AMS Meeting #1079
University of South Florida
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Evaluations of the Tutte - Krushkal - Renardy polynomial for cell complexes

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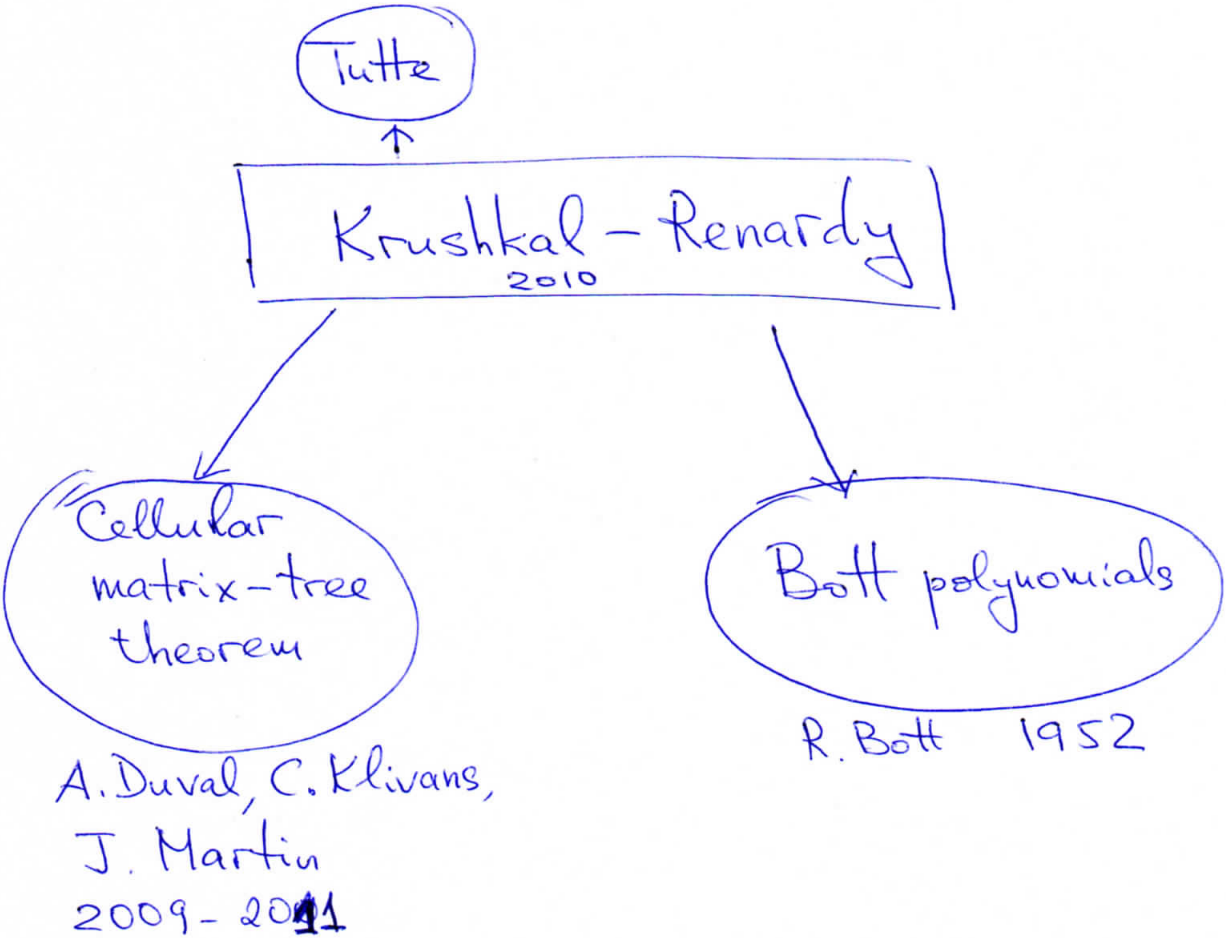
9:30 – 10:00

Joint work with Carlos Bajo

SU'2011 and Bradley Burdick



Tutte-Krushkal-Renardy polynomial

K finite CW complex of dimension k, $1 \le j \le k$

$$\boxed{T_K^j(X,Y) := \sum_{K_{(j-1)} \subseteq S \subseteq K_{(j)}} X^{\beta_{j-1}(S) - \beta_{j-1}(K)} \, Y^{\beta_j(S)}}$$

