Undergraduate research on Knots and Graphs

Sergei Chmutov

Ohio State University

SAMMS-2015

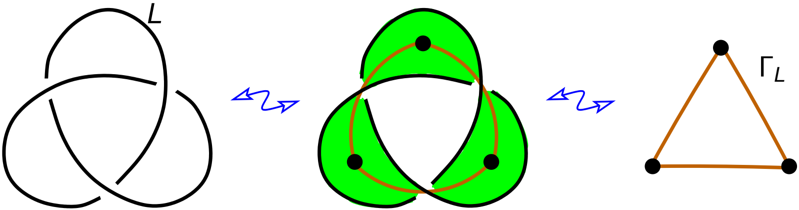
Friday, July 31, 2015

https://people.math.osu.edu/chmutov.1/wor-gr-su15/wor-gr.htm

2006. **Jeremy Voltz**, *Thistlethwaite's theorem for virtual links.*

M. B. Thistlethwaite, L. Kauffman, K.Murasugi, F.Jaeger

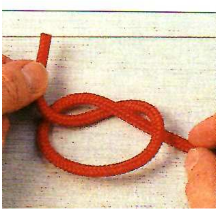
Up to a sign and a power of t the Jones polynomial $V_L(t)$ of an alternating link L is equal to the Tutte polynomial $T_{\Gamma_L}(-t,-t^{-1})$.

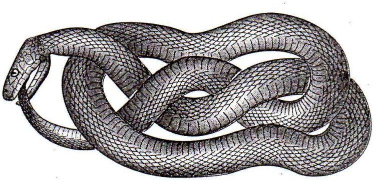


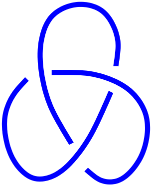
$$\begin{aligned} V_L(t) &= t + t^3 - t^4 \\ &= -t^2(-t^{-1} - t + t^2) \end{aligned}$$

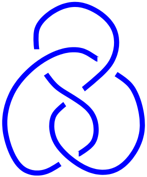
$$\begin{aligned} T_{\Gamma_L}(x,y) &= y + x + x^2 \\ T_{\Gamma_L}(-t,-t^{-1}) &= -t^{-1} - t + t^2 \end{aligned}$$

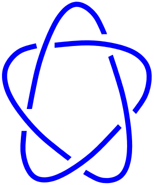
Knots

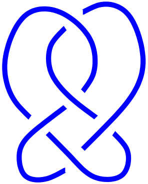




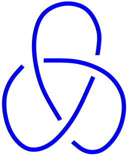




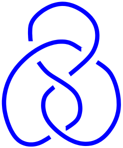




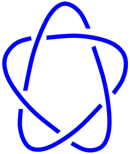
Knot Table



3_1



4_1



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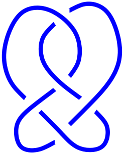
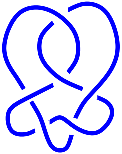
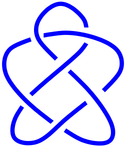


