In memory of Sergei Duzhin (1956—2015)

Sergei Chmutov

Ohio State University, Mansfield

Knots in Washington XL

Tuesday, March 10, 2015
10:00–11:00am

Sergei Duzhin (1956—2015).

June 17, 1956 —- February 1, 2015

Born: June 17, 1956. Davyd Haradok, Belarus.

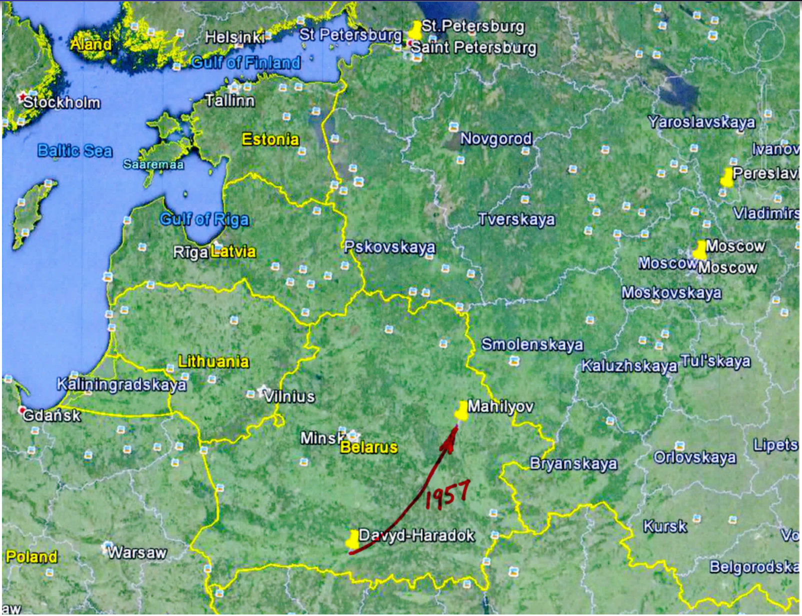
1957

1959

Vasilii Nikanorovich Duzhin, 1921-1998

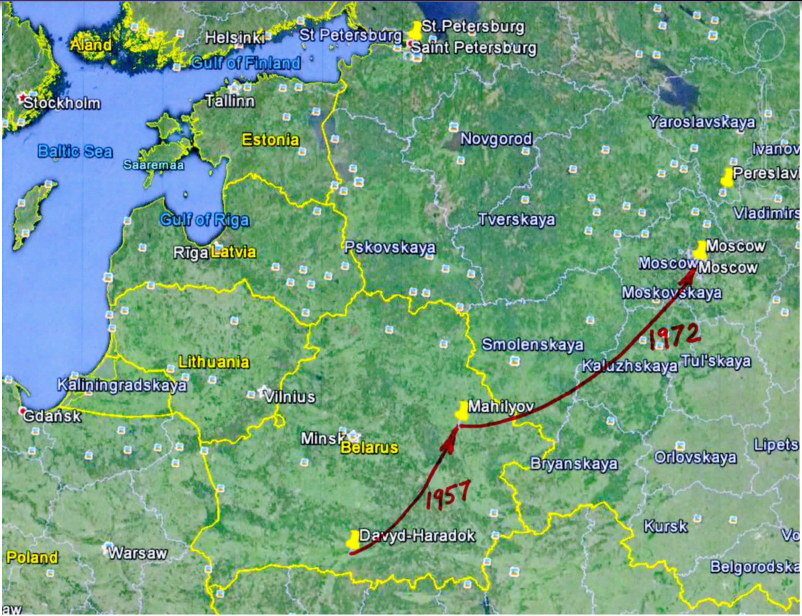
Nadezhda Vasilievna Silivestrova, 1923-1985

Davyd Haradok → Mahilyov, 1957.



Mahilyov, Belarus, 1960.

Mahilyov → Moscow, 1972.



