Linking numbers and the Conway polynomial of virtual links.

Knots in Washington XXIX. Saturday, Dec. 5, 2009 ~~3:45 – 4:10.~~ 4:08 –

Joint with Z. Cheng, T. Dokos, J. Lindquist.

I. Classical links L, m components

$\nabla(L)$ Conway polynomial $= c_0 + c_1 z + c_2 z^2 + \dots$

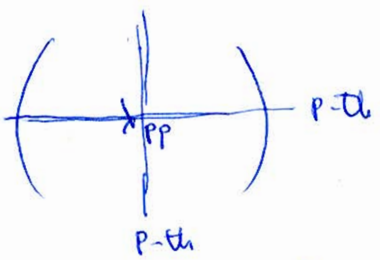
I.1

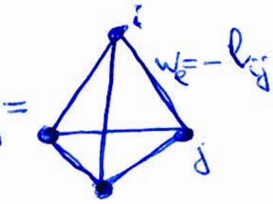
Th. (F. Hosokawa '58, R. Hartley '83, J. Hoste '85)

$$c_0 = c_1 = \dots = c_{m-2} = 0$$

$$c_{m-1} = \det \Lambda^{(p)}$$

$$\Lambda = (\lambda_{ij}), \quad \lambda_{ij} = \begin{cases} -l_{ij}(L) & \text{if } i \neq j \\ \sum_{k \neq i} l_{ik}(L) & \text{if } i = j \end{cases}$$

$\Lambda^{(p)} =$ 

$K_m =$ 

Matrix-Tree Theorem. $\det \Lambda^{(p)} = \sum_{T} \prod_{e \in T} w_e$

spanning tree

I.2 $l_{ij} = 0$ for all i, j, algebraically split links.

Th. (L. Traldi '84, J. Levine '97)

$$c_{m-1}(L) = c_m(L) = \dots = c_{2m-3}(L) = 0$$

$$c_{2m-2}(L) = \det \Lambda^{(p)}$$

$$\Lambda = (\lambda_{ij}), \quad \lambda_{ij} = \sum_k \mu_{ijk}(L)$$

↑ triple Milnor numbers

$$\mu_{ijk}(L) = -\mu_{jik}(L) = \mu_{jki}(L)$$

G. Masbaum, A. Vaintrob '02: Pfaffian Matrix-Tree Theorem

II. Virtual links.

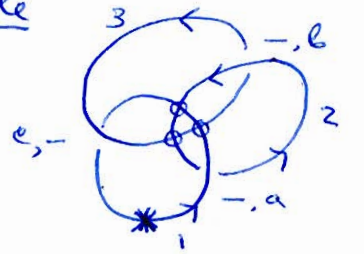
II.1. Linking numbers $l_{i/j} \neq l_{j/i}$

$$l_{i/j} = \sum_{\substack{\text{crossings } x \\ i \text{ over } j}} \varepsilon_x ,$$

Conway polynomial (S.Ch., M. Khoury, A. Rossi '09)

$$\nabla_{asc}(L) = \sum_{\substack{S \\ \text{ascending} \\ \text{one-component}}} \Big(\prod_{x \in S} \varepsilon_x \Big) z^{|S|} , \quad \nabla_{des}(L) = \ldots$$

Example

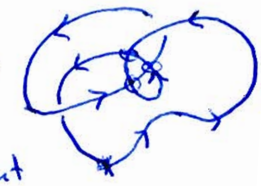


$l_{1/2} = -1, \ l_{2/1} = 0$

$l_{1/3} = 0, \ l_{3/1} = -1$

$l_{2/3} = -1, \ l_{3/2} = 0$

$S = \{a\}$: not one-component but ascending



$S = \{a, l\}$

Smooth the crossings from S according to orientation

S is one-component if the link obtained is a knot

S is ascending if at the first approach to each crossing of S we ~~jump~~ jump down to smooth it

Th.

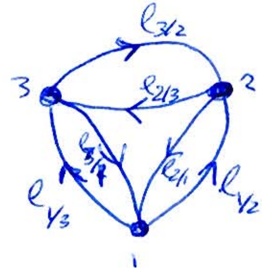
$$c_{m-1} = \det \Lambda^{(i)}$$

$$\Lambda = (\lambda_{ij}), \qquad \lambda_{ij} = \begin{cases} -l_{j/i} & \text{if } i \neq j \\ \sum_{k \neq i} l_{k/i} & \text{if } i = j \end{cases}$$

In the example:

$$\Lambda = \begin{pmatrix} l_{3/1}+l_{2/1} & -l_{2/1} & -l_{3/1} \\ -l_{1/2} & l_{1/2}+l_{3/2} & -l_{3/2} \\ -l_{1/3} & -l_{2/3} & l_{1/3}+l_{23} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\det \Lambda^{(1)} = l_{1/2}\, l_{1/3} + l_{1/2} l_{2/3} + l_{3/2}\, l_{1/3}$$



<u>Oriented Matrix-Tree Theorem</u>

$$\det \Lambda^{(1)} = \sum_{T} \prod_{e \in T} w_e$$

T: oriented spanning trees growing from 1

II.2. work in progress