Lecture 3: Ribbon graphs and virtual knots

Thursday June 2

Thistlethwaite's theorem:

$L =$  $\to$  $= \Gamma_L \Rightarrow J_L(t) = \pm t^N T_L(-t, -t^{-1})$

Ribbon graphs vertices - discs, edge - ribbons

- discs and ribbons intersect by disjoint line segments,
- each line segment lies on the boundary of precisely one vertex and precisely one edge
- every edge contain exactly two line segments

Examples

 = 

F ribbon graph. $r(F) = |V(F)| - k(F)$ ← # connected components.

$n(F) = |E(F)| - r(F)$

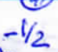
$bc(F) = $ # boundary components of F

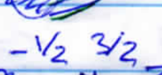
Bollobás–Riordan polynomial

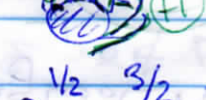
$$R_G(x,y,z) := \sum_{F \subseteq E(G)} x^{r(G)-r(F)+s(F)}\, y^{n(F)-s(F)}\, z^{k(F)-bc(F)+n(F)}$$

$s(F) := \dfrac{e_-(F) - e_-(F)}{2}$ | Example 

, $r(F) = 0$, $k(F) = 1$

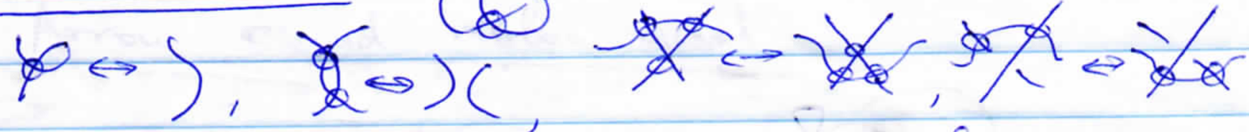
 $x^{-1/2} y^{1/2}$ | 

$x^{1/2} y^{1/2}$ | 

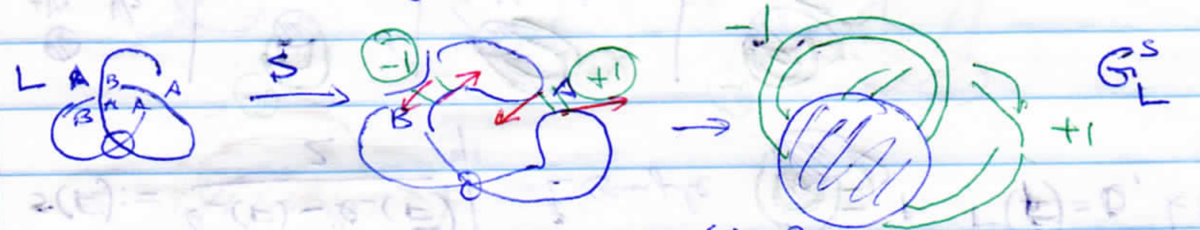
$x^{-1/2} y^{3/2} z$ | 

$x^{1/2} y^{3/2} z^2$

Virtual links



Virtual diagram ⟶ signed ribbon graph



$$[L](A,B,d) = \sum_s A^{\alpha(s)} B^{\beta(s)} d^{\delta(s-1)}$$

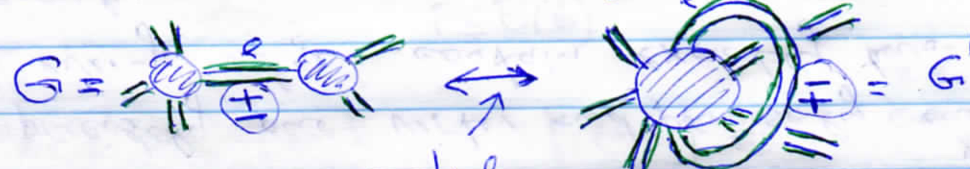
$$\underline{(\text{Theorem})} = A^e\left(x^k y^v z^{v+1} R_{G_L^s}(x,y,z)\right)\Bigg|_{\substack{x=Ad/B \\ y=Bd/A \\ z=1/d}}$$

Example

$$[\;\cdot\;] = A^2\left(\frac{Ad}{B}\cdot\frac{Bd}{A}\cdot\frac{1}{d^2}\cdot\left[\frac{B}{A}+d+\frac{B^2d}{A^2}\cdot\frac{1}{d}+\frac{Bd^2}{Ad^2}\right]\right)$$

$$= AB + A^2d + B^2 + AB$$

Partial duality $G_L^s \overset{\text{p.d.}}{\underset{e}{\longleftrightarrow}} G_L^{s'}$



Theorem

$$(yz)^{v(G)} B_G(x,y,z)\Big|_{xyz^2=1} = (yz)^{v(G')} B_{G'}(x,y,z)\Big|_{xyz^2=1}$$

Contraction/Deletion

	$e = +$	$e = -$
Not a loop, not a bridge	$R_G = R_{G/e} + R_{G-e}$	$R_G = \sqrt{y/x}\, R_{G-e} + \sqrt{x/y}\, R_{G/e}$
bridge	$R_G = (x+1) R_{G/e}$	$R_G = \sqrt{\frac{x}{y}}\,(y+1) R_{G/e}$