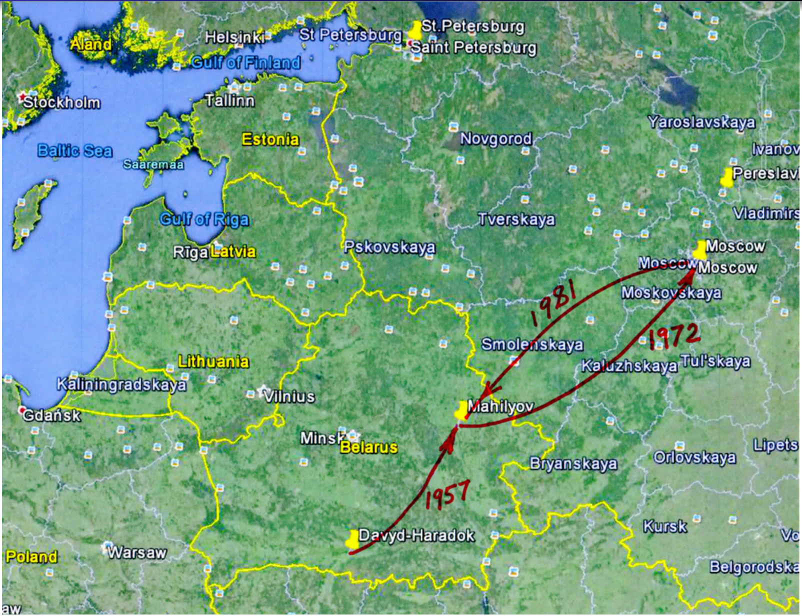
Moscow, 1972-1981.

Moscow State University.

Undergraduate: 1972 —1978.
Ph.D. program: 1978 — 1981.
Ph.D. thesis: “On some versions of the C-spectral sequence”.

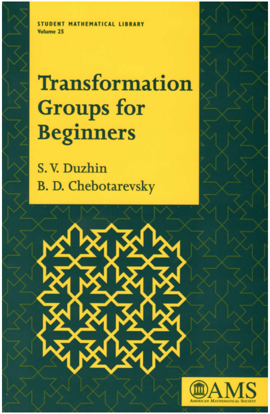
A.M.Vinogradov
University of Salerno (Italy)

Moscow → Mahilyov, 1981.



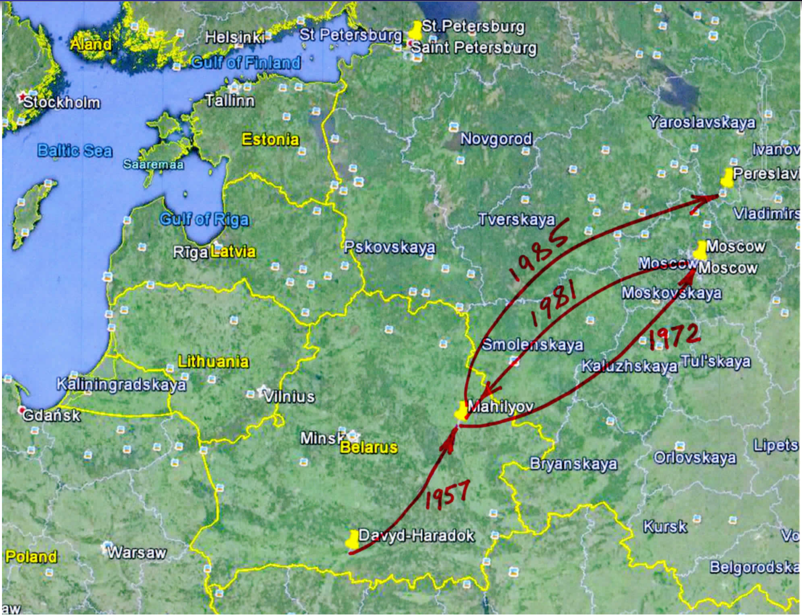
Mahilyov Pedagogical Institute.

1988



2004

Mahilyov → Pereslavl-Zalessky, 1985.



Pereslavl-Zalessky, 1985-1998.

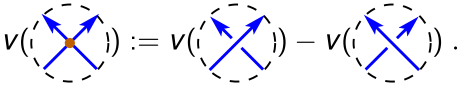
Program Systems Institute of the Russian Academy of Sciences.
A.M.Vinogradov: Laboratory of Problems of Big Dimension.
S.V.Duzhin: Laboratory of Problems of Small Dimension.

Pereslavl-Zalessky, 1985-1998.

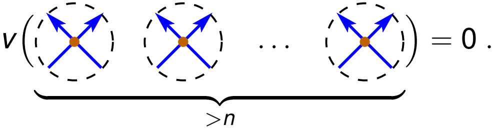
Sergei Duzhin

Vassiliev knot invariants.

