

ICTP — Trieste — ITALY

Advanced School and Conference on Knot Theory and its Applications to Physics and  
Biology

## Vassiliev invariants

*Sergei Chmutov*

The Ohio State University, Mansfield

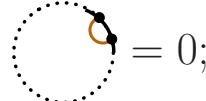
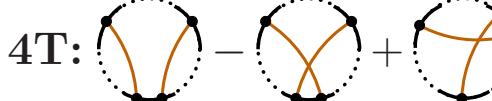
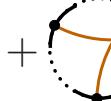
May 11 –29, 2009

Vassiliev skein relation:  $v(\text{Diagram A}) = v(\text{Diagram B}) - v(\text{Diagram C})$

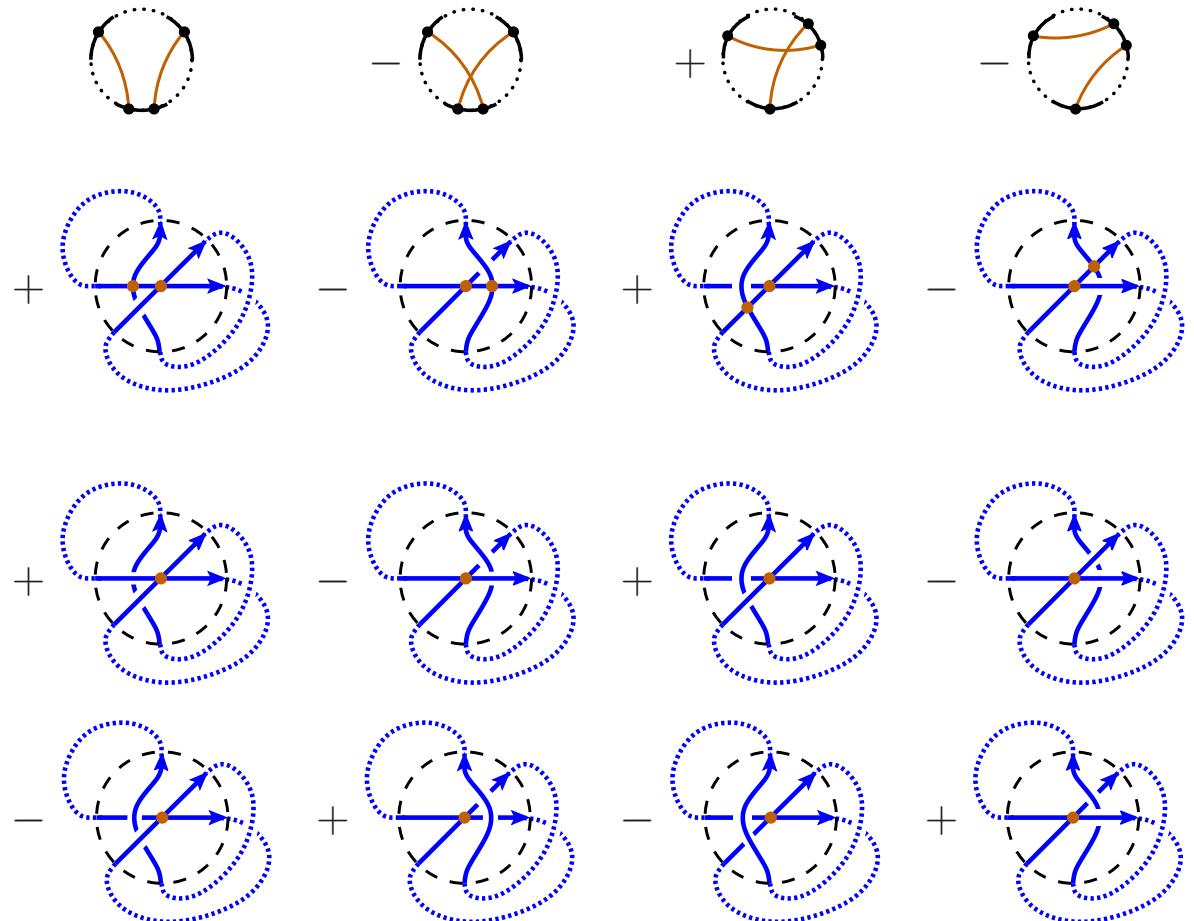
**Def.** Vassiliev invariant of order  $\leq n$ :  $v|_{\mathcal{K}_{n+1}} \equiv 0$ .

