# Symmetric chromatic polynomial and the Vassiliev invariants

AMS sectional meeting @ OSU
Sunday, March 18, 2018
9:00 – 9:30

$\boxed{1}$ R. Stanley '95 $\kappa : V(G) \to \mathbb{N}$, proper $\kappa(v_1) \neq \kappa(v_2)$ if $(v_1, v_2) \in E(G)$

$$X_G(x_1, x_2, \dots) = \sum_{\substack{\kappa \\ \text{proper}}} \prod_{v \in V(G)} x_{\kappa}, \qquad p_m = \sum_{i=1}^{\infty} x_i^m$$

Example $X_{\bullet\!-\!\bullet}(x_1, x_2, \dots) = \cancel{x_1x_1 +} x_1x_2 + x_1x_3 + \dots + x_1x_n + \dots$

$+ x_2x_1 \cancel{+ x_2x_2} + x_2x_3 + \dots x_2x_n + \dots$

$\cancel{+ x_3x_3}$

$+ x_3x_1 + x_3x_2 + x_3x_4 + \dots\dots$

$= x_1(p_1 - x_1) + x_2(p_1 - x_2) + x_3(p_1 - x_3) = p_1^2 - p_2$

Conjecture $X_G$ distinguishes trees.

$\left[ X_G(\underbrace{1, \dots, 1}_{q}, 0, 0, \dots) = \chi_G(q) \right.$ chromatic polynomial

$\boxed{2}$ Vassiliev invariants $v : \mathcal{K} \to \underset{\text{abelian group}}{\mathbb{A}}$ a knot invariant

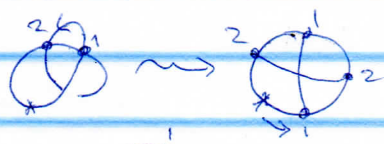
Singular knots $\mathcal{K}_n = \{ \underbrace{\times \dots \times}_{n} \}$

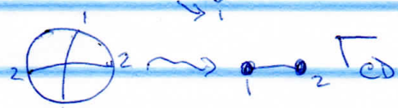
$v(\times) := v(\nearrow\!\!\!\nwarrow) - v(\nwarrow\!\!\!\nearrow)$

Def. $v$ is VI of order $\leq n$ if $v|_{\mathcal{K}_{>n}} \equiv 0$

$\mathcal{V}_0 \leq \mathcal{V}_1 \leq \mathcal{V}_2 \leq \dots$

$\mathcal{V}_n / \mathcal{V}_{n-1}$ is finite-dimensional $\Rightarrow [v]$ is determined by $v|_{\mathcal{K}_n}$

Chord diagram  Fact $v(K)$ depends on CD of $K$ only, $K \in \mathcal{K}_n$

Intersection graph  $\Gamma_{CD}$

$\mathcal{A}$ = Hopf algebra of chord diagrams modulo 4T

Intersection graph Conjecture (–, S. Duzhin, S. Lando '94 Advanced in Soviet Mathematics)

$v(K)$ depends only on $\Gamma_{CD(K)}$

wrong (counterexample in order 11)

# Weighted chromatic polynomial [CDL'94]

Weight $v(G) \to \mathbb{N}$

G is simple (without multiple edges and loop's)

$W_G(q_1, q_2, \ldots)$ possible multiple edges are reduced to a single one

- $W_G = W_{G \setminus e} + W_{G/e}$


- $W_{\underset{n}{\bullet}} = q_n$, $\{ W_{G_1 \sqcup G_2} = W_{G_1} W_{G_2}$

$\Downarrow$

Forest Hopf algebra

$\langle \Gamma_{CD}$ is a forest $\rangle \subset \mathcal{A}$

$\mathcal{F}$ " All weights $= 1$

$W_\bullet : \mathcal{F} \to \mathbb{C}[q_1, \ldots q_n, \ldots]$

graded Hoph algebra homomorphism

S. Noble, D. Welsh '99:

- $(-1)^{|V(G)|} W_G(q_j = -p_j) = \chi_G(p_1, p_2, \ldots)$
- $W_G(q_1, q_2, \ldots) = \sum_{E' \subseteq E(G)} q_{v_1} \cdots q_{v_k} (-1)^{|E'| - |V(G)| + k(E')}$

$k = k(E')$ = # of connected components of $E'$ as a spanning forest.

$v_1, \ldots, v_k$ = # of vertices in the connected components.

Example $W_{\bullet\!-\!\bullet} = W_{\underset{1}{\bullet}\!-\!\underset{1}{\bullet}} = W_{\underset{1}{\circ}\,\underset{1}{\circ}} + W_{\underset{2}{\bullet}} = q_2 + q_1^2 = (-1)^2 X_{\bullet\!-\!\bullet}(p_1 = -q_1, p_2 = -q_2)$

## Stanley's Conjecture

Let $D_1$ and $D_2$ be two CD such that $\Gamma_{D_1} \neq \Gamma_{D_2}$ ~~are~~ trees

Then $D_1 \neq D_2$ mod 4T relation

$\Leftrightarrow$ $\exists$ Vassiliev invariant $v$, such that $v(K_{D_1}) \neq v(K_{D_2})$

(of order $n = |V(\Gamma_{D_i})|$)