Université Lyon 1

Lecture 1: Graphs and their polynomials

Graphs
- Coloring: \( c: V \to \{1, \ldots, q\} \)
- Chromatic polynomial
  \[ f_G(q) = \# \text{ proper colorings of } G \text{ in } q \text{ colors} \]
  Properties
  - \( X_G = f_G(q) - f_G(0) \)  
  - \( f_{G_1 \cup G_2} = f_{G_1} \cdot f_{G_2} \)
  - \( X_G(q) = q^k F_k(q) \)
  - \( f_{G}(q) = \sum_{F \subseteq E(G)} (-1)^{|F|} q^{|F|} \)

Dichromatic polynomial
- \( Z_G(q, \nu) = \sum_{c \subseteq \mathcal{C}(G)} \# \text{ edge colored and properly bye. } \)
  Properties
  - \( Z_G(q, -1) = X_G(q) \)
  - \( Z_G(q, 0) = q^k \)
  - \( Z_G(q, \nu) = \sum_{F \subseteq E(G)} q^{|F|} \)

Potts model
- Energy of interaction
  - \( E = -J_\infty \text{ (coupling constant) } \)
- Energy of state: Hamiltonian
  - \( H(\epsilon) = -\sum e \epsilon(\{c\}) \)
- Boltzmann weight
  - \( e^{\beta H(\epsilon)} = \prod_{c \in \mathcal{C}(G)} e^{\beta \epsilon(\{c\})} \)
- Inverse temperature \( \beta = \frac{1}{kT} \)
- \( e^{\beta} = \text{Boltzmann constant} \)
Potts partition function: 
\[ Z_V(q, v_e) = \sum_{c \in \text{CL}(G)} e^{-\beta H(c)} = \sum_{c \in \text{CL}(G)} \prod_{e \in E(G)} \frac{1 + v_e \delta(c(e), c'(e))}{\prod_{e \in \text{EE}(G)} e^{\beta E(e)}} \]

If \( v_e = v \)

\[ Z_G(q, v) \] dichromatic polynomial of \( G \)

Fendley-Kasteleyn 1972: 
\[ Z_G(q, v_e) = \sum_{c \in \text{CL}(G)} e^{-\beta H(c)} \prod_{e \in E(G)} e^{\beta F(e)} \]

Probability of a state \( c \):
\[ P(c) = e^{-\beta H(c)} \prod_{e \in E(G)} e^{\beta F(e)} \]

Expected value of a function \( f(c) \)
\[ \langle f \rangle = \sum_{c} f(c) P(c) = \sum_{c} \frac{f(c)}{Z_G} \]

of the energy
\[ \langle H \rangle = \sum_{c} H(c) P(c) = \frac{1}{Z_G} \sum_{c} H(c) e^{-\beta H(c)} \prod_{e \in E(G)} e^{\beta F(e)} \]

Tutte polynomial:
\[ T_0(x, y) = \sum_{c \in \text{CL}(G)} \frac{r(c) - r(F)}{r(F)} \frac{n(F)}{n(F)} \]

Where
\[ r(F) = |V(G)| - k(F), \quad n(F) = |F| - r(F) \]

\[ T_0(x, y) = (x-1)^k(G) (y-1)^{1-|V(G)|} Z_G (x-1) (y-1) \]

\[ Z_G(q, v) = q^{k(G)} \prod_{e \in E(G)} e^{\beta F(e)} \]

Doubly weighted Tutte polynomial:
\[ T_G(x, y, x', y') = \sum_{c \in \text{CL}(G)} \frac{T_0(x, y)}{T_0(x', y')} T_0(x', y') \]

Properties:
- \( T_G = v_e T_0 + v_e T_G \) if \( e \) is either a bridge or a loop.
- \( T_G = (x_e (x-1) + v_e) T_0 \) if \( e \) is a bridge.
- \( T_G = (u_e + (y-1) v_e) T_0 \) if \( e \) is a loop.

For signed graphs:
\[ u_+ = u_+ = 1 \]
\[ u_+ = v_+ = 1 \]
\[ v_- = v_- = 1 \]
\[ u_- = u_- = 1 \]
\[ v_+ = v_+ = 1 \]

Godsil-Rowe
\[ u_+ = u_+ = 1 \]