Thistlethwaite's theorem:

\[ L = \sum \rightarrow \Delta = \sum L \iff \sum_L(\pm t^n) = \sum L(-t,-t^{-1}) \]

Ribbon graphs: vertices - discs, edge - ribbons,
- discs and ribbons intersect by disjoint line segments,
- each line segment lies on the boundary of precisely one vertex and precisely one edge,
- every edge contains exactly two line segments.

Examples:

- $F$, ribbon graph:
  \[ r(F) = |V(F)| - k(F) \]
  \[ n(F) = |E(F)| - r(F) \]
  \[ bc(F) = \# \text{boundary components of } F \]

Bollabás-Riordan polynomial:

\[ R_n(x,y,z) = \sum_{F \in E(n)} x^{r(F)} y^{s(F)} z^{n(F) - s(F) - k(F) - bc(F) + n(F)} \]

\[ s(F) = \frac{e_-(F) - e_+(F)}{2} \]

Example:

- $F$, ribbon graph:
  \[ r(F) = 0, \quad k(F) = 1 \]
Virtual links

Virtual diagram $\rightarrow$ signed ribbon graph

$[L](A, B, d) = \sum A^x B^y d^{z-1}$

(Theorem) $= A^x B^y d^{z-1} R_{G^L}^S (x, y, z)$

Example

$[G]$ = $A^2 \left( \frac{A^4 - Bd}{A^2} \cdot \frac{1}{d} \cdot \left[ \frac{B + d + B^2 d}{A^2} \cdot \frac{1}{d^2} \right] \right)$

$= AB + A^2 d + B^2 + AB$

Partial duality

$G_L \leftrightarrow G'_L$

Partial duality relative to the edge $e$

Theorem

$v(G) B_G^d (x, y, z) = v(G') B_{G'}^d (x, y, z)$

Contraction/Deletion

$x = \pm$

$R_G = R_{G/e} + R_{G \times e}$

$R_G = (x+1) R_{G/e}$

$R_G = \sqrt{x} R_{G/e} + \sqrt{y} R_{G/e}$

$R_G = \sqrt{x} (y+1) R_{G/e}$