1058th AMS Meeting
Macalester College
St. Paul, Minnesota

Virtual linking numbers and the Conway polynomial

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10:30 – 11:00

Joint work with

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I. Classical links

L link with m components

$$\nabla(L) = c_0 + c_1 z + c_2 z^2 + \ldots$$

the Conway polynomial

I.1 Theorem

(F. Hosokawa '58, R. Hartley '83, J. Hoste '85)

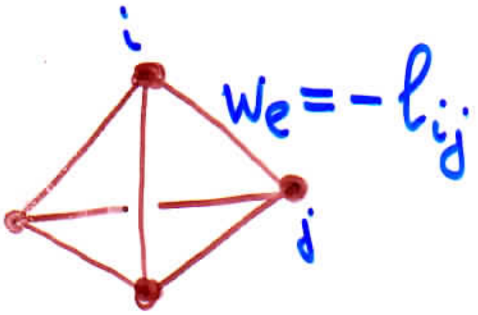
$$c_0 = c_1 = \ldots = c_{m-2} = 0$$

$$c_{m-1} = \det \Lambda^{(p)}$$

$$\Lambda = (\lambda_{ij}), \quad \lambda_{ij} = \begin{cases} -lk_{ij}(L) & \text{if } i \neq j \\ \sum_{k \neq i} lk_{ik}(L) & \text{if } i = j \end{cases}$$

$$\Lambda^{(p)} = \begin{array}{cc} \begin{pmatrix} \ddots & \vdots & \cdots \\ \cdots & \lambda_{pp} & \cdots \\ \cdots & \vdots & \ddots \end{pmatrix} & p\text{-th} \\ \quad p\text{-th} & \end{array}$$

Matrix-Tree Theorem



$$\det \Lambda^{(p)} = \sum_{\substack{T \\ \text{spanning} \\ \text{tree}}} \prod_{e \in T} w_e$$

I.2 algebraically split links

$$\ell_{ij} := lk_{ij}(L) = 0$$
for all i, j.

Theorem (L. Traldi '84, J. Levine '97)

$$c_{m-1} = c_m = \dots = c_{2m-3} = 0$$

$$c_{2m-2} = \det M^{(p)}$$

$$M = (m_{ij}), \quad m_{ij} = \sum_k \mu_{ijk}(L)$$

triple Milnor numbers

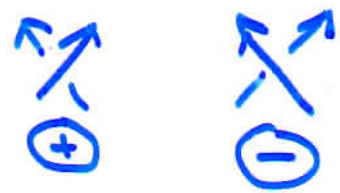
$$+\mu_{ijk}(L) = -\mu_{jik}(L) = +\mu_{jki}(L)$$

G. Masbaum, A. Vaintrob '02
Pfaffian Matrix-Tree Theorem

II Virtual links

II.1 Linking numbers $\ell_{i/j} \neq \ell_{j/i}$

$$\ell_{i/j} := \sum_{{}_i \times {}_j} \varepsilon_{\times} ,$$



Conway polynomial

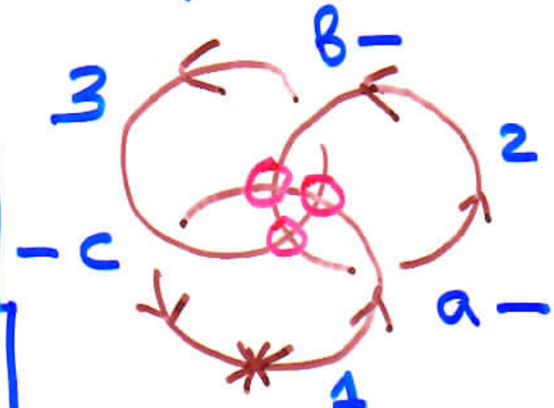
(—, M. Khoury, A. Rossi '09)

$$\nabla_{asc}(L_*) = \sum_{\substack{S \\ \text{ascending} \\ \text{one-component}}} \left(\prod_{\times \in S} \varepsilon_{\times} \right) \cdot z^{|S|}$$

- Smooth the crossings from S according to the orientation.
- S is **one-component** if the link obtained is a <u>knot</u>.
- S is **ascending** if at the first approach to each crossing of S we <u>jump</u> <u>down</u> to smooth it.