$\mathcal{V}_n := \{\text{Vassiliev invariants of order } \leq n\}$ .  $\text{symb}(v) := v|_{\mathcal{K}_n}$

$$\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \cdots \subseteq \mathcal{V}_n \subseteq \cdots \subseteq \cdots$$

1T:  = 0;      4T:  -  +  -  = 0

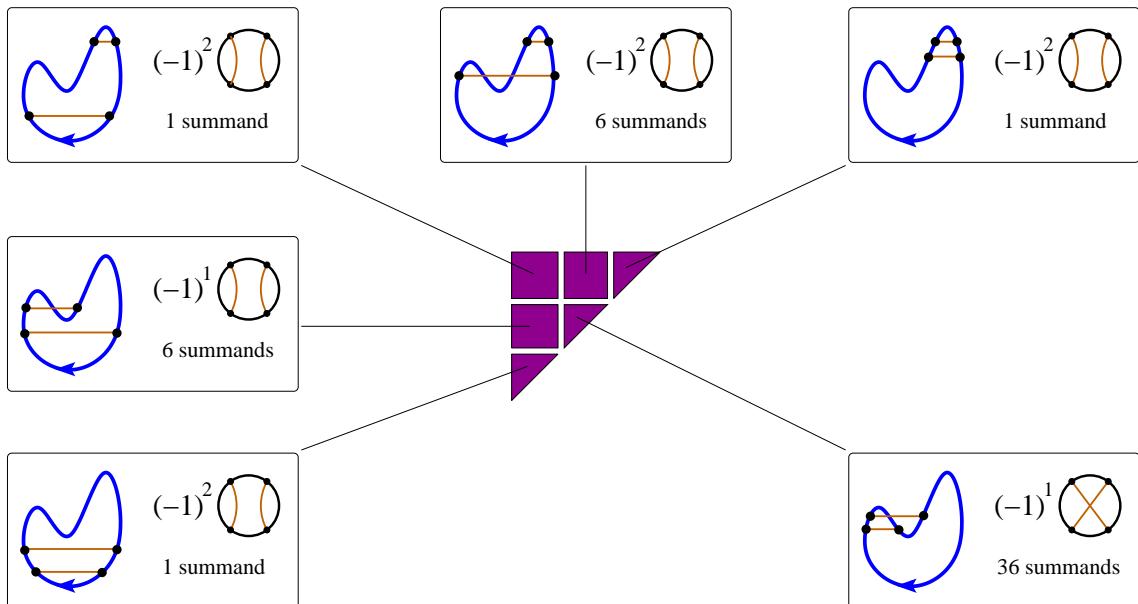
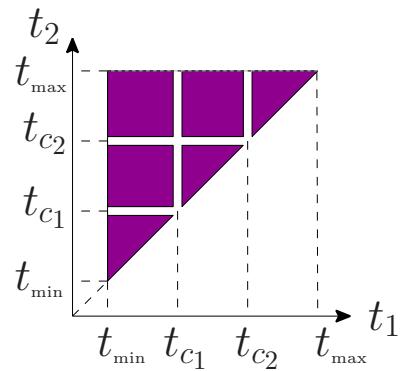
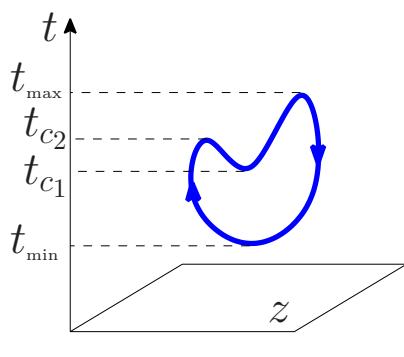
$\mathcal{A}_n := \text{span}(\text{chord diagrams with } n \text{ chords}) / (1T, 4T)$