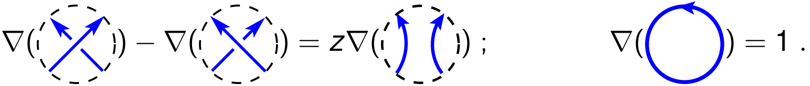
Vassiliev skein relation:



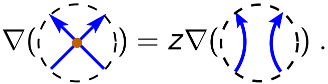
Definition. A knot invariant v is a *Vassiliev invariant* of order (or degree) $\leqslant n$ if its extension vanishes on all singular knots with more than n double points:



$$\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \cdots \subseteq \mathcal{V}_n \subseteq \mathcal{V}_{n+1} \subseteq \cdots \subseteq \mathcal{V} := \bigcup_{n=0}^{\infty} \mathcal{V}_n \,.$$

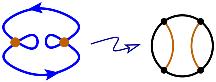
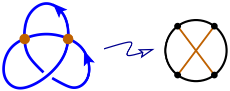


$$\nabla(K) = 1 + c_2(K)z^2 + c_4(K)z^4 + \dots .$$



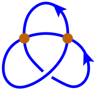
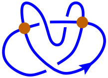
$$\mathrm{ord}(c_{2k}) \leqslant 2k$$

Vassiliev invariants. Chord diagrams.

  .

Lemma. *The value of $v(K)$ of a a Vassiliev invariant v of order $\mathrm{ord}(v) \leqslant n$ on a knot K with n double points depends only on the chord diagram of K.*

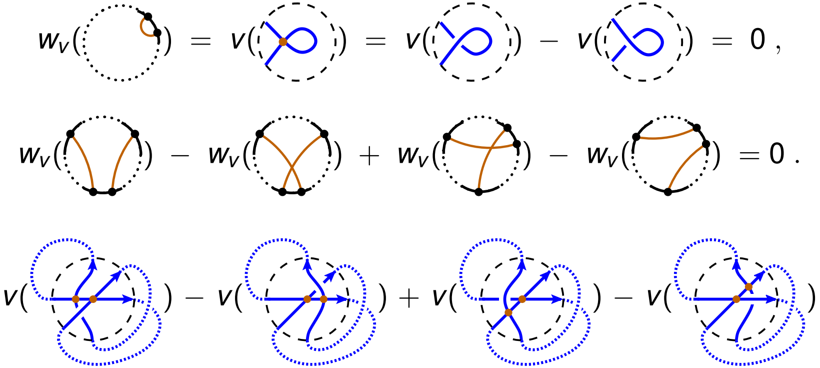
Example. n=2

$K_1 =$  $K_2 =$  ,

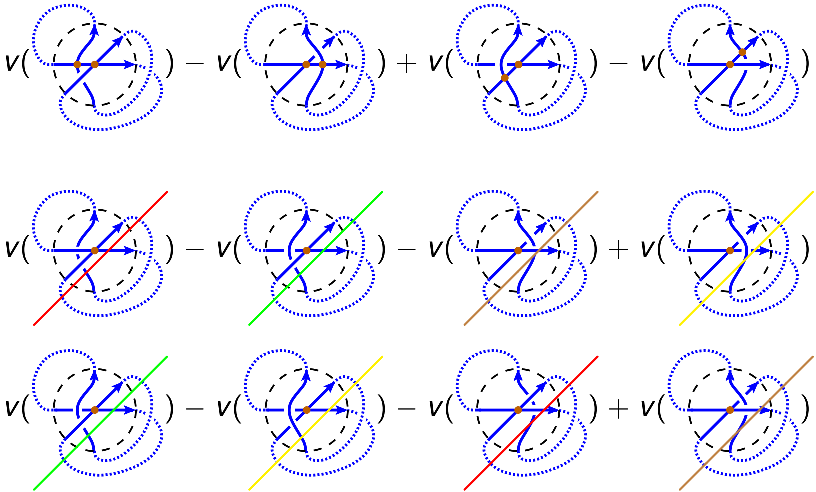
$$v(K_1) = v(K_2)$$

Definition. The *symbol* w_v of v is a restriction of v to the set of knots with precisely n double points, considered as a function on the set of chord diagrams.

Vassiliev invariants. One- and four-term relations.

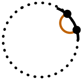


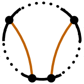
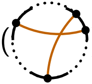
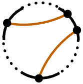
Vassiliev invariants. Proof of (4T).



$= 0$

Definition. A *weight system* w of order n is a function on the set of chord diagrams with n chords which satisfies (1T) and (4T) relations.

