Pell's equations

Math circles

University of South Alabama
Monday, April 19, 2010
7:00–8:30 pm
ILB 410

Diophantine equation

$$x^2 - my^2 = 1$$

$m=1$
$x^2-y^2=1$
$(x+y)(x-y)=1$
$y=0,\ x=\pm 1$

$m=2$ $\quad x, y$ integers >0

$$x^2-2y^2=\pm 1$$

n	0	1	2	3				
$(-1)^n = x^2-2y^2$	1	-1	$+1$	-1	1	-1	1	-1
x_n	1	1	3	7	17	41	99	239
y_n	0	1	2	5	12	29	70	169
n	0							

Express in terms of n.

$X = x+2y,\ Y = x+y,\quad x^2-2y^2=\pm 1 \Rightarrow X^2-2Y^2=\mp 1$

Check!

Exercise

$x_0=1,\ y_0=0;\ x_1=1,\ y_1=1,\quad x_2=3,\ y_2=2,\quad x_3=7,\ y_3=5\ \ldots$

$$x_{n+1}=x_n+2y_n$$
$$y_{n+1}=x_n+y_n$$

$$U = X+2Y = x+2y+2(x+y) = 3x+4y$$
$$V = X+Y = (x+2y)+(x+y) = 2x+3y$$

$$\boxed{\text{if } x^2-2y^2=1 \Rightarrow U^2-2V^2=1}$$

Powers of $(1+\sqrt{2})$

$$(1+\sqrt2)^2 = 1+2\sqrt2+2 = 3+2\sqrt2$$

$$(1+\sqrt2)^3 = 1+3\sqrt2+\underbrace{3\cdot(\sqrt2)^2}_{6}+\underbrace{(\sqrt2)^3}_{2\sqrt2} = 7+5\sqrt2$$

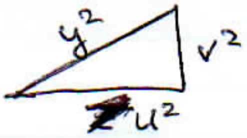
$$(1+\sqrt2)^4 = (3+2\sqrt2)^2 = 9+12\sqrt2+8 = 17+12\sqrt2$$

$$(1+\sqrt2)(1+\sqrt2)(1+\sqrt2) = 1\cdot1\cdot1+1\cdot1\cdot\sqrt2+1\cdot\sqrt2\cdot1+1\cdot\sqrt2\cdot\sqrt2$$
$$+\sqrt2\cdot1\cdot1+\sqrt2\cdot1\cdot\sqrt2+\sqrt2\cdot\sqrt2\cdot1+\sqrt2\cdot\sqrt2\cdot\sqrt2$$
$$=1+3\sqrt2+6+2\sqrt2 = 7+5\sqrt2$$

$$(1+\sqrt2)^n = x_n+y_n\sqrt2$$
$$(1-\sqrt2)^n = x_n-y_n\sqrt2$$
$$(1+\sqrt2)^n(1-\sqrt2)^n = (-1)^n = x_n^2-2y_n^2$$

$$\boxed{x_n = \frac{(1+\sqrt2)^n-(1-\sqrt2)^n}{2},\quad y_n = \frac{(1+\sqrt2)^n-(1-\sqrt2)^n}{2}}$$

Pythagorean triangles



3, 4, 5 ;
u, v, y ;

$v = u+1$

$$u^2+(u+1)^2=y^2$$
$$2u^2+2u+1=y^2$$
$$4u^2+4u+2=2y^2$$
$$(4u^2+4u+1)+1=2y^2$$
$$(2u+1)^2+1=2y^2$$

$x=2u+1$

$$x^2-2y^2=-1$$

$u=3 \Rightarrow x=7,\ y=5$

$u=20,\ v=21,\ y=29$ $\quad x=41,\ y=29$

$x^2-my^2=1$

$(1,0)$ $\quad (x_1, y_1)$ fundamental solution (the smallest possible $x>1$)

after continued fractions & Chebyshev

$$(x_2,y_2)=(x_1^2+my_1^2,\ 2x_1y_1)$$

$$(x_n,y_n) \rightsquigarrow (x_{n+1}, y_{n+1}) = (x_1x_n + m\,y_1y_n,\ x_1y_n + y_1x_n)$$

Continued fractions

$$\alpha = a_0 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots \cfrac{1}{a_n}}}} = [a_0, a_2, \ldots, a_n]$$

partial quotients

a_0 integer,
a_i integer positive.