## The Kontsevich integral.

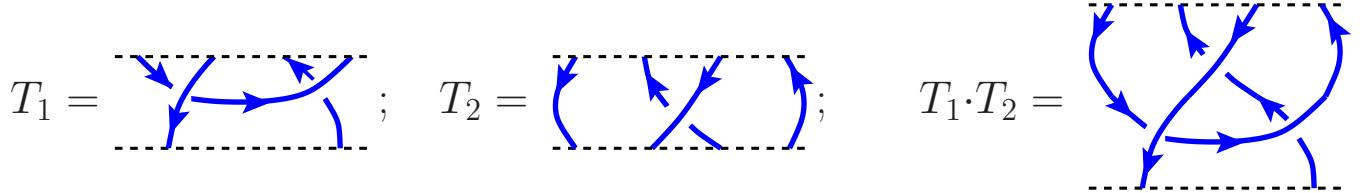
$$Z(K) := \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int_{t_{\min} < t_1 < \dots < t_m < t_{\max}} \sum_{P=\{(z_j, z'_j)\}} (-1)^{\downarrow} D_P \bigwedge_{j=1}^m \frac{dz_j - dz'_j}{z_j - z'_j}$$

$t_j$  are noncritical



## Horizontal invariance

### Tangle multiplication



Tangled chord diagrams.

Tangled 1T relation:

$$\text{Diagram: } \begin{array}{c} \text{arc} \\ \diagdown \quad \diagup \end{array} = 0 \quad \text{or} \quad \text{Diagram: } \begin{array}{c} \text{arc} \\ \diagup \quad \diagdown \end{array} = 0 .$$

Tangled 4T relation:

$$t_{ij} := (-1)^{\downarrow} \left| \begin{array}{c|c|c} \cdots & \cdots & \cdots \\ \diagdown & \diagup & \diagdown \\ i & j & k \end{array} \right| ,$$

$$[t_{ij} + t_{ik}, t_{jk}] = 0$$

$$(-1)^{\downarrow} \left| \begin{array}{c|c|c} \cdots & \cdots & \cdots \\ \diagdown & \diagup & \diagdown \\ i & j & k \end{array} \right| - (-1)^{\downarrow} \left| \begin{array}{c|c|c} \cdots & \cdots & \cdots \\ \diagup & \diagdown & \diagup \\ i & j & k \end{array} \right| + (-1)^{\downarrow} \left| \begin{array}{c|c|c} \cdots & \cdots & \cdots \\ \diagup & \diagdown & \diagup \\ i & j & k \end{array} \right| - (-1)^{\downarrow} \left| \begin{array}{c|c|c} \cdots & \cdots & \cdots \\ \diagup & \diagup & \diagup \\ i & j & k \end{array} \right| = 0 .$$

Example.

$$\text{Diagram: } \begin{array}{c} \text{arc} \\ \diagup \quad \diagdown \end{array} - \text{Diagram: } \begin{array}{c} \text{arc} \\ \diagdown \quad \diagup \end{array} - \text{Diagram: } \begin{array}{c} \text{arc} \\ \diagup \quad \diagup \end{array} + \text{Diagram: } \begin{array}{c} \text{arc} \\ \diagdown \quad \diagdown \end{array} = 0$$

$$\omega_{ij} := \frac{dz_i - dz_j}{z_i - z_j} , \quad \Omega_{ij} := t_{ij} \frac{dz_i - dz_j}{z_i - z_j} .$$

Horizontal deformation:  $T_\lambda$ .

$$\Delta = \Delta_0 \times [0, 1] = \begin{array}{c} \text{Diagram of a parallelepiped } \Delta \\ \text{with faces } \Delta_0 \text{ and } \Delta_1 \text{ shaded purple.} \end{array} .$$

Stokes' theorem:  $\int_{\partial\Delta} \Omega = \int_{\Delta} d\Omega = 0$ , since  $d\Omega = 0$ .

$\partial\Delta = \Delta_0 - \Delta_1 + \sum \{\text{faces}\}$ . We prove that  $\Omega|_{\{\text{face}\}} = 0$ .