$K_{(j)}$ j^{th} skeleton of K, $S \subseteq \{ j\text{-cells} \}$

$\beta_j(S) = \operatorname{rank} H_j(S; \mathbb{Z})$

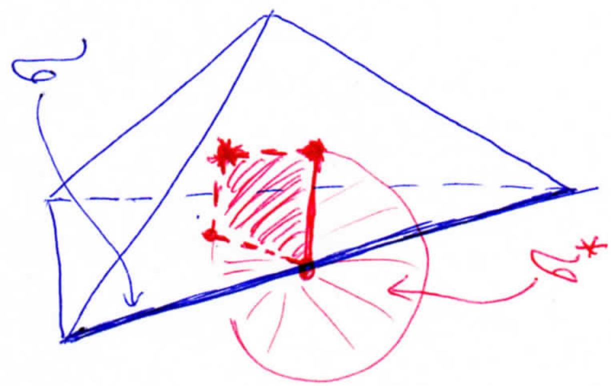
Properties 1) $T_K^1(X,Y) = T_{K_{(1)}}(X+1, Y+1)$ Tutte polynomial of the graph

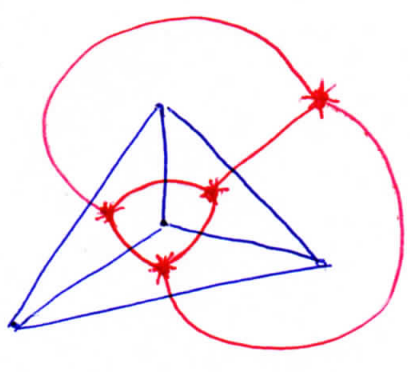
2) $T_K^j(X,Y) = T_{K^*}^{n-j}(Y,X)$

dual cell structures on an n-mnfld. $M = S^n$

$$\{ j\text{-cells of } K \} \xleftrightarrow{1-1} \{ (n-j)\text{-cells of } K^* \}$$

$$\sigma \longleftrightarrow \sigma^*$$





Cellular Spanning Trees (CST)

Def $S \subseteq K_{(j)}$ is a j-CST if

$S \supset K_{(j-1)}$ and $\quad 1 \le j \le k$

1) $H_j(S) = 0$ $\quad$ 2) $\tilde{\beta}_{j-1}(S) = 0$

Gil Kalai: $K = \Delta^n$ n-simplex, $n+1$ vertices

$$\sum_{S \text{ } j\text{-CST of } K} |\tilde{H}_{j-1}(S)|^2 = (n+1)^{\binom{n-1}{j}}$$

$K_{(1)} = K_{n+1}$

Cayley's formula

$\#(\text{span. trees of } K_{n+1}) = (n+1)^{n-1}$

A. Duval, C. Klivans, J. Martin:

cellular matrix-tree theorem

$$\sum_{S \text{ } j\text{-CST}} |\tilde{H}_{j-1}(S)|^2 \approx \det(\text{Laplacian of } K)$$

for K APC $\tilde{\beta}_j(K) = 0$ $j < k$

Observations 1) $T^j_K(0,0) = \#\{j\text{-CST}\}$

2) Modified Tutte-Krushkal-Renardy polynomial

$$\tilde{T}^j_K(X,Y) := \sum_{K_{(j-1)} \subseteq S \subseteq K_{(j)}} |\mathrm{tor}(H_{j-1}(S))|^2 X^{\beta_{j-1}(S) - \beta_{j-1}(K)} Y^{\beta_j(S)}$$

$$\tilde{T}^1_K(X,Y) = T_{K(1)}(X,Y)$$

$$\tilde{T}^j_K(X,Y) = \tilde{T}^{n-j}_{K^*}(Y,X)$$ for dual cell structures K, K^* on S^n

Bott polynomial

$$R_K(\lambda) := \sum_{S \subseteq \{k\text{-cells}\}} (-1)^{c_k(K) - c_k(S)} \lambda^{\beta_k(S)}$$

$\dim K = k$

combinatorial invariant $\equiv$ invariant under subdivisions

Zhenghan Wang 1994 : series of polynomials

Observation 3)

$$R_K(\lambda) = (-1)^{\beta_k(K)} T_K^k(-1, -\lambda)$$

Skein relation

$\sigma \in \{k\text{-cells}\}$ open, $\bar{\sigma}$ closure of σ in K.

, $\partial\sigma \subset K_{(k-1)}$ boundary of $\bar{\sigma}$

Definitions σ is a loop if $H_k(\bar{\sigma}) \cong \mathbb{Z}$.

σ is a bridge if $\cancel{H_{k-1}(\partial\sigma) \cong \mathbb{Z}}$ $\beta_{k-1}(K - \sigma) = \beta_{k-1}(K) + 1$

σ is contractible if $H_{k-1}(\partial\sigma) \cong \mathbb{Z}$

Observation 4)

i) if σ is neither a loop nor a bridge and contractible,

$$T_K^k(X, Y) = T_{K-\sigma}^k(X, Y) + T_{K/\bar{\sigma}}^k(X, Y)$$

ii) if σ is a loop, $T_K^k(X, Y) = (Y+1)\, T_{K-\sigma}^k(X, Y)$

iii) if σ is a bridge and contractible $T_K^k(X, Y) = (X+1)\, T_{K/\bar{\sigma}}^k(X, Y)$