$$a_0 = \lfloor \alpha \rfloor$$

$$\alpha = a_0 + \underbrace{\{\alpha\}}_{\alpha_1}$$

$$\left\lfloor \frac{1}{\alpha} \right\rfloor = a_2 \qquad \frac{1}{\{\alpha\}} = a_2 + \left\{\frac{1}{\{\alpha\}}\right\}$$

$$\{\alpha\} = \frac{1}{a_2+\left\{\frac{1}{\{\alpha\}}\right\}}$$

Exercises

1) $$17/3 = 5 + \frac{2}{3} = 5 + \frac{1}{3/2} = 5 + \frac{1}{1+\frac{1}{2}} =$$

$$3/2 = 1 + \frac{1}{2}$$

$$= [5, 1, 2]$$

2) $$3/17 = 0 + \frac{1}{17/3} =$$

$$= 0 + \frac{1}{5 + \frac{1}{1+\frac{1}{2}}}$$

3) $$\cancel{\frac{8}{6} = 1 + \frac{2}{6} = 1 + \frac{1}{6/2} = 1 + \frac{1}{3}}$$

$$(\sqrt{2}-1)(\sqrt{2}+1) = 1$$

3) $$\sqrt{2} = 1 + (\sqrt{2}-1) = 1 + \frac{1}{\sqrt{2}+1}$$

$$\sqrt{2}+1 = 2 + \frac{1}{\sqrt{2}+1}$$

$$= 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 \ddots}}}} = [1, 2, 2, 2, \ldots 2 \ldots]$$

4) $$\sqrt{5} = [2, 4, 4, 4 \ldots\ldots]$$

5) $$\sqrt{3} = 1 + (\sqrt{3}-1) = 1 + \frac{1}{\left(\frac{\sqrt{3}+1}{2}\right)}$$

$$(\sqrt{3}-1)(\sqrt{3}+1) = 3 - 1 = 2$$

$$(\sqrt{3}-1) = \frac{2}{\sqrt{3}+1}$$

$$= 1 + \frac{1}{1 + \frac{\sqrt{3}-1}{2}} = 1 + \frac{1}{1 + \frac{1}{\sqrt{3}+1}} = 1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{\left(\frac{\sqrt{3}+1}{2}\right)}}} = [1, 1 2\ 1 2\ 1 2 \ldots]$$

period $= 2$

Ex. Find

[1, 1, 1, ...]
[2, 1, 1 ...]
[2, 3, 1, 1, ...]

[2, 2, 2, ...]
[1, 2, 1, 2, 1, 2 ...]
[2, 1, 2, 1, 2, 1, ...
[1, 3, 1, 2, 1, 2, 1 2 ...]

Let $\alpha = [a_1, a_2, a_3, \dots, a_n, \dots]$ be infinite continued fraction.

The rational number $r_n = [a_1, a_2, \dots, a_n] = \frac{p_n}{q_n}$ is called n-th convergent to α.

Example Consider $\alpha = \sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}} = [1, 2, 2, \dots]$

Ask ~~the~~ students:

$r_1 = 1 = \frac{1}{1}$

$r_2 = 1 + \frac{1}{2} = \frac{3}{2}$

$r_3 = 1 + \cfrac{1}{2 + \frac{1}{2}} = 1 + \cfrac{1}{5/2} = 1 + \frac{2}{5} = \frac{7}{5}$

$r_4 = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \frac{1}{2}}} = 1 + \cfrac{1}{12/5} = 1 + \frac{5}{12} = \frac{17}{12}$, $\quad r_5 = \frac{41}{29}$, $r_6 = \frac{99}{70}$,

$2/5$ $\qquad r_7 = \frac{239}{169}$

Theorem Let $r_n = \frac{p_n}{q_n}$ be the n-th convergent of $\sqrt{2}$ then $x = p_n$, $y = q_n$ is a solution of Pell's equation $x^2 - 2y^2 = (-1)^n$

Proof. $r_2 + 1 = \frac{p_n + q_n}{q_n}$ is the n-th convergent of $\sqrt{2} + 1 = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}} = [2, 2, \dots$

So the (n+1)-st convergent of $\sqrt{2}$ will be $1 + \frac{1}{r_2 + 1} = 1 + \frac{q_n}{p_n + q_n} = \frac{p_n + 2q_n}{p_n + q_n}$