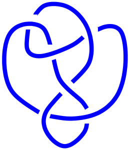
5_2



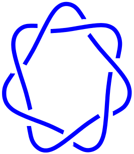
6_1



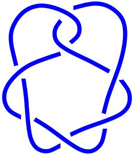
6_2



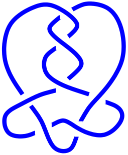
6_3



7_1

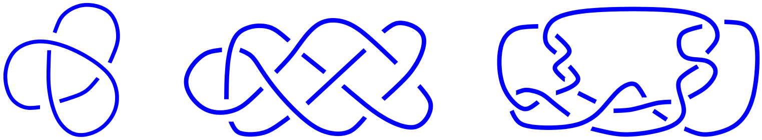


7_2

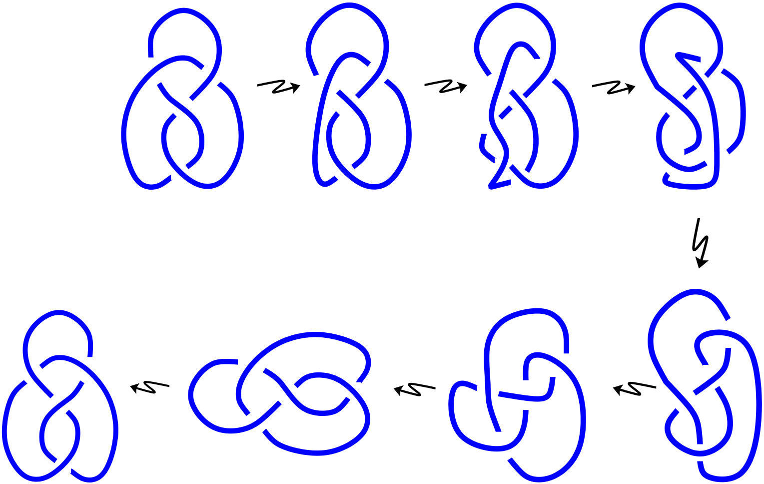


7_3

Unknots = Trivial Knots

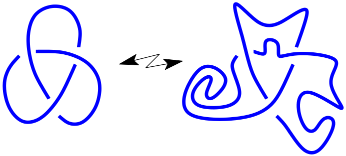


Knot isotopy

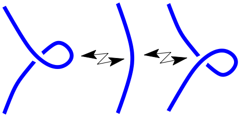


Reidemeister moves

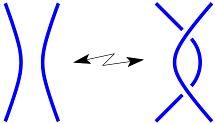
Planar isotopy:



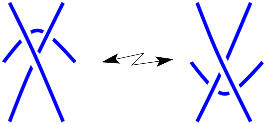
RI:



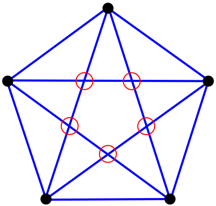
RII:



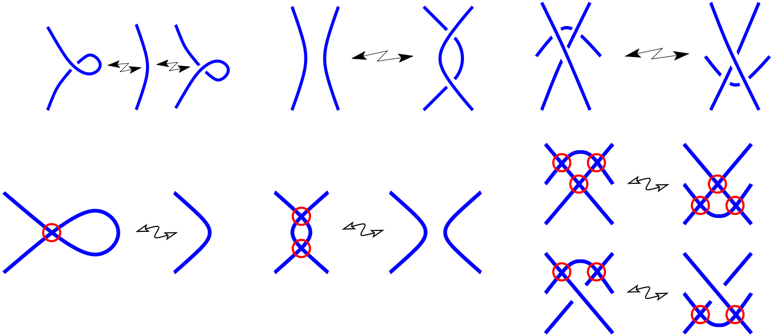
RIII:



Virtual crossings



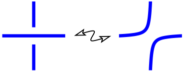
Reidemeister moves



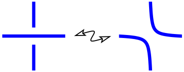
The Kauffman bracket.

Let L be a virtual link diagram.

A-splitting:



B-splitting:



A *state* S is a choice of either A- or B-splitting at every classical crossing.

$\alpha(S) :=$ #(of A-splittings in S)

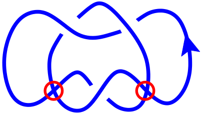
$\beta(S) :=$ #(of B-splittings in S)

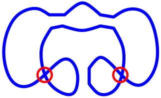
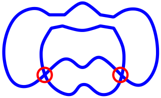
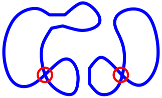
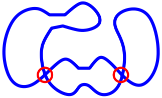
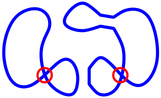
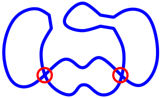
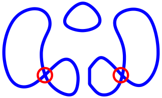
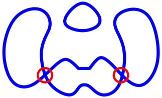
$\delta(S) :=$ #(of circles in S)

$$[L](A,B,d) \;:=\; \sum_S A^{\alpha(S)}\, B^{\beta(S)}\, d^{\delta(S)-1}$$

$$J_L(t) \;:=\; (-1)^{w(L)} t^{3w(L)/4} [L](t^{-1/4}, t^{1/4}, -t^{1/2} - t^{-1/2})$$

The Kauffman bracket. Example.

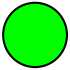
$L =$  (α, β, δ)

			
$(3,0,1)$	$(2,1,2)$	$(2,1,2)$	$(1,2,1)$
			
$(2,1,2)$	$(1,2,1)$	$(1,2,3)$	$(0,3,2)$

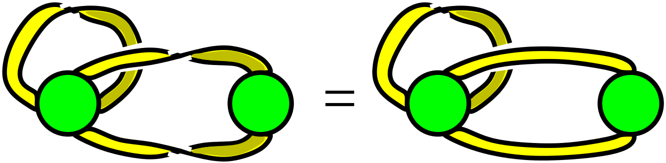
$[L] = A^3 + 3A^2Bd + 2AB^2 + AB^2d^2 + B^3d; \qquad J_L(t) = 1$

Ribbon graphs

A ribbon graph R is a surface represented as a union of

vertices-discs  and edges-ribbons 

- discs and ribbons intersect by disjoint line segments,
- each such line segment lies on the boundary of precisely one vertex and precisely one edge;
- every edge contains exactly two such line segments.