Restriction to the face  $\{t_k = t_{k+1}\}$ :

$$\begin{aligned} & (-1)^\downarrow \begin{array}{c} | \\ \bullet - \bullet \\ | \end{array} \omega_{12} \wedge \omega_{23} + (-1)^\downarrow \begin{array}{c} | \\ \bullet - \bullet \\ | \end{array} \omega_{12} \wedge \omega_{13} \\ & + (-1)^\downarrow \begin{array}{c} | \\ \bullet - \bullet \\ | \end{array} \omega_{13} \wedge \omega_{12} + (-1)^\downarrow \begin{array}{c} | \\ \bullet - \bullet \\ | \end{array} \omega_{13} \wedge \omega_{23} \\ & + (-1)^\downarrow \begin{array}{c} | \\ \bullet - \bullet \\ | \end{array} \omega_{23} \wedge \omega_{12} + (-1)^\downarrow \begin{array}{c} | \\ \bullet - \bullet \\ | \end{array} \omega_{23} \wedge \omega_{13} \end{aligned}$$

$$\begin{aligned}
&= \left( (-1)^{\downarrow} \begin{array}{|c|c|c|} \hline & & \\ \hline \bullet & \bullet & \bullet \\ \hline & & \\ \hline \end{array} - (-1)^{\downarrow} \begin{array}{|c|c|c|} \hline & & \\ \hline \bullet & \bullet & \bullet \\ \hline & \bullet & \\ \hline \end{array} \right) \omega_{12} \wedge \omega_{23} \\
&\quad + \left( (-1)^{\downarrow} \begin{array}{|c|c|c|} \hline & & \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} - (-1)^{\downarrow} \begin{array}{|c|c|c|} \hline & & \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \right) \omega_{23} \wedge \omega_{31} \\
&\quad + \left( (-1)^{\downarrow} \begin{array}{|c|c|c|} \hline & & \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline & & \\ \hline \end{array} - (-1)^{\downarrow} \begin{array}{|c|c|c|} \hline & & \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline & & \\ \hline \end{array} \right) \omega_{31} \wedge \omega_{12} \\
\\
&= \left( (-1)^{\downarrow} \begin{array}{|c|c|c|} \hline & & \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline & & \\ \hline \end{array} - (-1)^{\downarrow} \begin{array}{|c|c|c|} \hline & & \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline & & \\ \hline \end{array} \right) (\omega_{12} \wedge \omega_{23} + \omega_{23} \wedge \omega_{31} + \omega_{31} \wedge \omega_{12}) = 0,
\end{aligned}$$

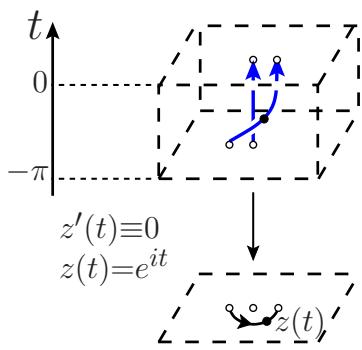
because of the *Arnold identity*:

$$f + g + h = 0 \implies \frac{df}{f} \wedge \frac{dg}{g} + \frac{dg}{g} \wedge \frac{dh}{h} + \frac{dh}{h} \wedge \frac{df}{f} = 0$$

(in our case  $f = z_1 - z_2$ ,  $g = z_2 - z_3$ ,  $h = z_3 - z_1$ )

□

**Example.**  $R := Z\left(\begin{array}{c} \text{blue X} \\ \text{dashed box} \end{array}\right) = \exp\left(\frac{\text{orange vertical double arrow}}{2}\right) \cdot \text{black X}$



Indeed, for one chord:

$$\begin{aligned} \frac{1}{2\pi i} \int_{-\pi}^0 \text{black X} \cdot \frac{dz - dz'}{z - z'} &= \left( \frac{1}{2\pi i} \int_{-\pi}^0 \frac{de^{it}}{e^{it}} \right) \cdot \text{black X} \\ &= \frac{1}{2} \cdot \text{black X} = \left( \frac{\text{orange vertical double arrow}}{2} \right) \cdot \text{black X}. \end{aligned}$$

For two chords:

$$\frac{1}{4\pi^2} \int_{-\pi}^0 \int_{-t_1}^0 \text{black X} \cdot dt_2 dt_1 = \left( \frac{1}{4\pi^2} \int_{-\pi}^0 -t_1 dt_1 \right) \cdot \text{black X} = \frac{1}{2} \left( \frac{\text{orange vertical double arrow}}{2} \right)^2 \cdot \text{black X}.$$