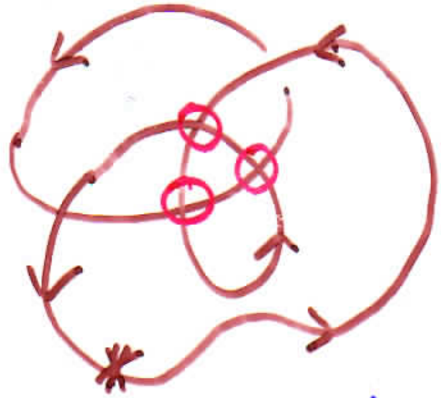
<u>Example.</u>

$l_{1/2}=-1,\ l_{2/1}=0,$
$l_{1/3}=0,\ l_{3/1}=-1,$
$l_{2/3}=-1,\ l_{3/2}=0$



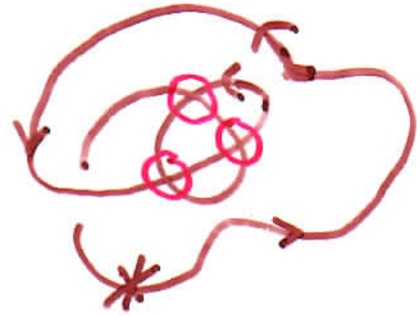
$\nabla_{asc}(L)=z^2$

$S=\{a\}$



<u>not</u> one-component
<u>but</u> ascending

$S=\{a,B\}$



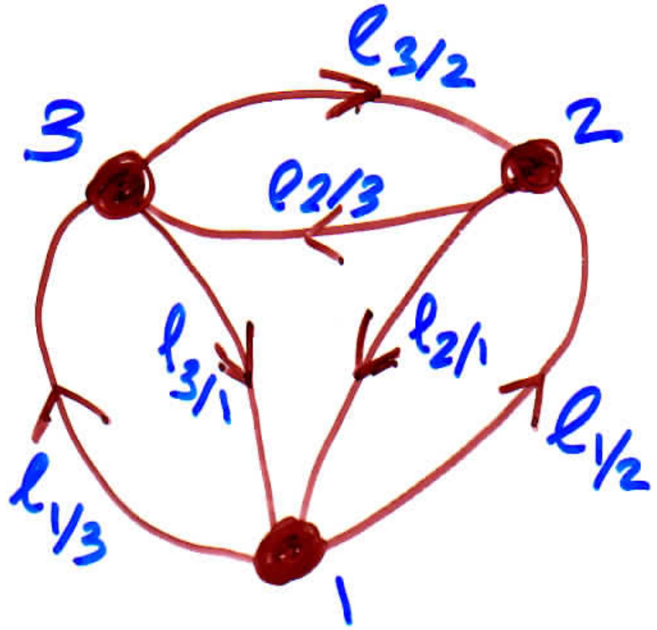
one-component
ascending

Theorem

$$c_{m-1} = \det \Lambda^{(1)}$$

$$\Lambda = (\lambda_{ij}), \quad \lambda_{ij} = \begin{cases} -\ell_{j/i}, & \text{if } i \neq j \\ \sum_{k \neq i} \ell_{k/i}, & \text{if } i = j \end{cases}$$

Oriented matrix-tree Theorem



$$\det \Lambda^{(1)} = \sum_{T} \prod_{e \in T} w_e$$

T: oriented spanning tree growing from vertex 1

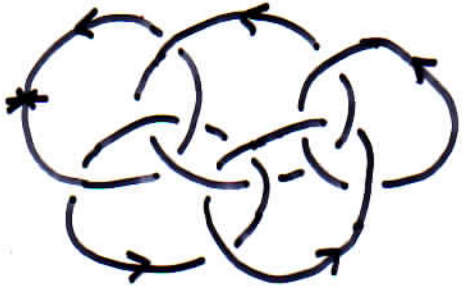
Example $m = 3$

$$\Lambda = \begin{pmatrix} \ell_{3/1}+\ell_{2/1} & -\ell_{2/1} & -\ell_{3/1} \\ -\ell_{1/2} & \ell_{1/2}+\ell_{3/2} & -\ell_{3/2} \\ -\ell_{1/3} & -\ell_{2/3} & \ell_{1/3}+\ell_{2/3} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\det \Lambda^{(1)} = \ell_{1/2}\ell_{1/3} + \ell_{1/2}\ell_{2/3} + \ell_{3/2}\ell_{1/3}$$

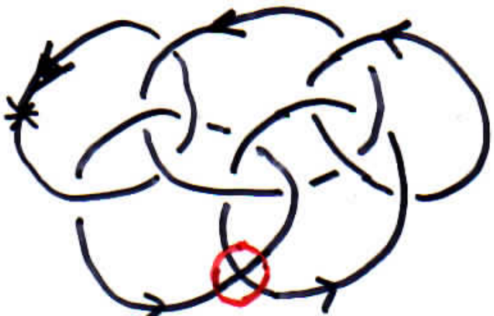
II.2 Work in progress

$L =$ 

$$\nabla_{asc}(L) = z^8$$

$m = 5, \quad 2m-2 = 8$

$m-1 = 4$

$L =$ 

$$\nabla_{asc}(L) = 2 \cdot z^8$$