Carlos Bajo, Bradey Burdick

Tutte polynomial

Università di Pisa
Wednesday, May 16, 2012
14:30 – 15:30

1

$$T_G(x,y) = \sum_{\substack{S \text{ spanning subgraph} \\ V(S)=V(G) \\ E(S)\subseteq E(G)}} (x-1)^{\beta_0(S)-\beta_0(G)} (y-1)^{\beta_1(S)}$$

$k(G) = \#$ conn. comp

$r(G) = v(G) - k(G)$

$T_G(x,y) =$ partition function of the Potts model

$$Z_G(q,y) = q^{k(G)}(y-1)^{r(G)}\, T_G\left(1+\frac{q}{y-1},\, y\right)$$

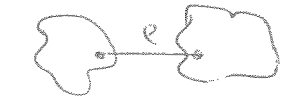
$$T_G(x,y) = (x-1)^{-k(G)}(y-1)^{-v(G)}\, Z_G\big((x-1)(y-1),\, y\big)$$

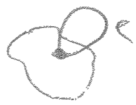
$$Z_G(q,y) = \sum_{c\in Col_q(G)} y^{\#(\text{non proper colored edges})}$$

$$\chi_G(q) = Z_G(q,0) = q^{k(G)}(-1)^{r(G)}\, T_G(1-q,\,0) \quad \text{chromatic polynomial}$$

Properties

$T_G = T_{G-e} + T_{G/e}$, if e is regular

$T_G = x\, T_{G/e}$ if e is a bridge 

$T_G = y\, T_{G-e}$ if e is a loop 

$T_{G_1 \sqcup G_2} = T_{G_1} \cdot T_{G_2}$

$T_{\circ} = 1$

$T_G(1,1) = \#(\text{spanning trees in } G)$

for planar G: $T_G(x,y) = T_{G^*}(y,x)$

Bollobás–Riordan polynomial

$$BR_G(X,Y,Z) := \sum_{F\subseteq E(G)} X^{r(G)-r(F)}\, Y^{n(F)} \cdot Z^{k(F)-bc(F)+n(F)}$$

Tutte-Krushkal-Renardy polynomial

K finite CW complex of dimension k, $1 \leq j \leq k$

$$T_K^j(X,Y) := \sum_{K_{(j-1)} \subseteq S \subseteq K_{(j)}} X^{\beta_{j-1}(S)-\beta_{j-1}(K)}\, Y^{\beta_j(S)}$$

$S \subseteq \{j\text{-cells}\}$

$K_{(j)}$ j^{th} skeleton of K

$\beta_j(S) = \operatorname{rank} H_j(S;\mathbb{Z})$

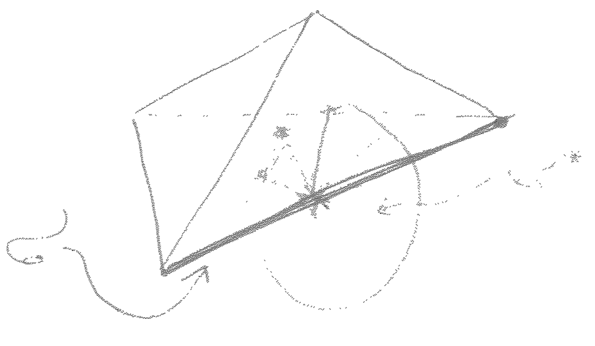
Properties 1) $T^1_K(X,Y) = T_{K_{(1)}}(X+1, Y+1)$ Tutte polynomial of the graph $K_{(1)}$

