Beraha numbers

Beraha numbers, Tutte polynomial and its topological generalizati[on]

Université Claude Bern[ard] Lyon 1, Tuesday April 29, 2012, 10:

$$B_n = 2 + 2\cos\left(\frac{2\pi}{n}\right) = 4\cos^2\left(\frac{\pi}{n}\right)$$

n	1	2	3	4	5	6	7	8	9	10	$n \to \infty$
B_n	4	0	1	2	$\varphi+1$	3	≈ 3.247	$2+\sqrt{2}$	≈ 3.53	$\frac{5+\sqrt{5}}{2}$	$\longrightarrow 4$

golden section

$\varphi^2 = \varphi + 1$, $\varphi = \frac{1+\sqrt{5}}{2}$

Chromatic polynomial G graph

q-coloring $\quad Col_q(G) \ni c : V(G) \to \{1, \dots, q\}$

c is proper if $c(v_1) \neq c(v_2)$ for any $(v_1, v_2) \in E(G)$

v_1 $\quad$ v_2

Def $\chi_G(q) := \#(\text{proper } q\text{-colorings of } G)$

Properties
$$\chi_G = \chi_{G-e} - \chi_{G/e}$$
$$\chi_{G_1 \sqcup G_2} = \chi_{G_1}\chi_{G_2}$$
$$\chi_\bullet = q$$

4-color Theorem: G planar $\Longrightarrow$ $\chi_G(4) \neq 0$

$\chi_G(\varphi+1) \neq 0$ (Saaty & Kainen's book)

Dichromatic polynomial

$$Z_G(q, y) = \sum_{c \in Col_q(G)} y^{\#(\text{non proper colored edges})}$$

$$Z_G = Z_{G-e} + (y-1) Z_{G/e}$$
$$Z_{G_1 \sqcup G_2} = Z_{G_1} Z_{G_2}$$
$$Z_\bullet = q$$

$$\chi_G(q) = Z_G(q, 0)$$

Th. (Fortuin-Kasteleyn '72)

$$Z_G(q,y) = \sum_{F \subseteq E(G)} q^{k(F)} (y-1)^{e(F)}, \quad e(F) = |F|, \quad k(F) = \#\text{ conn. components of } F$$

Potts model: energy of interaction along the edge $-J_e$ coupling con[stant]

partition function: $Z_G(q, y)$, $y = e^{J\cdot\beta}$, $\beta = \frac{1}{\varkappa T}$

Tutte polynomial

$$T_G(x,y)=\sum_{F\subseteq E(G)}(x-1)^{k(F)-k(G)}(\ldots \; \mathcal{Z}_{(y-1)^{n(F}\ldots$$

$$T_G(x,y)=(x-1)^{-k(G)}(y-1)^{-v(G)}\mathcal{Z}_G\big((x-1)(y-1),\, y\big)$$

$$\mathcal{Z}_G(q,y)=q^{k(G)}(y-1)^{r(G)}\,T_G\left(1+\frac{q}{y-1},\, y\right)$$

$$\chi_G(q)=q^{k(G)}(-1)^{r(G)}\,T_G(1-q,\,0) \qquad r=v(G)-k(G$$

Properties

$T_G=T_{G-e}+T_{G/e}$, e regular

$T_G=x\,T_{G/e}$, e bridge

$T_G=y\,T_{G-e}$, e loop.

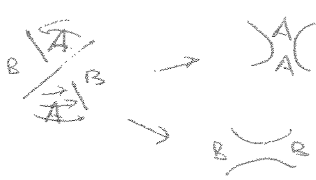
$T_{G_1\sqcup G_2}=T_{G_1\cdot G_2}=T_{G_1}\cdot T_{G_2}$, $T_\bullet=1$

$$T_G(x,y)=\sum_T x^{i(T)}y^{j(T)} \quad \text{activitie } i,j$$

Example $T_\Delta=T_\lrcorner+T_{\ominus}=x^2+T_{\smile}+T_{\circ}=x^2+x+y$

Relation to the knot theory

Jones polynomial



A-splitting

B-splitting

A *state* S is a choice of either A or Bsplitting at each crossing

$\alpha(S)=\#(A\text{-splittings in } S)$

$\beta(S)=\#(B\text{-splitting in } S)$

$\delta(S)=\#(\text{circles in } S)$

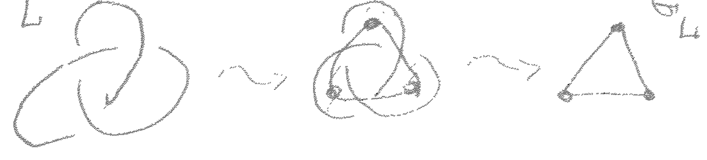
Kauffman bracket $[L](A,B,d):=\sum_S A^{\alpha(S)}B^{\beta(S)}d^{\delta(S)-1}$

$+1$ $\quad -1$ $\quad w(L):=\sum_X \pm 1$ writhe of L

Jones polynomial $J_L(t):=(-1)^{w(L)}t^{3w(L)/4}[L](t^{-1/4},t^{1/4},-t^{1/2}-t^{-1/2}$

$J_L=t+t^3-t^4$

$=-t^3(-t-t^{-1}+t^2)$



$T_{G_L}(-t,-t^{-1})=-t-t^{-1}+t^2$

Thistlethwaite's Theorem $J_L(t)=\pm t^N\cdot T_G(-t,-t^{-1})$