Two approaches to virtual Thistlethwaite's theorem

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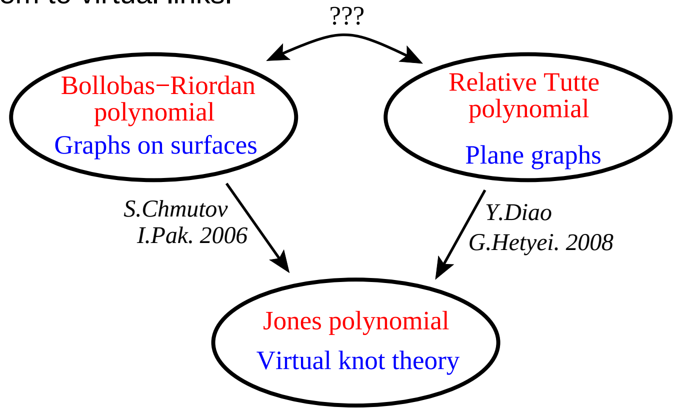
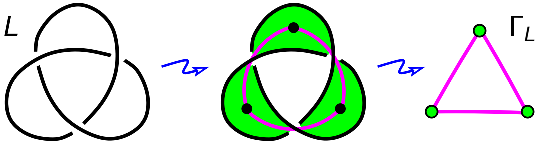
Friday, December 3, 2010
4:55 — 5:20 p.m.

Motivation

Knot theory: two different generalizations of Thistlethwaite's theorem to virtual links.



Up to a sign and a power of t the Jones polynomial $V_L(t)$ of an alternating link L is equal to the Tutte polynomial $T_{\Gamma_L}(-t,-t^{-1})$.



$$V_L(t) = t + t^3 - t^4 = -t^2(-t^{-1} - t + t^2)$$

$$T_{\Gamma_L}(x,y) = y + x + x^2$$

$$T_{\Gamma_L}(-t,-t^{-1}) = -t^{-1} - t + t^2$$

The Bollobás–Riordan polynomial

For a doubly weighted ribbon graph G with weights (x_e, y_e) of an edge $e \in G$ we have

$$B_G(X,Y,Z) = \sum_{F\subseteq G} (\prod_{e\in F} x_e)(\prod_{e\in G\setminus F} y_e) X^{k(F)-k(G)} Y^{n(F)} Z^{k(F)-bc(F)+n(F)},$$

where

- $k(F)$ *be the number of components of* F*;*
- $n(F) := e(F) - v(F) + k(F)$ *be the nullity of* F*;*
- $\mathrm{bc}(F)$ *be the number of boundary components of* F*.*

Virtual Thistlethwaite's theorem (J. Voltz, — '06)

Let L be a virtual link diagram, G_L be the corresponding signed ribbon graph, and $n := n(G_L)$, $k := k(G_L)$,

$$x_+ := y_+ := 1,\ x_- := \frac{B}{A},\ y_- := \frac{A}{B}\ .$$

Then

$$[L](A,B,d) = A^n B^{e-n} d^{k-1} R_{G_L}\left(\frac{Ad}{B}, \frac{Bd}{A}, \frac{1}{d}\right)\ .$$

Construction of G_L

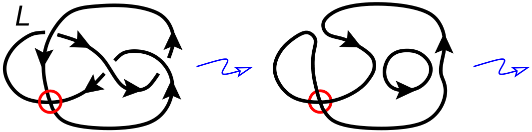
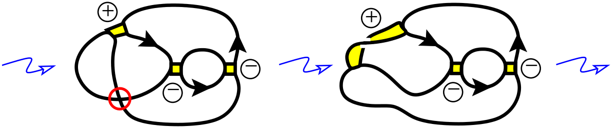
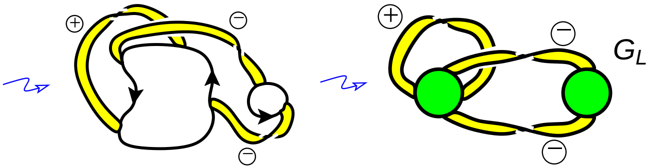


Diagram Seifert state



Attaching bands to Seifert circles

Untwisting Seifert circles



Pulling Seifert circles apart

Glue in the vertex-discs

The relative Tutte polynomial

$$T_{\Gamma,H} := \sum_{F\subseteq\Gamma\setminus H} (\prod_{e\in F} x_e)(\prod_{e\in\overline{F}} y_e) X^{k(F\cup H)-k(\Gamma)} Y^{n(F)} \psi(H_F)$$

where $\overline{F} := \Gamma \setminus (F \cup H)$, and $H_F := (F \cup H)/F$. Our choice of ψ is

$$\psi(H_F) := d^{\delta(H_F)-k(H_F)} w^{v(H_F)-k(H_F)} ,$$

$\delta(H_F)$ is the number of circles that immerse to the medial graph of H_F.

