Higher dimensional graph theory

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4:00–5:00pm
A coloring of \( G \) with \( x \) colors is a map \( c : V(G) \to \{1, \ldots, x\} \). A coloring \( c \) is proper if for any edge \( e = (v_1, v_2) \): \( c(v_1) \neq c(v_2) \).

\[
\chi_G(x) := \text{# of proper colorings of } G \text{ in } x \text{ colors.}
\]

Properties .

- \( \chi_G = \chi_{G-e} - \chi_{G/e} \);
- \( \chi_{G_1 \sqcup G_2} = \chi_{G_1} \cdot \chi_{G_2} \); for a disjoint union \( G_1 \sqcup G_2 \);
- \( \chi_{\bullet} = x \);
- \( \chi_G(x) = \sum_{F \subseteq E(G)} (-1)^{|F|} x^{k(F)} \),
  where the sum runs over all spanning subgraphs \( F \) and \( k(F) \) is the number of connected components of \( F \).
Dichromatic polynomial $Z_G(x, t)$ of graphs.

\[
Z_G(x, t) := \sum_{c \in Col_G(x)} t^{\text{\# edges colored not properly by } c}
\]

Properties.

- $\chi_G(x) = Z_G(x, 0)$ ;
- $Z_G = Z_{G-e} + (t - 1)Z_{G/e}$ ;
- $Z_{G_1 \sqcup G_2} = Z_{G_1} \cdot Z_{G_2}$, for a disjoint union $G_1 \sqcup G_2$ ;
- $Z_{\cdot} = x$ ;
- $Z_G(x, t) = \sum_{F \subseteq E(G)} x^{k(F)}(t - 1)^{|F|}$ ;
- $Z_G(x, t)$ is the partition function of the Potts model in statistical mechanics.
Tutte polynomial $T_G(x, v)$ of graphs.

$$T_G(x, y) := (x - 1)^{-k(G)}(y - 1)^{-v(G)}Z_G((x - 1)(y - 1), y).$$

Properties.

- $T_G = T_{G-e} + T_{G/e}$ if $e$ is neither a bridge nor a loop;
- $T_G = xT_{G/e}$ if $e$ is a bridge;
- $T_G = yT_{G-e}$ if $e$ is a loop;
- $T_{G_1 ∪ G_2} = T_{G_1 · G_2} = T_{G_1} · T_{G_2}$ for a disjoint union $G_1 ∪ G_2$ and a one-point join $G_1 · G_2$;
- $T_\bullet = 1$;

$$T_G(x, y) := \sum_{F \subseteq E(G)} (x - 1)^{k(F) - k(G)}(y - 1)^{e(F) - v(F) + k(F)}.$$
Specializations of $T_G(x, v)$.

- $\chi_G(x) = (-1)^{|V(G)|}(-x)^{k(G)} T_G(1-x, 0)$;
- $T_G(1, 1) = \# \text{ of spanning trees of } G$;
- $T_G(2, 1) = \# \text{ of spanning forests of } G$;
- $T_G(1, 2) = \# \text{ of spanning connected subgraphs of } G$;
- $T_G(2, 2) = 2^{|E(G)|} = \# \text{ of spanning subgraphs of } G$;

*Flow polynomial:*

$F_G(y) = (-1)^{|E(G)|E+|V(G)|+k(G)} T_G(0, 1-y)$;

- For planar $G$: $T_G(x, y) = T_G^*(y, x)$
Cayley’s and Kalai’s formulas for \# of spanning trees.

A. Cayley, 1889 (C. Borchardt, 1860): \# of spanning trees of $K_n$
\[
= n^{n-2}.
\]

G. Kalai, 1983: \# of $j$ dimensional spanning trees of an $(n-1)$ dimensional simplex
\[
= n \binom{n-2}{j}.
\]
Cellular spanning trees.

$K$ finite cell (CW) complex of dimension $k$.

$K_{(j)}$ $j$-skeleton of $K$.

**Spanning subcomplex** $S$ of dimension $j$: $K_{(j-1)} \subseteq S \subseteq K_{(j)}$.

$S_j$ set of all spanning subcomplexes of dimension $j$.

$f_j(S)$ # of $j$-cells of $S$.

$\tilde{\beta}_j(S)$ reduced $j$-th Betti number $= rank(\tilde{H}_j(S; \mathbb{Z}))$.