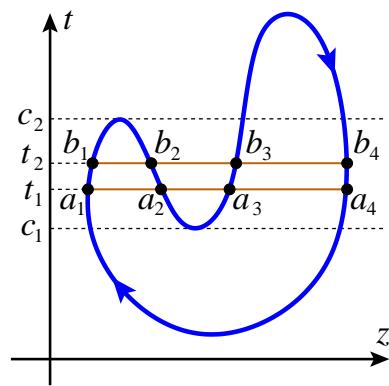
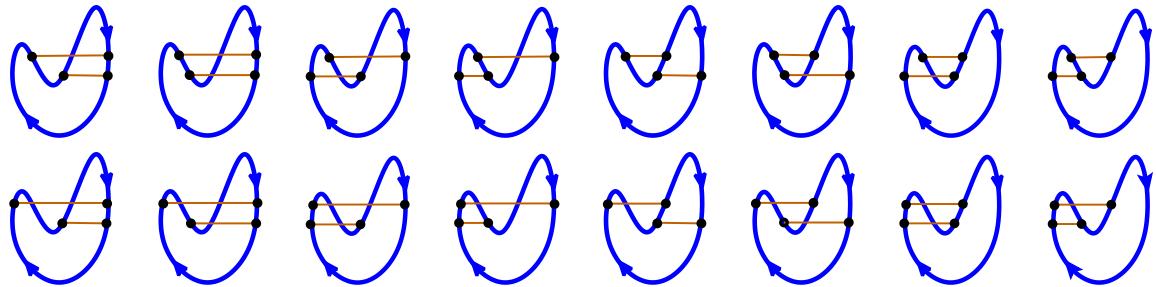
For  $n$  chords we will have:  $\frac{1}{n!} \left( \frac{\text{orange vertical double arrow}}{2} \right)^n \cdot \text{black X}.$

**Exercise.**  $R^{-1} := Z\left(\begin{array}{c} \text{blue X} \\ \text{dashed box} \end{array}\right) = \exp\left(-\frac{\text{orange vertical double arrow}}{2}\right) \cdot \text{black X}$

## Example.

The coefficient of the chord diagram  in  $Z\left(\text{Diagram}\right)$ .

Out of the total number of 51 pairings the following 16 contribute to the coefficient:



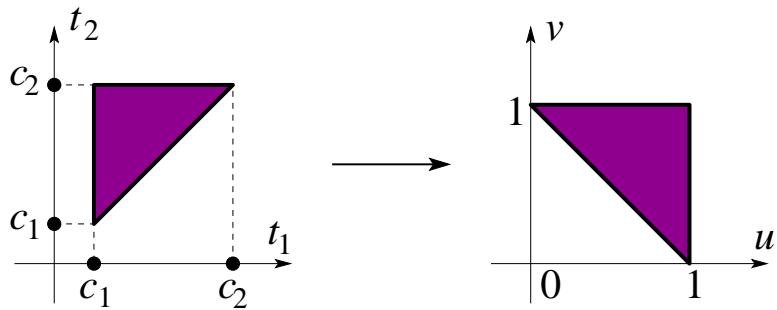
$$\begin{aligned} a_{jk} &:= a_k - a_j; \\ (jk) \in A &:= \{(12), (13), (24), (34)\}; \\ b_{lm} &:= b_m - b_l; \\ (lm) \in B &:= \{(13), (23), (14), (24)\}. \end{aligned}$$

The coefficient of  is equal to

$$\frac{1}{(2\pi i)^2} \int_{\Delta} \sum_{(jk) \in A} \sum_{(lm) \in B} (-1)^{j+k+l+m} d \ln a_{jk} \wedge d \ln b_{lm}$$

$$= -\frac{1}{4\pi^2} \int_{\Delta} \sum_{(jk) \in A} (-1)^{j+k+1} d \ln a_{jk} \wedge \sum_{(lm) \in B} (-1)^{l+m-1} d \ln b_{lm}$$

$$= -\frac{1}{4\pi^2} \int_{\Delta} d \ln \frac{a_{12}a_{34}}{a_{13}a_{24}} \wedge d \ln \frac{b_{14}b_{23}}{b_{13}b_{24}} =$$



