Physics and mathematics of knots

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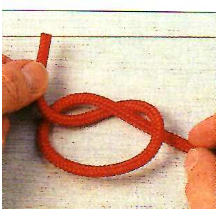
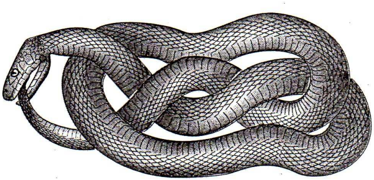
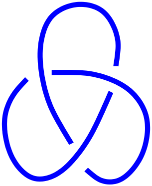
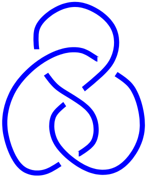
12:15 — 12:55 PM

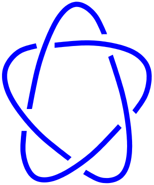
Knots

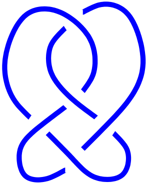




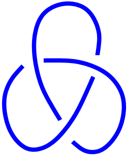




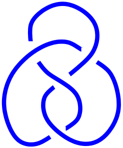




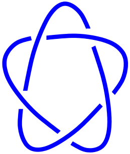
Knot Table



3_1



4_1



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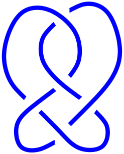

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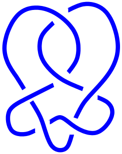

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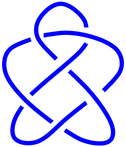

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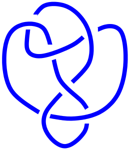

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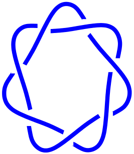

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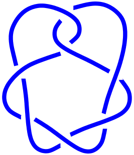

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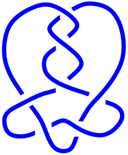
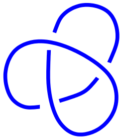
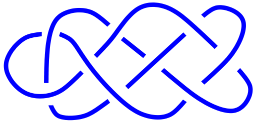


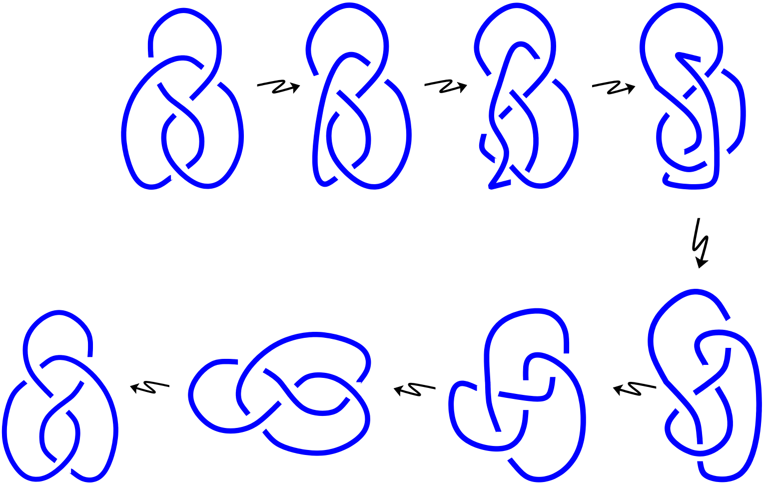
7_3

Unknots = Trivial Knots



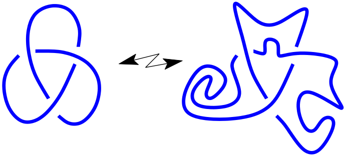


Knot isotopy

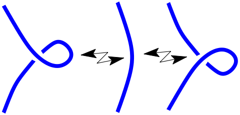


Reidemeister moves

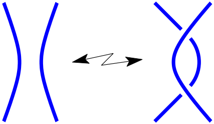
Planar isotopy:



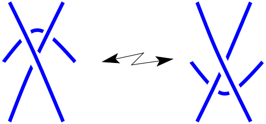
RI:



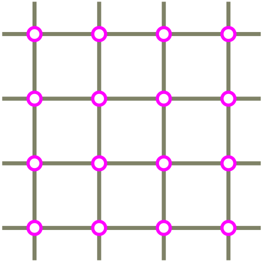
RII:



RIII:



Potts model (C.Domb 1952)



$q = 2$ the Ising model (W.Lenz, 1920)

Let G be a graph.
Particles are located at vertices of G.
Each particle has a *spin*, which takes
q different values .
A *state*, $\sigma \in \mathcal{S}$, is an assignment of spins
to all vertices of G.
Neighboring particles interact with each other only is their spins are the same. The energy of the interaction along an edge e is $-J_e$ (*coupling constant*).
The model is called *ferromagnetic* if $J_e > 0$ and
antiferromagnetic if $J_e < 0$.