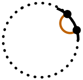
$w($  $) = 0$

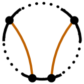
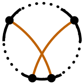
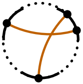
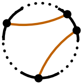
$w($  $) - w($  $) + w($  $) - w($  $) = 0\,.$

Theorem. [M. Kontsevich'93] *Any weight system is a symbol of an appropriate Vassiliev invariant of order $\leqslant n$.*

Vassiliev invariants. Algebra of chord diagrams.

Definition. $\mathcal{A}_n$ is $\mathbb{C}$-vector space spanned by chord diagrams modulo (1T) and (4T) relations:

(1T)  $= 0$

(4T)  $-$  $+$  $-$  $= 0$.

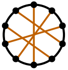
$$\mathcal{V}_n/\mathcal{V}_{n-1} \cong (\mathcal{A}_n)^*$$

Vassiliev invariants. Upper bound.

Journal of Knot Theory and Its Ramifications, Vol. 3 No. 2 (1994) 141–151

AN UPPER BOUND FOR THE NUMBER OF VASSILIEV KNOT INVARIANTS

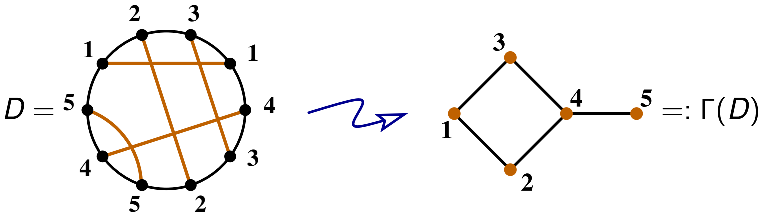
S. V. CHMUTOV,* S. V. DUZHIN†

Theorem. $\dim \mathcal{A}_n \leqslant (n-1)!$ **Proof.** $D \sim$ 

Vassiliev invariants. Upper bound.

- S. Chmutov and S. Duzhin'94:
  $\dim \mathcal{A}_n < (n-1)! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.
- K. Ng'98: $\dim \mathcal{A}_n < (n-2)!/2$.
- A. Stoimenow'98: $\dim \mathcal{A}_n < n!/a^n$, where $a = 1.1$.
- B. Bollobás and O. Riordan'00:
  $\dim \mathcal{A}_n < n!/(2\ln(2) + o(1))^n \approx n!/1.38^n$.
- D. Zagier'01; $\dim \mathcal{A}_n < \frac{6^n\sqrt{n}\cdot n!}{\pi^{2n}} \lesssim n!/a^n$ for
  $a < \pi^2/6 = 1.644...$.

Vassiliev invariants. Intersection graph.



Intersection Graph Conjecture. *If D_1 and D_2 are two chord diagrams with $\Gamma(D_1) = \Gamma(D_2)$, then $D_1 = D_2$ as elements of $\mathcal{A}$ (that is, modulo 4T).*

Observation. Chromatic polynomial of $\Gamma(D)$ is a weight system.

Vassiliev invariants. CDL.

ADVANCES IN SOVIET MATHEMATICS
Volume **21**, 1994

Vassiliev Knot Invariants
I. Introduction

S. V. CHMUTOV, S. V. DUZHIN, AND S. K. LANDO

Vassiliev Knot Invariants
II. Intersection Graph Conjecture for Trees

S. V. CHMUTOV, S. V. DUZHIN, AND S. K. LANDO

Vassiliev Knot Invariants
III. Forest Algebra and Weighted Graphs

S. V. CHMUTOV, S. V. DUZHIN, AND S. K. LANDO

Sergei Lando

Theorem. *The intersection Graph Conjecture holds when* $\Gamma(D)$ *is a tree (forest).*

B. Mellor, *The intersection graph conjecture for loop diagrams*, J. Knot Theory Ramifications **9** (2000) 187–211.

Counterexample to the IGC in order 11: T. Le and H. Morton, P. Cromwell'96.

S. Chmutov, S. Lando, *Mutant knots and intersection graphs*, Algebr. Geom. Topol. **7** (2007) 1579–1598.

CDL3. Chromatic polynomial $\chi_\Gamma(x)$.

Definition 1. $\chi_\Gamma(x) =$ # of proper colorings of Γ in x colors.

Definition 2.

