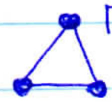
Polynomials of graphs on surfaces

OSU Top. seminar
Nov. 30, 2010
3:30pm.

Motivation: Thistlethwaite's theorem '87

L  $\sim$  Γ_L $\quad V_L(t) = \pm t^k\, T_{\Gamma_L}(-t, -t^{-1})$

Tutte polynomial

$$T_\Gamma(x,y) := \sum_{H \subseteq \Gamma} (x-1)^{k(H)-k(\Gamma)} (y-1)^{n(H)} \quad \left(\prod_{e\in H} x_e \prod_{e \notin H} y_e\right)$$

spanning subgraph of Γ:
$V(H) = V(\Gamma)$, $E(H) \subseteq E(\Gamma)$

$k(H) := b_0(H)$
$n(H) := b_1(H)$

Potts model (stat. mechanics)
T_Γ partition function.

Properties:
- $T_\Gamma = T_{\Gamma - e} + T_{\Gamma/e}$, e is ordinary
- $T_\Gamma = x\, T_{\Gamma/e}$, e is a bridge
- $T_\Gamma = y\, T_{\Gamma - e}$, e is a loop
- $T_{\Gamma_1 \sqcup \Gamma_2} = T_{\Gamma_1} \cdot T_{\Gamma_2}$
- $T_\circ = 1$

Thistlethwaite
- I. Pak '04 → Bollobás–Riordan polynomial (Virtual links)
- Y. Diao, G. Hetyei '08 → Relative Tutte polynomial

Clark Butler

Virtual links
L. Kauffman '99: virtual Reidemeister;
M. Goussarov, M. Polyak, O. Viro '00
G. Kuperberg: $L \hookrightarrow \Sigma \times I$
$$\downarrow$$
$$\Sigma \longrightarrow \mathbb{R}^2$$

isotopy +  = 

Bollobás–Riordan polynomial

$\mathbb{R}$ ribbon graph

$$B_{\mathbb{R}}(X,Y,Z) = \sum_{H \subseteq \mathbb{R}} \Big(\prod_{e\in H} x_e\Big)\Big(\prod_{e\notin H} y_e\Big) X^{c(H)-c(\mathbb{R})} Y^{n(H)} Z^{c(H)+n(H)-f(H)}$$

$f(H)$ = # of boundary components of H (faces)

$\hat{\mathbb{R}} = \Sigma_{\mathbb{R}}$

if H is orientable

$$2c(H) - \chi(\hat{H}) = c(H) + n(H) - f(H)$$
$= 2g$
$= \#\ \mu$ (Möbius bands)

Virtual Thistlethwaite

Theorem (J. Voltz, – '06)

$$[L](A,B,d) = A^{n} B^{e-n} d^{k-1} B_{\mathbb{R}_L}\left(\frac{Ad}{B}, \frac{Bd}{A}, \frac{1}{d}\right)$$

$x_+ = y_+ = 1$
$x_- = B/A$,
$y_- = A/B$

(Y. Diao, G. Hetyei '08)

Relative Tutte polynomial of plane graphs

$$T_{G,F} = \sum_{H \subseteq G \setminus F} \Big(\prod_{e\in H} x_e\Big)\Big(\prod_{e\notin (H\cup F)} y_e\Big) X^{c(H\cup F)-c(G)} Y^{n(H)} \psi(F_H)$$

0-edges

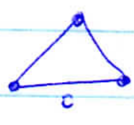
(M. Carnovale '09
Y. Dong
J. Jeffries)

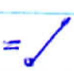
F_H is obtained from $H \cup F$ by contracting edges of H

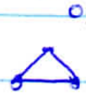
$$F_H := (H \cup F)/H$$

$$\psi(F_H) := d^{\delta(F_H) - c(F_H)} \cdot w^{v(F_H) - c(F_H)}$$

C. Butler

G, F: 

$H =$ , $H \cup F$: , $F_H =$  $\delta(F_H) = 1$

$H =$ , $H \cup F$: , F_H  $\delta(F_H) = 1$

Theorem

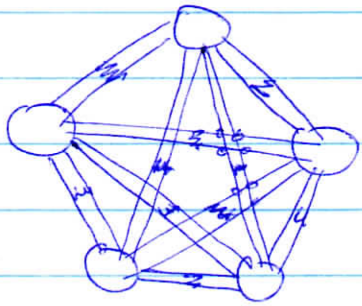
$$[L](A,B,d) = A^{v-c} B^{|E(G\setminus F)| - v(G) + c(G)} d^{c(G)-1} T_{G,F}$$