Potts model

Energy of a state σ (*Hamiltonian*),

$$H(\sigma) = -\sum_{(a,b)=e\in E(G)} J_e\, \delta(\sigma(a),\sigma(b)).$$

Boltzmann weight of σ:

$$e^{-\beta H(\sigma)} = \prod_{(a,b)=e\in E(G)} e^{J_e\beta\delta(\sigma(a),\sigma(b))} = \prod_{(a,b)=e\in E(G)} \Big(1+(e^{J_e\beta}-1)\delta(\sigma(a),\sigma(b))\Big),$$

where the *inverse temperature* $\beta = \frac{1}{\kappa T}$, T is the temperature, $\kappa = 1.38\times 10^{-23}$ joules/Kelvin is the *Boltzmann constant*.
The Potts partition function (for $x_e := e^{J_e\beta} - 1$)

$$Z_G(q,x_e) := \sum_{\sigma\in\mathcal{S}} e^{-\beta H(\sigma)} = \sum_{\sigma\in\mathcal{S}} \prod_{e\in E(G)} (1 + x_e\delta(\sigma(a),\sigma(b)))$$

Potts model (properties)

Probability of a state σ: $P(\sigma) := e^{-\beta H(\sigma)}/Z_G$.
Expected value of a function $f(\sigma)$:

$$\langle f\rangle := \sum_{\sigma} f(\sigma)P(\sigma) = \sum_{\sigma} f(\sigma)e^{-\beta H(\sigma)}/Z_G \ .$$

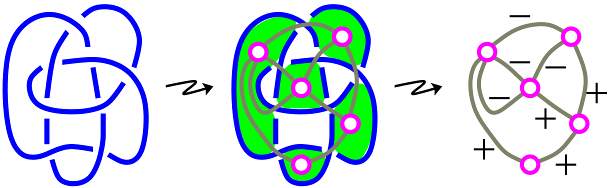
Expected energy: $\langle H\rangle = \sum_{\sigma} H(\sigma)e^{-\beta H(\sigma)}/Z_G = -\frac{d}{d\beta}\ln Z_G$.

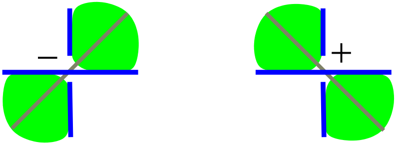
Fortuin—Kasteleyn'1972: $Z_G(q,x_e) = \sum_{F\subseteq E(G)} q^{k(F)} \prod_{e\in F} x_e$,

where $k(F)$ is the number of connected components of the spanning subgraph F.

$Z_G = Z_{G\setminus e} + x_e Z_{G/e}$.

From knots to graphs

$8_{21} =$  $= G_{8_1}$



Kauffman bracket

Let L be a diagram of a link,
  G_L be the corresponding signed graph,
  e_- be the number of negative edges of G_L,
  e_+ be the number of positive edges of G_L,
  v be the number of vertices of G_L.

Then

$$[L](A,B,d) = \frac{A^{e_+}B^{e_-}}{d^{v+1}} Z_{G_L}(q,x_e) \ ,$$

where $q = d^2$, $x_+ = \frac{Bd}{A}$, and $x_- = \frac{Ad}{B}$.

Jones polynomial

$$J_L(t) = (-1)^{w(L)} t^{3w(L)/4} [L](t^{-1/4}, t^{1/4}, -t^{1/2} - t^{-1/2})$$
$$= \frac{(-1)^{w(L)} t^{(3w(L)+2e_- - 2e_+)/4}}{(-t^{1/2} - t^{-1/2})^{v+1-e_- - e_+}} Z_{G_L}(q, x_e) \,,$$

where $q = t + t^{-1} + 2$, $x_+ = -t - 1$, and $x_- = -t^{-1} - 1$.