$\chi_\Gamma = \chi_{\Gamma - e} - \chi_{\Gamma/e}$;

$\chi_{\Gamma_1 \sqcup \Gamma_2} = \chi_{\Gamma_1} \cdot \chi_{\Gamma_2}$, for a disjoint union $\Gamma_1 \sqcup \Gamma_2$;

$\chi_\bullet = x$.

Example.

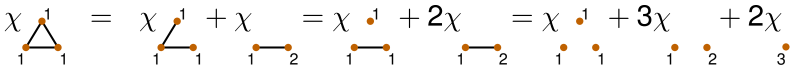
$$\chi_{\triangle} = x(x-1)(x-2).$$

CDL3. Weighted Chromatic Polynomial $\widetilde{\chi}_{\widetilde{\Gamma}}$.

Weighted graph $\widetilde{\Gamma}$ is a vertex-weighted by natural numbers graph without loops and multiple edges.

Definition. $\widetilde{\chi}_{\widetilde{\Gamma}} = \widetilde{\chi}_{\widetilde{\Gamma}-e} + \widetilde{\chi}_{\widetilde{\Gamma}/e}$;
$\widetilde{\chi}_{\widetilde{\Gamma}_1 \sqcup \widetilde{\Gamma}_2} = \widetilde{\chi}_{\widetilde{\Gamma}_1} \cdot \widetilde{\chi}_{\widetilde{\Gamma}_2}$, for a disjoint union $\widetilde{\Gamma}_1 \sqcup \widetilde{\Gamma}_2$;
$\widetilde{\chi}_{\bullet_w} = s_w$.

Example.



$$= s_1^3 + 3s_1s_2 + 2s_3 \Big|_{\substack{s_1 = x \\ s_2 = -x \\ s_3 = x}} = x(x-1)(x-2) .$$

R.Stanley'95. Symmetric Chromatic Polynomial X_Γ.

Richard Stanley, *A symmetric function generalization of the chromatic polynomial of a graph,* Advances in Math. **111**(1) (1995) 166–194.

Definition. $X_\Gamma(x_1, x_2, \dots) = \sum_{\substack{\varkappa: V(\Gamma)\to\mathbb{N} \\ \text{proper}}} \prod_{v\in V(\Gamma)} x_{\varkappa(v)}$.

$$p_n := \sum_{i=1}^{\infty} x_i^n \,.$$

Example. $X_{\triangle} = \sum_{i\neq j\neq k\neq i} x_i x_j x_k = \sum_{i\neq j} x_i x_j (p_1 - x_i - x_j)$

$$= p_1 \sum_{i\neq j} x_i x_j - \sum_{i\neq j} x_i^2 x_j - \sum_{i\neq j} x_i x_j^2$$

$$= p_1 \sum_i x_i(p_1 - x_i) - \sum_i x_i^2(p_1 - x_i) - \sum_i x_i(p_2 - x_i^2)$$

$$= p_1^3 - p_1p_2 - p_1p_2 + p_3 - p_2p_1 + p3 = \boxed{p_1^3 - 3p_1p_2 + 2p_3}\,.$$

Stanley's conjecture.

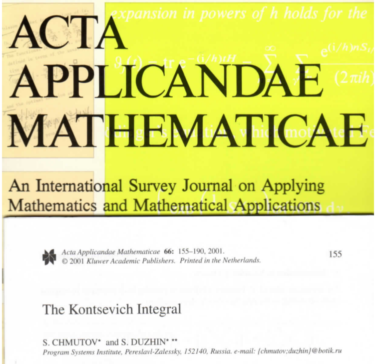
Conjecture. *Symmetric Chromatic Polynomial X_Γ distinguishes trees.*

S. Noble, D. Welsh, *A weighted graph polynomial from chromatic invariants of knots*, Annales de l'institut Fourier **49**(3) (1999) 1057–1087.

Theorem. $X_\Gamma(p_1, p_2, ...) = (-1)^{|V(\Gamma)|} \widetilde{\chi}_{\widetilde{\Gamma}}|_{s_i=-p_i}$.

Generalization of the weighted chromatic polynomial to graphs with loops in a sense of Tutte adding another variable responsible for loops.