**Definition.** A $j$-dimensional **Cellular Spanning Tree** ($j$-CST) $S$ of $K$ is a $j$-dimensional spanning subcomplex such that:

$$\tilde{H}_j(S) = 0, \quad \tilde{\beta}_{j-1}(S) = 0, \quad (|\tilde{H}_{j-1}(S)| < \infty).$$

$T_j(K)$ set of all $j$-CST’s of $K$.

$$\tilde{\tau}_j(K) := \sum_{S \in T_j(K)} |\tilde{H}_{j-1}(S)|^2$$
Kalai’s theorem (1983). If $K$ is a simplex with $n$ vertices, $k = n - 1$, then

$$\sum_{S \in \mathcal{T}_j(K)} |\tilde{H}_{j-1}(S)|^2 = \tilde{\tau}_j(K) = n^\binom{n-2}{j}.$$ 

Example. $n = 6, j = 2$.  
46608 contractible 2-CST’s; 12 homeomorphic to $\mathbb{R}P^2$.  
$H_1(\mathbb{R}P^2) = \mathbb{Z}_2 \implies 46608 + 12 \times 4 = 46656 = 6^6$.  

R. Bott, 1952: \[ R_K(\lambda) := \sum_{S \in \mathcal{S}_k} (-1)^{f_k(K) - f_k(S)} \lambda^{\beta_k(S)}. \]

Z. Wang, 1994: For \( k = 1 \), the Bott polynomial is essentially the flow polynomial of the graph \( K \).
V. Krushkal, D. Renardy, 2010: For $1 \leq j \leq k$, 

$$T^j_K(x, y) := \sum_{S \in S_k} x^{\beta_{j-1}(S) - \beta_{j-1}(K)} y^{\beta_j(S)}.$$ 

- $T^1_K(x, y) = T_{K(1)}(x + 1, y + 1)$. 
- For dual cellulations $K$ and $K^*$ of the sphere $S^k$,

$$T^j_K(x, y) = T^{k-j}_{K^*}(y, x).$$
Dual cellulations.

\[ \{ j \text{ - cells of } K \} \leftrightarrow \{ (k - j) \text{ - cells of } K^* \} \]
C. Bajo, B. Burdick, S. Ch., 2014:

\[ \tilde{T}_j^i(K, x, y) := \sum_{S \in S_k} |\text{tor}(H_{j-1}S)|^2 x^{\beta_{j-1}(S)-\beta_{j-1}(K)} y^{\beta_j(S)}. \]

- If \( \tilde{\beta}_j(K) = 0, 1 \leq j < k \), then \( \tilde{T}_j^i(0, 0) = \tilde{\tau}_j(K) \).
- For dual cellulations \( K \) and \( K^* \) of the sphere \( S^k \),

\[ \tilde{T}_j^i(K, x, y) = \tilde{T}_{k-j}^{k}(y, x) \]

\[ R_K(\lambda) = (-1)^{\beta_k(K)} T_k^k(-1, -\lambda). \]
Let $\sigma \in K$ be a $k$-cell, $\overline{\sigma}$ be its closure in $K$, and $\partial \sigma := \overline{\sigma} - \sigma$ be its boundary. $\overline{\sigma}$, $\partial \sigma$, $K - \sigma$, and $K/\overline{\sigma}$ inherit the cellular structure from $K$.

**Definition.**

- $\sigma$ is a *loop* in $K$ if $H_k(\overline{\sigma}) \cong \mathbb{Z}$;
- $\sigma$ is a *bridge* in $K$ if $\beta_{k-1}(K - \sigma) = \beta_{k-1}(K) + 1$;
- $\sigma$ is *boundary regular* if $\tilde{H}_{k-1}(\partial \sigma) \cong \mathbb{Z}$. 

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Example.

\[ K = \Sigma_e \dim \begin{array}{c|c|c|c} \text{cell} & 0 & 1 & 2 \\
\hline p & e & \sigma \\
\end{array} \]

\[ K \sim S^2 \lor S^1 \]

\[ H_0(K) = H_1(K) = H_2(K) = \mathbb{Z} \]

\[ \bar{\sigma} = K \implies \sigma \text{ is a loop.} \]
\[ \partial \sigma = e \cup p = S^1. \text{ So } H_1(\partial \sigma) = \mathbb{Z}, \text{ and } \sigma \text{ is boundary regular.} \]
\[ T_K^2(x, y) = 1 + y. \]
(i) If $\sigma$ is neither a bridge nor a loop and is boundary regular, then

$$T^k_K(X, Y) = T^k_{K/\sigma}(X, Y) + T^k_{K-\sigma}(X, Y).$$

(ii) If $\sigma$ is a loop, then

$$T^k_K(X, Y) = (Y + 1)T^k_{K-\sigma}(X, Y).$$

(iii) If $\sigma$ is a bridge and boundary regular, then

$$T^k_K(X, Y) = (X + 1)T^k_{K/\sigma}(X, Y).$$
Example.

\[ K = \]

\[ \sigma' \]

\[ \partial \sigma' = e \cup p = S^1. \]

So \( H_1(\partial \sigma') = \mathbb{Z} \), and \( \sigma' \) is boundary regular.

\[ T^2_K(x, y) = (x + 1) T^2_{K/\sigma'}(x, y). \]

\[ T^2_{K/\sigma'}(x, y) = y + 1. \]

\[ T^2_K(x, y) = (x + 1)(y + 1) = xy + x + y + 1 \]
THANK YOU!