So $p_{n+1} = p_n + 2q_n$

$q_{n+1} = p_n + q_n$

$$\sqrt{3} = 1 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\dots}}}}}} = [1,1,2,1,2,1,2,1,2\dots]$$

$$B \begin{cases} x \to 2x+3y \\ y \to x+2y \end{cases}$$

$$r_1 = 1 = \frac{1}{1} \qquad x^2 - 3y^2 = -2$$

$$r_2 = 1 + \frac{1}{1} = 2 = \frac{2}{1} \qquad x^2 - 3y^2 = 1$$

$$r_3 = 1 + \cfrac{1}{1+\cfrac{1}{2}} = 1 + \frac{1}{3/2} = 1 + \frac{2}{3} = \frac{5}{3}, \qquad x^2 - 3y^2 = -2$$

B

$$r_4 = 1 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1}}} = 1 + \frac{1}{4/3} = 1 + \frac{3}{4} = \frac{7}{4}, \qquad x^2 - 3y^2 = +1$$

$$r_5 = 1 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2}}}} = 1 + \cfrac{1}{1+\cfrac{1}{8/3}} = 1 + \cfrac{1}{1+\cfrac{3}{8}} = 1 + \frac{1}{11/8} = 1 + \frac{8}{11} = \frac{19}{11}$$

$2/3$

$$x^2 - 3y^2 = -2$$

$r_n - 1$ = n-th convergent for

$$= \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\dots}}}} \Bigg\} n \text{ levels.}$$

$$\begin{array}{r} 19 \\ 19 \\ \hline 171 \\ 19 \\ \hline 361 \\ -363 \\ \hline -2 \end{array}$$

$$\boxed{x^2 - 3y^2 = \frac{3(-1)^n - 1}{2}}$$

$$r_{n+1} = 1 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\dots}}}}$$

$x \to ax+by$

$y \to cx+dy$

$$\overbrace{(a^2+bc)}^{2} x + \overbrace{(ab+bd)}^{\substack{3 \\ b(a+d)}} y$$

$$\underbrace{(ca+dc)}_{c(a+d)=1} x + \underbrace{(bc+d^2)}_{2} y$$

$b = 3c$

$a^2+3c^2=2$

$$a^2 + \frac{3}{4a^2} = 2 \qquad 4a^4 - 4a^2 + 3 = 0$$

$$4a^4 - 2(2a^2-1)^2 + 2 = 0$$

$$\begin{cases} a^2 + 3c^2 = 2 \\ 2ac = 1 \\ c = \frac{1}{2a} \end{cases}$$

$$(2a^2-1)^2 = -2$$

Chebyshev polynomials

$\cos\alpha - \cos\beta = 2\cos\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$

First kind.

$T_n(x) = \cos(n\varphi)$, where $x = \cos\varphi$

$$\cos(n+1)\varphi + \cos(n-1)\varphi = 2\cos\varphi\cos n\varphi$$

$$\boxed{T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)}$$

$$T_0(x) = 1$$
$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x$$
$$T_4(x) = 8x^4 - 8x^2 + 1 = 2(2x^2-1)^2 - 1 = 2(4x^2 - 4x + 1) - 1$$

$$T_k(T_m(x)) = T_m(T_k(x)) = T_{k \cdot n}(x)$$

Second kind.

$U_n(x) = \dfrac{\sin(n\varphi)}{\sin\varphi}$, if $x = \cos\varphi$ $\qquad U_n(x) = \frac{1}{n}T_n'(x)$

$$U_0(x) = 0$$
$$U_1(x) = 1 \qquad U_{n+1}(x) = 2x\,U_n(x) - U_{n-1}(x)$$

$$U_2(x) = 2x$$
$$U_3(x) = 4x^2 - 1$$
$$U_4(x) = 8x^3 - 4x^2$$

$$\frac{1}{n+1}T_{n+1}' = 2x\frac{1}{n}T_n' - \frac{1}{n-1}T_{n-1}'$$

$$T_{n+1}' = 2T_n + 2xT_n' - T_{n-1}'$$

?

$\sin(n+1)\varphi + \sin(n-1)\varphi =$
$= \sin n\varphi\cos\varphi + \cancel{\cos n\varphi\sin\varphi}$
$+ \sin n\varphi\cos\varphi - \cancel{\cos n\varphi\sin\varphi}$
$= 2\sin n\varphi\cos\varphi$

$$\cos^2 n\varphi + \sin^2 n\varphi = 1$$
$$T_n^2(x) + (1-x^2)U_n^2(x) = 1$$

If (x_1, y_1) is a (fundamental) solution of $x^2 - my^2 = 1 \Rightarrow -m = \frac{1-x_1^2}{y_1^2}$ $\qquad (2x_1^2 - 1, 2x_1y_1)$

Then $(T_n(x_1), y_1U_n(x_1))$ is also a solution.