Kontsevich integral



ACTA APPLICANDAE MATHEMATICAE

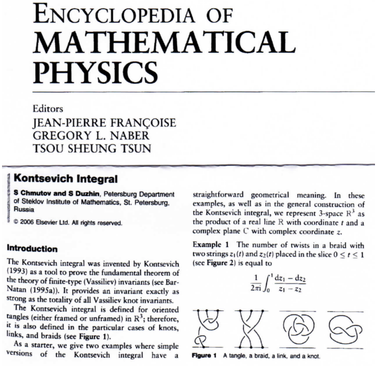
An International Survey Journal on Applying Mathematics and Mathematical Applications

Acta Applicandae Mathematicae **66:** 155–190, 2001. 155

The Kontsevich Integral

S. CHMUTOV* and S. DUZHIN* **
Program Systems Institute, Pereslavl-Zalessky, 152140, Russia. e-mail: {chmutov;duzhin}@botik.ru

2001



ENCYCLOPEDIA OF MATHEMATICAL PHYSICS

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TSOU SHEUNG TSUN

Kontsevich Integral

S Chmutov and S Duzhin, Petersburg Department of Steklov Institute of Mathematics, St. Petersburg, Russia

Introduction

The Kontsevich integral was invented by Kontsevich (1993) as a tool to prove the fundamental theorem of the theory of finite-type (Vassiliev) invariants (see Bar-Natan (1995a)). It provides an invariant exactly as strong as the totality of all Vassiliev knot invariants.

The Kontsevich integral is defined for oriented tangles (either framed or unframed) in $\mathbb{R}^3$; therefore, it is also defined in the particular cases of knots, links, and braids (see **Figure 1**).

As a starter, we give two examples where simple versions of the Kontsevich integral have a straightforward geometrical meaning. In these examples, as well as in the general construction of the Kontsevich integral, we represent 3-space $\mathbb{R}^3$ as the product of a real line $\mathbb{R}$ with coordinate t and a complex plane $\mathbb{C}$ with complex coordinate z.

Example 1 The number of twists in a braid with two strings $z_1(t)$ and $z_2(t)$ placed in the slice $0 \le t \le 1$ (see **Figure 2**) is equal to

$$\frac{1}{2\pi i}\int_0^1 \frac{dz_1 - dz_2}{z_1 - z_2}$$

Figure 1 A tangle, a braid, a link, and a knot.

2006

TOPOLOGY AND ITS APPLICATIONS

Topology and its Applications 92 (1999) 201–223

A lower bound for the number of Vassiliev knot invariants

S. Chmutov [1], S. Duzhin [*,1,2]

Program Systems Institute, 152140 Pereslavl-Zalessky, Russia

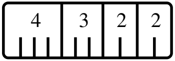
Received 27 February 1997; received in revised form 8 September 1997

1999

M. Kontsevich [Ko]: "... using [BN] (Exercise 6.14) one can obtain the estimate

$$\dim(\mathcal{P}_n) > e^{c\sqrt{n}}, \quad n \to +\infty\ ."$$

D. Bar-Natan [BN]:"*Exercise* 6.14. (Kontsevich, [Ko]). Using the correspondence like



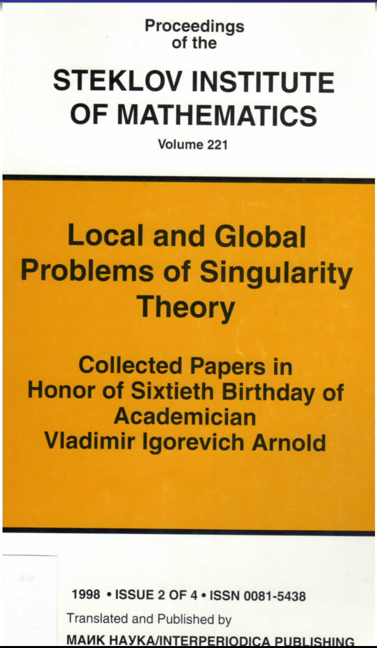
$\longleftrightarrow$ 10+1=4+3+2+2

show that $\dim(\mathcal{P}_n)$ is greater or equal to the number of partitions of $n+1$."

Vassiliev invariants. Lower bound.

