatorial part $\Longrightarrow$ the multivariable Bollobás-Riordan polynomial.

Iain Moffatt (University of South Alabama, Mobile).

Partial duality and Bollobas and Riordan's ribbon graph polynomial, preprint `arXiv:math.CO/0809.3014`.

The duality theorem $\Longleftarrow$ the HOMFLYPT polynomial.

References

[BR] B. Bollobás and O. Riordan, *A polynomial of graphs on surfaces*, Math. Ann. **323** (2002) 81–96.

[Ch] S. Chmutov, *Generalized duality for graphs on surfaces and the signed Bollobás-Riordan polynomial*, preprint `arXiv:math.CO/0711.3490`. To appear in Journal of Combinatorial Theory Ser.B.

[CP] S. Chmutov, I. Pak, *The Kauffman bracket of virtual links and the Bollobás-Riordan polynomial*, Moscow Mathematical Journal **7**(3) (2007) 409–418. Preprint `arXiv:math.GT/0609012`,

[CV] S. Chmutov, J. Voltz, *Thistlethwaite's theorem for virtual links*, Journal of Knot Theory and its Ramifications , **17**(10) (2008) 1189-1198. Preprint `arXiv:math.GT/0704.1310`.

[DFKLS] O. Dasbach, D. Futer, E. Kalfagianni, X.-S. Lin, N. Stoltzfus, *The Jones polynomial and graphs on surfaces*, Journal of Combinatorial Theory, Ser.B **98** (2008) 384–399. Preprint `math.GT/0605571`.

[Th] M. Thistlethwaite, *A spanning tree expansion for the Jones polynomial*, Topology **26** (1987) 297–309.