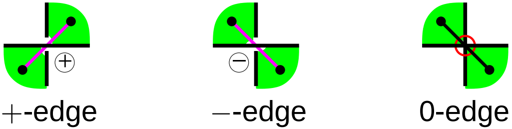
Virtual Thistlethwaite's theorem (Y. Diao, G.Hetyei'08)

Let L be a virtual link diagram, and Γ *the relative plane Tait graph of L. Then, under the substitution*

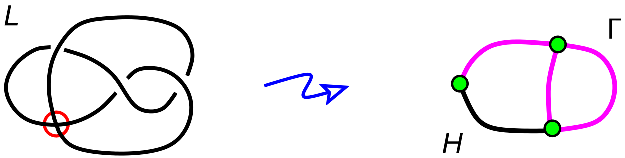
$$X=\frac{Bd}{A},\ Y=\frac{Ad}{B},\ w=\frac{B}{A},\ x_+=y_+=1,\ x_-=\frac{B}{A},\ y_-=\frac{A}{B}$$

we have,

$$[L](A,B,d)=A^{v(\Gamma)-k(\Gamma)}B^{|E(\Gamma\setminus H)|-v(\Gamma)+k(\Gamma)}d^{k(\Gamma)-1}T_{\Gamma,H}\ .$$

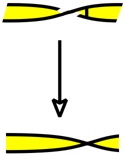


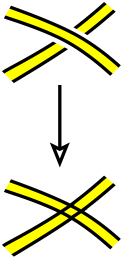
Example.



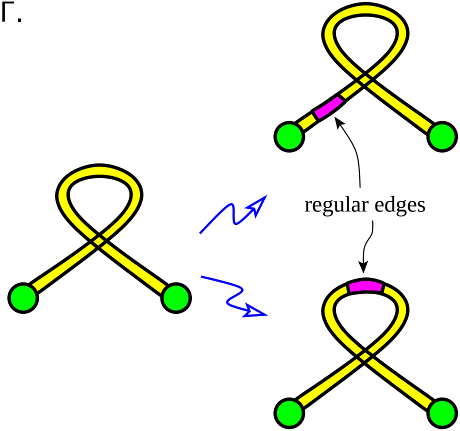
From ribbon graphs to relative plane graphs

Let G be a ribbon graph. Consider a planar projection of G which is 1-to-1 except the points of singularities. These singularities are restricted to two types.
The first occurs when a ribbon twists over itself; in this case a whole line interval on the ribbon is projected to a single point.
The second type occurs when the images of two edge ribbons cross. In this case, the projection is 2-to-1 over the disc of the intersection.



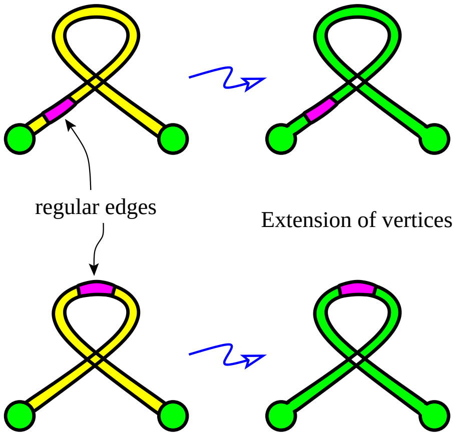


On each edge of G we choose a portion of the ribbon on which the projection is 1-to-1. We will call it a regular edge. The regular edges are the non-zero edges of the relative plane graph Γ.



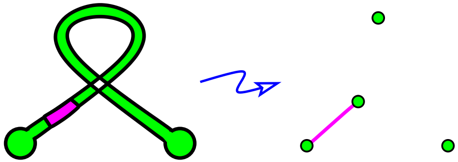
Construction of Γ. (Step 2.)