- S. Chmutov, S. Duzhin, S Lando'94: $\dim \mathcal{P}_n \geqslant 1$.
- S.Chmutov, A. Varchenko'95: $\dim \mathcal{P}_n \geqslant [n/2]$.
- S. Duzhin'96: $\dim \mathcal{P}_n \gtrsim n^2/96$.
- S. Chmutov and S. Duzhin'99: $\dim \mathcal{P}_n \gtrsim n^{\log n}$.
- O. Dasbach'00: $\dim \mathcal{P}_n \gtrsim e^{c\sqrt{n}}$ for any constant $c < \pi\sqrt{2/3}$.
  $\dim \mathcal{A}_n \gtrsim e^{n/\log_b n}$ for any constant $b < \pi^2/6$

S. Duzhin, A. Kaishev, S.Chmutov, 1998


Proceedings of the

STEKLOV INSTITUTE OF MATHEMATICS

Volume 221

Local and Global Problems of Singularity Theory

Collected Papers in Honor of Sixtieth Birthday of Academician Vladimir Igorevich Arnold

1998 • ISSUE 2 OF 4 • ISSN 0081-5438

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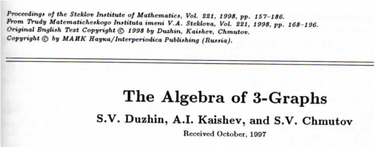
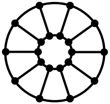

Proceedings of the Steklov Institute of Mathematics, Vol. 221, 1998, pp. 157–186.
From Trudy Matematicheskogo Instituta imeni V.A. Steklova, Vol. 221, 1998, pp. 168–196.

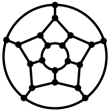
The Algebra of 3-Graphs

S.V. Duzhin, A.I. Kaishev, and S.V. Chmutov

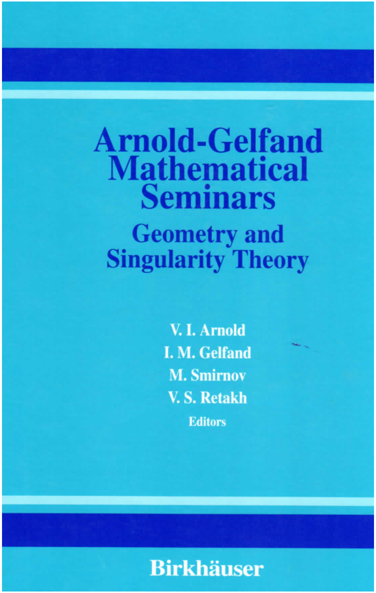
Received October, 1997

1999

$w_{10} =$ 

$d_{10} =$ 

Arnold's invariants of curves 1997



Explicit formulas for Arnold's generic curve invariants

S. Chmutov and S. Duzhin[1]

Abstract

We review the explicit formulas for Arnold's generic curve invariants due to Viro, Shumakovich and Polyak and add some remarks concerning the invariants of spherical curves and curves immersed in arbitrary orientable surfaces.

1. Statement of the problem

A generic curve is a smooth immersion of the circle into the plane whose only singularities are transversal double points. Up to a diffeomorphism of the plane, how many generic closed curves are there?

One can immediately invent a host of invariants for generic plane curves, for example, the total number of double points, the Whitney index or the (unordered) sequence of edge numbers of the regions into which the curve divides the plane.

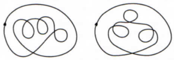
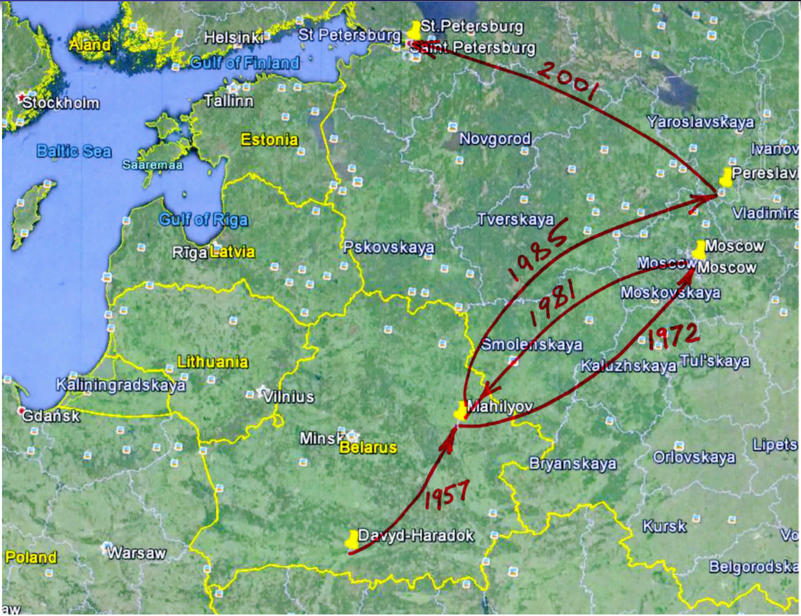


Figure 1: Two different curves with an equal number of double points $(= 4)$, equal Whitney numbers $(= -3)$ and equal sequences of edge numbers $(6, 6, 1, 1, 1, 1)$.

