Tutte polynomial

\[ T_G(x, y) = \sum_{S \text{ spanning subgraph}} (x-1)^{E(S) - \nu(S) - \nu(G)} (y-1)^{\kappa(G) - \nu(G)} \]

where \( \nu(S) \) is the number of vertices of \( S \), \( E(S) \) is the number of edges of \( S \), \( \kappa(G) \) is the number of connected components of \( G \), \( \nu(G) \) is the number of vertices of \( G \), and \( \kappa(G) \) is the number of connected components of \( G \).

The partition function of the Potts model is given by:

\[ Z_G(q, y) = q^{\kappa(G)} (y-1)^{\nu(G)} T_G(1+\frac{q}{y-1}, y) \]

\[ T_G(q, y) = (x-1)^{E(G)} (y-1)^{\nu(G)} \sum_{\text{properly colored edges}} y^k \]

\[ \chi_G(q) = Z_G(q, 0) = q^{\kappa(G)} (y-1)^{\nu(G)} T_G(1, 0) \]

Properties:
- \( T_G = T_{G-x} + T_{G/x} \) if \( x \) is regular
- \( T_G = x T_{G/x} \) if \( x \) is a bridge
- \( T_G = T_{G-x} \) if \( x \) is a loop
- \( T_G \cup G_2 = T_G \cdot T_{G_2} \)
- \( T_G(1, 1) = \# \text{(spanning trees in } G) \)

For planar \( G \):

\[ T_G(x, y) = T_{G^{\text{red}}}(y, x) \]

Bollobás-Riordan polynomial

\[ BR_G(x, y, z) = \sum_{E \in E(G)} x^{\nu(E)} y^{\kappa(E)} z^{k(E)} \]

for planar graphs.

\[ Z = \sum_{F \in E(G)} k(F) - \kappa(F) + n(F) \]
Tutte-Krushkal-Reurdv polynomial

\[ T^d_{k}(x, y) = \sum_{K_{i} \subseteq S \subseteq K^{(d)}} \beta_{d-1}(S) \beta_{d-1}(K) \cdot \beta_{d}(S) \]

\[ K^{(d)} \] \( d \)th skeleton of \( K \)

\[ \beta_{d}(S) = \text{rank} \ H_{d}(S; \mathbb{Z}) \]

Properties:
1) \( T^d_{k}(x, y) = T^d_{k+(x+1, y+1)} \)
2) \( T^d_{k}(x, y) = T^d_{n-d}(y, x) \)

Tutte polynomial of the graph \( K^{(1)} \)

dual cell structures on an \( n \)-dim manifold \( M = S \)

\( \{d\text{-cells}\} \leftrightarrow \{ (n-d)\text{-cells} \} \)

Cellular Spanning Trees (CST)

Definition: \( S \subseteq K^{(d)} \) is a \( d \)-CST if

1) \( H_{d}(S) = 0 \)
2) \( \beta_{d-1}(S) = 0 \)
Cayley's formula \[ K = \Delta^n \text{ n-simplex} \]

\[ K_{(i)} = K_{n+1} \]

\[ \# \text{ (span. trees of } K_{n+1} \text{)} = (n+1)^{n-1} \]

Gil Kalai \[ \sum_{S \text{ j-cst of } K} \left| \widetilde{H}_{d-1}(S) \right|^2 = \binom{n-1}{d-1} \]

A. Duval, C. Klivans, J. Martin:

\[ \sum_{S \text{ j-cst}} \left| \widetilde{H}_{j-1}(S) \right|^2 = \det \text{ (Laplacian of } K) \]

for \( K_{(i)^*} \text{ APC } \beta_j^* (K) = 0 \text{ if } k. \)

Observations joint with Carlos Bajo and Bradley Bardick

1) \( T^d_k(0,0) = \# \{ j \text{-cst}\} \)

2) Modified Tutte-Krushkal-Rennie polynomial

\[ \tilde{T}^d_k(X,Y) := \sum_{K_{(i)} \subseteq S \subseteq K_{(i)} \text{ (d-j)}} \left| \text{tor} (H_{j-1}(S)) \right|^2 \times \beta_{j-1}(S). \beta_j^* (K) \beta_j^* (S) \]

\[ \tilde{T}_k^d(X,Y) = T_{K,(i)}(X+1,Y+1) \]

\[ \tilde{T}_k^d(X,Y) = \tilde{T}^{n-d}_{K,(i)}(Y,X) \text{ for dual cell structures } \]

\[ K, K^* \text{ on } S^n \]
Bott polynomial (1952)

\[ \dim K = k \quad \Rightarrow \quad R_k(\lambda) := \sum_{S \subseteq \{k \text{-cells}\}} (-1)^{|S|} \beta_{|S|} \]

\[ \beta_{|S|} = \#(k \text{-cells of } S) \]

\[ R_k(\lambda) \text{ is a combinatorial invariant} \quad \Rightarrow \quad \text{invariant under subdivisions} \]

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Observation 3) \quad R_k(\lambda) = (-1)^{k-1} T_k(-1, -1)

Contraction / Deletion relations

\[ \mathcal{O} \text{ is an open } k \text{-cell, } \bar{\mathcal{O}} \text{ is the closure of } \mathcal{O} \text{ in } K \]

\[ \partial \mathcal{O} \subset K^{k-1} \text{ boundary of } \mathcal{O} \]

Definition \quad \mathcal{O} \text{ is a loop if } H_k(\mathcal{O}) = \mathbb{Z} \]

\[ \mathcal{O} \text{ is a bridge if } \beta_{k-1}(K - \mathcal{O}) = \beta_{k-1}(K) + 1 \]

\[ \mathcal{O} \text{ is contractible if } H_{k-1}(\partial \mathcal{O}) = 0 \]

Observation 4)

(i) if \( \mathcal{O} \) is neither a loop nor a bridge and contractible

\[ T_k^\mathcal{O}(X,Y) = T^{k-\mathcal{O}}_k(X,Y) + T^\mathcal{O}_k(X,Y) \]

(ii) if \( \mathcal{O} \) is a loop, \( T_k^\mathcal{O}(X,Y) = (X+1) T^{k-\mathcal{O}}_k(X,Y) \)

(iii) if \( \mathcal{O} \) is a bridge and contractible

\[ T_k(X,Y) = (X+1) T^{k-\mathcal{O}}_k(X,Y) \]