Change of variables  
(reversing orientation):

$$u = \frac{a_{12}a_{34}}{a_{13}a_{24}},$$

$$v = \frac{b_{14}b_{23}}{b_{13}b_{24}}.$$

$$= \frac{1}{4\pi^2} \int_{\Delta'} d \ln u \wedge d \ln v = \frac{1}{4\pi^2} \int_0^1 \left( \int_{1-u}^1 d \ln v \right) \frac{du}{u}$$

$$= - \frac{1}{4\pi^2} \int_0^1 \ln(1-u) \frac{du}{u} = \frac{1}{4\pi^2} \sum_{k=1}^{\infty} \int_0^1 \frac{u^k}{k} \frac{du}{u}$$

$$= \frac{1}{4\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\zeta(2)}{4\pi^2} = \frac{1}{24} .$$

## Goussarov Theorem

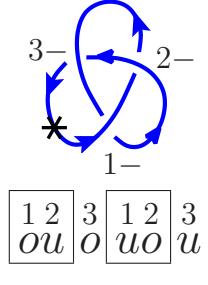
### The Conway polynomial

$$\nabla(\text{X}) - \nabla(\text{X}) = z \nabla(\text{U}) ; \quad \nabla(\text{O}) = 1 .$$

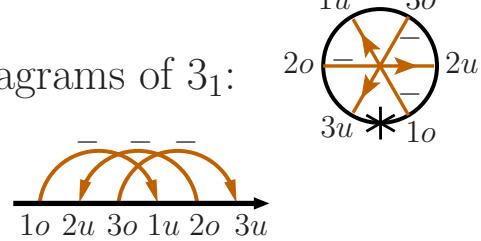
$$\nabla(K) = 1 + c_2(K)z^2 + c_4(K)z^4 + \dots$$

$$c_2(\text{X}) - c_2(\text{X}) = lk(\text{U}) ; \quad c_2(\text{O}) = 0 .$$

**Polyak–Viro formula:**  $c_2(K) = \sum_{\substack{i,j \\ ouuo}} \varepsilon_i \varepsilon_j .$



Gauss diagrams of  $3_1$ :



$$c_2(K) = \langle \text{X}, G_K \rangle := \langle \text{X} - \text{X} - \text{X} + \text{X}, G_K \rangle .$$

Map  $I : \mathbb{Z}[\text{GD}] \rightarrow \mathbb{Z}[\text{GD}], \quad I(D) := \sum_{D' \subseteq D} D'$

$$I\left(\text{X}\right) = \text{X} + \text{X} + \text{X} + \text{X} + \text{X} + \text{X}$$

**Theorem** (Goussarov). *For any  $v \in \mathcal{V}_n$  there is a function  $c : \mathbb{Z}[GD] \rightarrow \mathbb{Z}$  such that  $v = c \circ I$  and  $c(D) = 0$  for  $|D| > n$ .*

$$\text{Inverse map: } I^{-1}(D) := \sum_{D' \subseteq D} (-1)^{|D-D'|} D'$$

$$c = v \circ I^{-1}$$

However one should extend  $v$  to non-realizable Gauss diagrams.