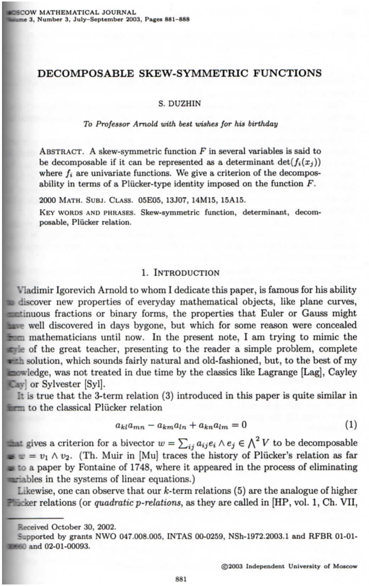
The problem we will discuss is the study of invariants and the classification of plane curves and also curves immersed in orientable surfaces. Although this problem has a venerable history (it was first studied by Gauss), the most basic invariants of plane curves, J^+, J^- and St, were only discovered in 1993 by V. I. Arnold who used V. Vassiliev's approach to topological invariants through discriminants and singularity theory.

[1] Both authors acknowledge financial support from the International Science Foundation and the Russian Foundation for Fundamental Research.

Pereslavl-Zalessky → St.Petersburg, 2001.



Moscow Mathematical Journal, 2003



MOSCOW MATHEMATICAL JOURNAL
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DECOMPOSABLE SKEW-SYMMETRIC FUNCTIONS

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To Professor Arnold with best wishes for his birthday

Abstract. A skew-symmetric function F in several variables is said to be decomposable if it can be represented as a determinant $\det(f_i(x_j))$ where f_i are univariate functions. We give a criterion of the decomposability in terms of a Plücker-type identity imposed on the function F.

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1. Introduction

Vladimir Igorevich Arnold to whom I dedicate this paper, is famous for his ability to discover new properties of everyday mathematical objects, like plane curves, continuous fractions or binary forms, the properties that Euler or Gauss might have well discovered in days bygone, but which for some reason were concealed from mathematicians until now. In the present note, I am trying to mimic the style of the great teacher, presenting to the reader a simple problem, complete with solution, which sounds fairly natural and old-fashioned, but, to the best of my knowledge, was not treated in due time by the classics like Lagrange [Lag], Cayley [Cay] or Sylvester [Syl].

It is true that the 3-term relation (3) introduced in this paper is quite similar in form to the classical Plücker relation

$$a_{kl}a_{mn} - a_{km}a_{ln} + a_{kn}a_{lm} = 0 \qquad (1)$$

that gives a criterion for a bivector $w = \sum_{ij} a_{ij} e_i \wedge e_j \in \bigwedge^2 V$ to be decomposable as $w = v_1 \wedge v_2$. (Th. Muir in [Mu] traces the history of Plücker's relation as far as to a paper by Fontaine of 1748, where it appeared in the process of eliminating variables in the systems of linear equations.)

Likewise, one can observe that our k-term relations (5) are the analogue of higher Plücker relations (or *quadratic p-relations*, as they are called in [HP, vol. 1, Ch. VII,

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881

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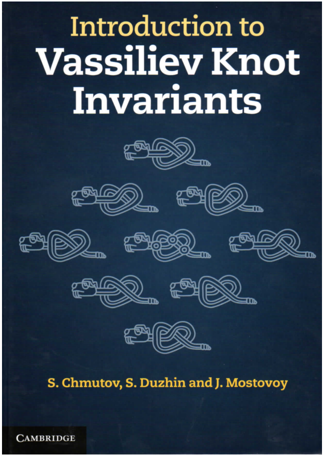

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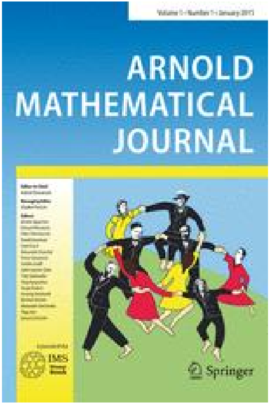

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