Physics and mathematics of knots

Sergei Chmutov

Ohio State University, Mansfield

Ohio Wesleyan University

Tuesday, March 1, 2011
12:15 — 12:55 PM
Knot Table

Sergei Chmutov

Physics and mathematics of knots
Unknots = Trivial Knots
Knot isotopy
Reidemeister moves

Planar isotopy:

RI:

RII:

RIII:
Let $G$ be a graph. Particles are located at vertices of $G$. Each particle has a spin, which takes $q$ different values.

A state, $\sigma \in S$, is an assignment of spins to all vertices of $G$. Neighboring particles interact with each other only if their spins are the same. The energy of the interaction along an edge $e$ is $-J_e$ (coupling constant).

The model is called ferromagnetic if $J_e > 0$ and antiferromagnetic if $J_e < 0$. 

$q = 2$ the Ising model (W. Lenz, 1920)
Energy of a state $\sigma$ (*Hamiltonian*),

$$H(\sigma) = - \sum_{(a,b) = e \in E(G)} J_e \delta(\sigma(a), \sigma(b)).$$

*Boltzmann weight* of $\sigma$:

$$e^{-\beta H(\sigma)} = \prod_{(a,b) = e \in E(G)} e^{J_e \beta \delta(\sigma(a), \sigma(b))} = \prod_{(a,b) = e \in E(G)} \left(1 + (e^{J_e \beta} - 1)\delta(\sigma(a), \sigma(b))\right),$$

where the *inverse temperature* $\beta = \frac{1}{\kappa T}$, $T$ is the temperature, $\kappa = 1.38 \times 10^{-23}$ joules/Kelvin is the *Boltzmann constant*. The Potts partition function (for $x_e := e^{J_e \beta} - 1$)

$$Z_G(q, x_e) := \sum_{\sigma \in S} e^{-\beta H(\sigma)} = \sum_{\sigma \in S} \prod_{e \in E(G)} \left(1 + x_e \delta(\sigma(a), \sigma(b))\right).$$
Probability of a state $\sigma$: \[ P(\sigma) := e^{-\beta H(\sigma) / Z_G} . \]

Expected value of a function $f(\sigma)$:

\[ \langle f \rangle := \sum_\sigma f(\sigma) P(\sigma) = \sum_\sigma f(\sigma) e^{-\beta H(\sigma) / Z_G} . \]

Expected energy: \[ \langle H \rangle = \sum_\sigma H(\sigma) e^{-\beta H(\sigma) / Z_G} = -\frac{d}{d\beta} \ln Z_G . \]

Fortuin—Kasteleyn’1972: \[ Z_G(q, x_e) = \sum_{F \subseteq E(G)} q^{k(F)} \prod_{e \in F} x_e , \]
where $k(F)$ is the number of connected components of the spanning subgraph $F$. 

$Z_G = Z_G \setminus e + x_e Z_G / e$. 

From knots to graphs

\[ 8_{21} = \begin{array}{c}
\text{Knot Diagram}
\end{array} \Rightarrow \begin{array}{c}
\text{Graph}
\end{array} \Rightarrow \begin{array}{c}
G_{8_1}
\end{array} = G_{8_1} \]
Let \( L \) be a diagram of a link, 
\( G_L \) be the corresponding signed graph,  
e\_ be the number of negative edges of \( G_L \), 
e\(+\) be the number of positive edges of \( G_L \),  
v be the number of vertices of \( G_L \).  
Then  
\[
[L](A, B, d) = A^{e_+} B^{e_-} \frac{d^v+1}{d^v+1} Z_{G_L}(q, x_e),
\]
where \( q = d^2 \), \( x_+ = \frac{Bd}{A} \), and \( x_- = \frac{Ad}{B} \).
\[ J_L(t) = (-1)^{w(L)} t^{3w(L)/4} [L](t^{-1/4}, t^{1/4}, -t^{1/2} - t^{-1/2}) \]

\[ = \frac{(-1)^{w(L)} t^{(3w(L)+2e_- -2e_+)/4}}{(-t^{1/2} - t^{-1/2})^{v+1-e_- -e_+}} \ Z_{G_L}(q, x_e), \]

where \( q = t + t^{-1} + 2 \), \( x_+ = -t - 1 \), and \( x_- = -t^{-1} - 1 \).