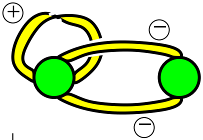
Bollobás-Riordan polynomial

Let F be a ribbon graph;

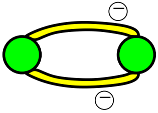
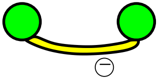
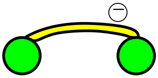
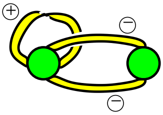
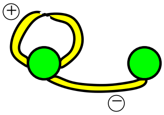
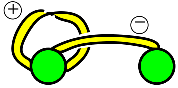
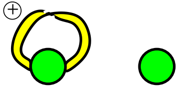
- $v(F)$ be the number of its vertices;
- $e(F)$ be the number of its edges;
- $k(F)$ be the number of components of F;
- $r(F) := v(F) - k(F)$ be the *rank* of F;
- $n(F) := e(F) - r(F)$ be the *nullity* of F;
- $bc(F)$ be the number of boundary components of F;
- $s(F) := (e_-(F) - e_-(\overline{F}))/2$.

$$R_G(x,y,z) \ := \sum_F x^{r(G)-r(F)+s(F)} y^{n(F)-s(F)} z^{k(F)-bc(F)+n(F)}$$

Bollobás-Riordan polynomial. Example.

$L =$ 

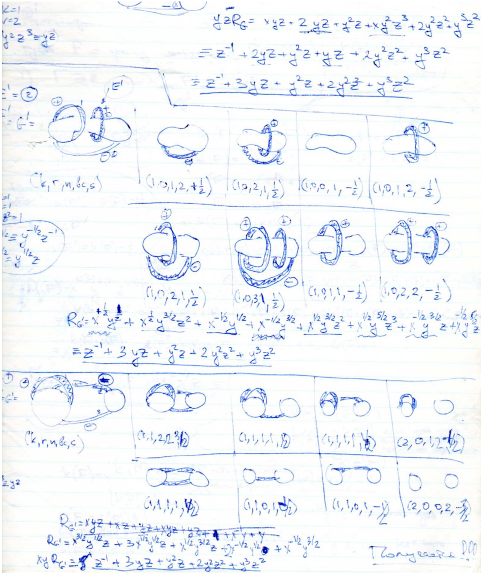
(k, r, n, bc, s)

			
$(1,1,1,2,1)$	$(1,1,0,1,0)$	$(1,1,0,1,0)$	$(2,0,0,2,-1)$
			
$(1,1,2,1,1)$	$(1,1,1,1,0)$	$(1,1,1,1,0)$	$(2,0,1,2,-1)$

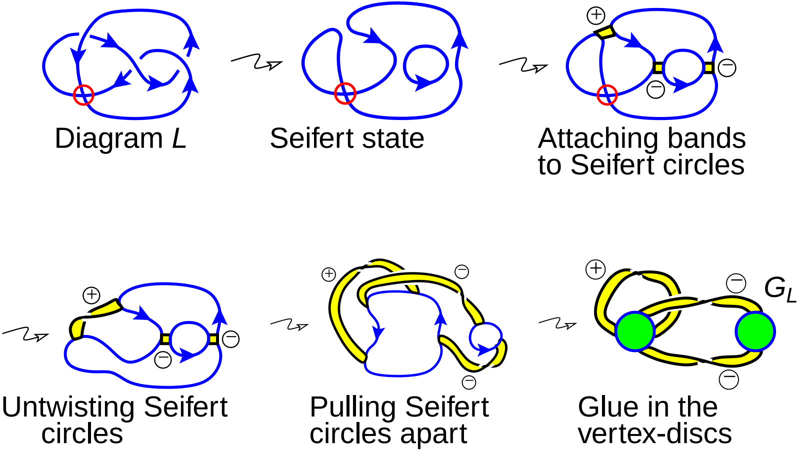
$$R_G(x,y,z) = x + 2 + y + xyz^2 + 2yz + y^2z \ .$$

Problem.

Make a construction of a ribbon graph from a link diagram and relate the parameters (α, β, δ) with parameters (k, r, n, bc, s) (possibly after some substitution).



Construction.



Main Theorem.

Let L be a virtual link diagram, G_L be the corresponding signed ribbon graph, and $n := n(G_L)$, $r := r(G_L)$, $k := k(G_L)$. Then

$$[L](A, B, d) = A^n B^r d^{k-1} R_{G_L}\left(\frac{Ad}{B}, \frac{Bd}{A}, \frac{1}{d}\right) .$$

THANK YOU!