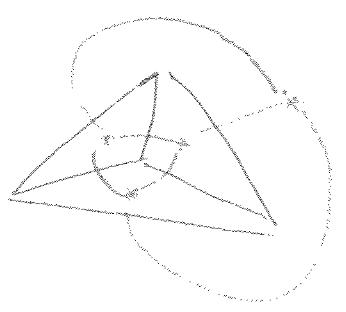
2) $T^j_K(X,Y) = T^{n-j}_{K^*}(Y,X)$

dual cell structures on an n-dim mfld $M = S$

$$\{j\text{-cells of } K\} \overset{1-1}{\longleftrightarrow} \{(n-j)\text{-cells of } K^*\}$$

$$\sigma \longleftrightarrow \sigma^*$$





Cellular Spanning Trees (CST)

Def. $S \subset K_{(j)}$ is a j-CST if ($1 \leq j \leq k$)

$S \supseteq K_{(j-1)}$ (spanning) and

1) $H_j(S) = 0$, 2) $\hat{\beta}_{j-1}(S) = 0$

Cayley's formula $\quad K = \Delta^n \quad$ n-simplex, $n+1$ vertices

$$K_{(1)} = K_{n+1}$$

$$\#(\text{span. trees of } K_{n+1}) = (n+1)^{n-1}$$

Gil Kalai

$$\sum_{\substack{S\ j\text{-CST}\\ \text{of } K}} |\tilde{H}_{j-1}(S)|^2 = (n+1)^{\binom{n-1}{j}}$$

A. Duval, C. Klivans, J. Martin:

$$\sum_{S\ j\text{-CST}} |\tilde{H}_{j-1}(S)|^2 = \det(\text{Laplacian of } K)$$

for K APC, $\tilde{\beta}_j(K) = 0$, $j < k$.

Observations joint with Carlos Bajo and Bradley Burdick

1) $T^j_K(0,0) = \#\{j\text{-CST}\}$

2) Modified Tutte-Krushkal-Renardy polynomial

$$\tilde{T}^j_K(X,Y) := \sum_{K_{(j-1)} \subseteq S \subseteq K_{(j)}} |\mathrm{tor}(H_j(S))|^2 X^{\beta_{j-1}(S) - \beta_{j-1}(K)} Y^{\beta_j(S)}$$

$$\tilde{T}^1_K(X,Y) = T_{K_{(1)}}(X+1, Y+1)$$

$$\tilde{T}^j_K(X,Y) = \tilde{T}^{n-j}_{K^*}(Y,X) \text{ for dual } \cancel{\text{cell}} \text{ structures } K, K^* \text{ on } S^n$$

Bott polynomial (1952)

$\dim K = k \qquad R_K(\lambda) := \sum_{S \subseteq \{k\text{-cells}\}} (-1)^{f_k(K) - f_k(S)} \lambda^{\beta_k(S)}$

$f_k(S) = \#(k\text{-cells of } S)$

$R_K(\lambda)$ is a combinatorial invariant $\equiv$ invariant under subdivisions

~~Zhenghan~~ Zhenghan Wang : series of polynomials
1994

Observation 3) $\quad R_K(\lambda) = (-1)^{\beta_k(K)} T_K^k(-1, -\lambda)$

Contraction / Deletion relations

σ (open) k-cell, $\bar{\sigma}$ closure of σ in K

$\partial\sigma \subset K_{(k-1)}$ boundary of σ

Definition σ is a loop if $H_k(\bar{\sigma}) \cong \mathbb{Z}$

σ is a bridge if $\beta_{k-1}(K - \sigma) = \beta_{k-1}(K) + 1$

σ is contractible if $\tilde{H}_{k-1}(\partial\sigma) \cong \mathbb{Z}$

Observation 4)

(i) if σ is neither a loop nor a bridge and contractible

$$T_K^k(X,Y) = T_{K-\sigma}^k(X,Y) + T_{K/\bar{\sigma}}^k(X,Y)$$

(ii) if σ is a loop, $T_K^k(X,Y) = (Y+1)\, T_{K-\sigma}^k(X,Y)$

(iii) if σ is a bridge and contractible $T_K^k(X,Y) = (X+1)\, T_{K/\bar{\sigma}}^k(X,Y)$