**Mixed Gauss diagrams with chords** representing singular knots.

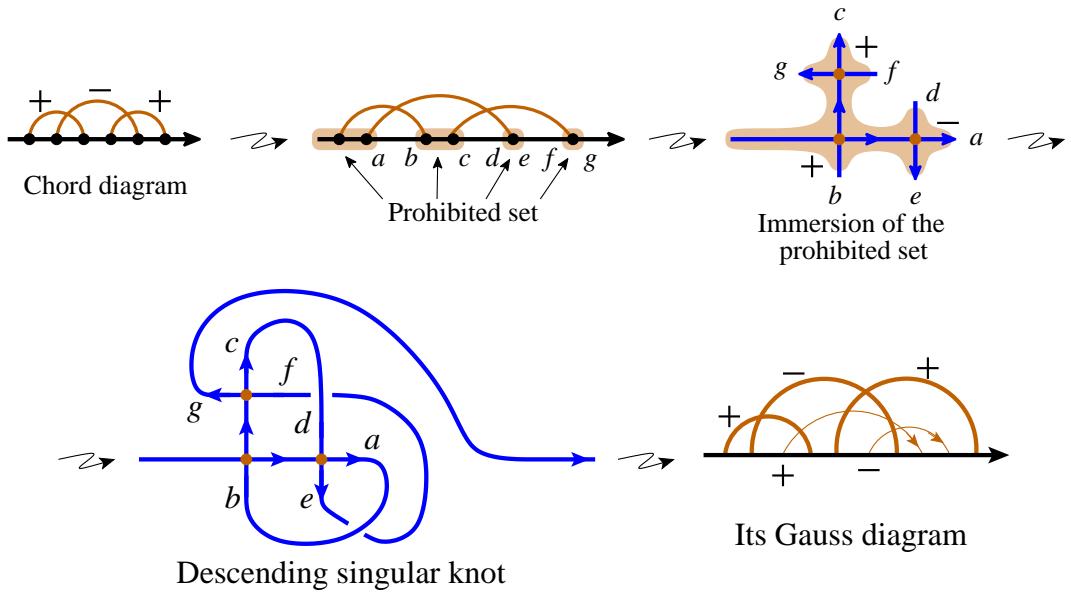
A Gauss diagram is *descending* if

- (1) *all the arrows are directed to the right, and*
- (2) *no endpoint of an arrow can be followed by the left endpoint of a chord.*

Forbidden situations:



**Lemma.** *Each long chord diagram with signed chords underlies a unique (up to isotopy) singular classical long knot that has a descending Gauss diagram.*



A map  $P$  making a diagram “more descending”:

- (1) Replace all the left-pointing arrows of  $D$  by the right-pointing according to the Vassiliev skein relation:

$$\text{Diagram with orange arrow pointing left} \xrightarrow{\quad} \text{Diagram with orange arrow pointing right} + \varepsilon \text{ Diagram with orange arrow pointing right}$$

- (2) Remove “prohibited pairs”:

$$\begin{array}{ccc} \text{Diagram with prohibited pair} & \xrightarrow{\quad} & \text{Diagram without prohibited pair} \\ \text{Diagram with prohibited pair} & \xrightarrow{\quad} & \text{Diagram without prohibited pair} \end{array}$$

For a (non-realizable) Gauss diagram  $D$  there is a number  $m$  such that  $P^m(D)$  is a linear combination of descending diagrams, modulo the diagrams with more than  $n$  chords.

### Extend of $v$ to non-realizable Gauss diagrams.

If  $D$  is a descending Gauss diagram with signed chords, there exists precisely one singular classical knot  $K$  which has a descending diagram with the same signed chords. We set

$v(D) := v(K)$ . Now, if  $D$  is an arbitrary diagram, then we apply the previous algorithm to obtain a linear combination  $\sum a_i D_i$  of descending diagrams. Set  $v(D) := \sum a_i v(D_i)$ .

□

## Vassiliev invariants coming from the HOMFLYPT polynomial

$$aP(\text{X}) - a^{-1}P(\text{X}) = zP(\text{Y}) ; \quad P(\text{O}) = 1 .$$

Make a substitution  $a = e^h$  and take the Taylor expansion  $P(K) = \sum_{k,l} p_{k,l}(K)h^k z^l$ .

### Goussarov's Lemma.

*The coefficient  $p_{k,l}$  is a Vassiliev invariant of order  $\leq k + l$ .*

$$p_{k,l}(K) =: \langle A_{k,l}, G_K \rangle$$

Combinations  $A_{k,l}$  for small  $k$  and  $l$ .

$$A_{0,2} = \text{Diagram} ; \quad A_{2,0} = 0 ;$$


---

$$A_{0,3} = 0 ; \quad A_{3,0} = -4A_{1,2} ;$$

$$A_{1,2} = -2 \left( \text{Diagram} + \text{Diagram} + \text{Diagram} + \text{Diagram} + \text{Diagram} - \text{Diagram} + \text{Diagram} \right. \\ \left. + \text{Diagram} - \text{Diagram} \right) ;$$


---

$$A_{0,4} = \text{Diagram} + \text{Diagram} + \text{Diagram} + \text{Diagram} + \text{Diagram} + \\ + \text{Diagram} + \\ + \text{Diagram} ;$$

$$A_{2,2} = 78 \text{ terms.}$$