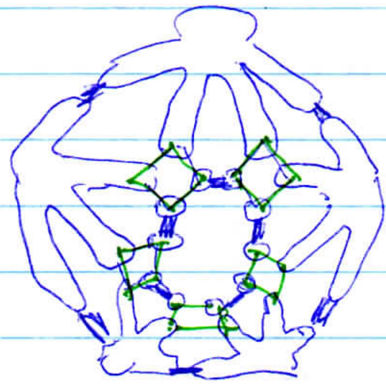
$X = \frac{Bd}{A}$, $Y = \frac{Ad}{B}$, $w = B/A$, $x_+ = y_+ = 1$, $x_- = B/A$, $y_- = A/B$

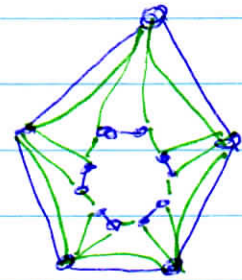
Theorem (C. Butler, — '10)

$$X^{\alpha} Y^{\beta} T_{G,F}(X,Y) = B_R\left(X, Y, \frac{1}{\sqrt{XY}}\right)$$

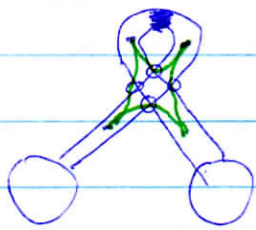
$$\alpha = c(G) - c(R) - \beta$$
$$\beta = -\tfrac{1}{2}\left(v(R) - v(G)\right).$$



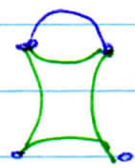




0-edges



R



o-edges

G

C. Estill, D. Grollmus, M. Ro '06

J. Belcher, R. Greene '07

M. Las Vergnas polynomial

V. Krushkal polynomial '09

$$P_{G,\Sigma}(X,Y,A,B) = \sum_{H \subseteq G} X^{c(H)-c(G)}\, Y^{k(H)}\, A^{s(H)/2}\, B^{s^{\perp}(H)/2}$$

orientable

$$k(H) := \dim \ker\left(H_1(H) \to H_1(\Sigma)\right)$$

$$s(H) := 2g(|H|) = \dim\left(V / V \cap V^{\perp}\right) \quad \Big|\ V := \operatorname{im}\left(H_1(H) \to H_1(\Sigma)\right)$$

$$s^{\perp}(H) := 2g(\Sigma - |H|) = \dim\left(V^{\perp} / (V \cap V^{\perp})\right)$$

Theorem (Krushkal '09)

$$B_G(X,Y,Z) = Y^{g} P_{G,\Sigma}\left(X, Y, YZ^2, Y^{-1}\right)$$

Th. (R. Askanazi, C. Estill, J. Michel, P. Stollenwerk '10)

$$LV_{G,\Sigma}(x,y,z) = z^{g} P_{G,\Sigma}\left(x-1, y-1, z, z^{-1}\right)$$

Matroids (M, r) H. Whitney '35

M, $r : 2^M \to \mathbb{Z}_{\geq 0}$

finite set; 2^M = subsets of M

Axioms

(R1) $r(\emptyset) = 0$

(R2) $H \subset M \ni y \notin H$

$$r(H \cup y) = \begin{cases} r(H) & \text{or} \\ r(H)+1 \end{cases}$$

(R3) if $r(H \cup y) = r(H \cup z) = r(H)$, then

$$r(H \cup \{y, z\}) = r(H)$$

Example

G graph, $M = E(G)\ (= \mathcal{C}(G))$

$$r(H) = v(G) - c(H)$$

nullity: $\boxed{n(H) := |H| - r(H)}$

Dual matroid M^*

$$r_{M^*}(H) := |H| + r_M(M \setminus H) - r_M(M)$$

$$r_{M^*}(M) + r_M(M) = |M|$$

Whitney duality theorem

$(\mathcal{C}(G))^*$ is graphical $\Leftrightarrow$ G is planar

$$\left(\mathcal{C}(G)\right)^* = \mathcal{C}(G^*)$$

For $G \hookrightarrow \Sigma$

$M := (\mathcal{C}(G^*))^* \longrightarrow M' := \mathcal{C}(G)$ is

a matroid perspective

Def. $r_M(X) - r_M(Y) \geq r_{M'}(X) - r_{M'}(Y)$ for all $Y \subseteq X$

$$\boxed{LV_{G,\Sigma}(x,y,z) = \sum_{H \subseteq M} (x-1)^{r(M') - r_{M'}(H)} (y-1)^{n_M(H)} \cdot z^{(r(M) - r_M(H)) - (r(M') - r_{M'}(H))}}$$