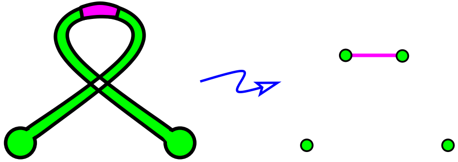
Extend the vertex discs of G through to the regular edge of each ribbon.



Construction of Γ. (Step 3.)

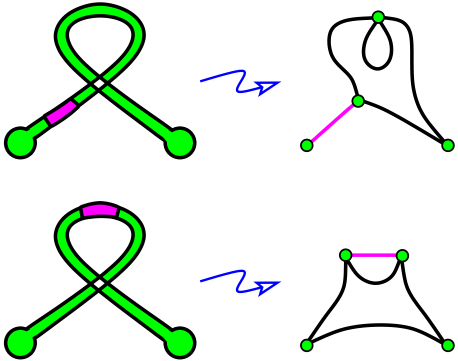
Each of these extended vertices is segmented by the regular edges and the singularities of the projection. These segments become the vertices of G.



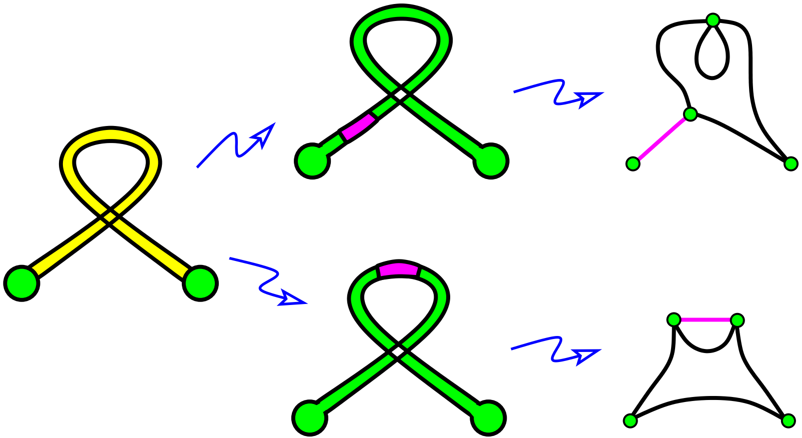


Construction of G. (Step 4.)

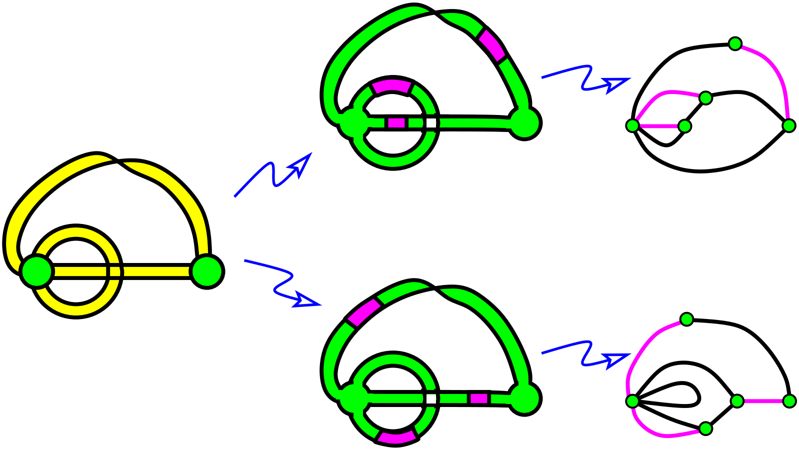
The 0-edges of Γ correspond to the double points of the restriction of the projection to the boundary of G. They connect the vertices of Γ which correspond to the extended regions sharing the same double point in a checkerboard manner.



Example



Another example



Theorem

Suppose G is a ribbon graph, and Γ is a relative plane graph of a projection of G.

Then under the substitution $w = \sqrt{\frac{X}{Y}}, d = \sqrt{XY}$,

$$X^{\alpha} Y^{\beta} T_{G,H}(X, Y) = B_R(X, Y, \frac{1}{\sqrt{XY}}) ,$$

where $\beta = -\frac{1}{2}(v(R) - v(G))$,

$$\alpha = k(G) - k(R) - \beta .$$