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Symmetric chromatic function in star basis

Sergei Chmutov

Ohio State University, Mansfield

with

Ishaan Shah

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Overview

Stanley's chromatic symmetric function.

R. Stanley, *A symmetric function generalization of the chromatic polynomial of a graph*, Advances in Math. **111**(1) 166–194 (1995).

$$\boxed{X_G(x_1,x_2,\dots) := \sum_{\substack{\varkappa:V(G)\to\mathbb{N}\\ \text{proper}}} \prod_{v\in V(G)} x_{\varkappa(v)}}$$

Power function basis. $\quad p_m := \sum_{i=1}^{\infty} x_i^m.$

Example.

$$\begin{aligned} X_{\bullet\!-\!\bullet} &= \widehat{x_1x_1} + x_1x_2 + x_1x_3 + \dots \\ & \quad x_2x_1 + \widehat{x_2x_2} + x_2x_3 + \dots \\ & \quad x_3x_1 + x_3x_2 + \widehat{x_3x_3} + \dots \\ & \quad \vdots \qquad\quad \vdots \qquad\qquad\quad \ddots \\ &= p_1^2 - p_2 . \end{aligned}$$

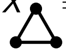
Chromatic symmetric function in power basis.

$$X_G = \sum_{S\subseteq E_G} (-1)^{|S|} p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_k},$$

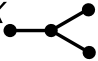
where $(\lambda_1, \dots, \lambda_k) =: \lambda(S) \vdash |V(G)|$ is a partition of the number of verticies according to the connected components of the spanning subgraph S.

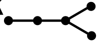
With shorter notation $p_{\lambda(S)} := p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_k}$, we have $X_G = \sum_{S\subseteq E_G} (-1)^{|S|} p_{\lambda(S)}$.

Examples. X  $= p_1^2 - p_2$, X  $= p_1^3 - 2p_1p_2 + p_3$,

X  $= p_1^3 - 3p_1p_2 + 2p_3$, $X_{K_n}(x_1, x_2, \dots) = n!e_n(x_1, x_2, \dots)$,

X  $= p_1^4 - 3p_1^2p_2 + p_2^2 + 2p_1p_3 - p_4$,

X  $= p_1^4 - 3p_1^2p_2 + 3p_1p_3 - p_4$.

X  $= p_1^5 - 4p_1^3p_2 + 4p_1^2p_3 + 2p_1p_2^2 - 3p_1p_4 - p_2p_3 + p5.$

Chromatic symmetric function. Conjectures.

Tree conjecture.
X_G distingushes trees.

A $(3+1)$ poset is the disjoint union of a 3-element chain and 1-element chain.
A poset P is $(3+1)$-*free* if it contains no induced $(3+1)$ posets.

Incomparability graph inc(P) of P: vertices are elements of P; (uv) is an edge if neither $u \leqslant v$ nor $v \leqslant u$.

***e*-positivity conjecture.**
The expansion of $X_{inc(P)}$ in terms of elementary symmetric functions has positive coefficients for $(3+1)$-free posets P.

Weighted graphs.

S. Chmutov, S. Duzhin, S. Lando, *Vassiliev knot invariants III. Forest algebra and weighted graphs*, Advances in Soviet Mathematics **21** 135–145 (1994).

Definition. A *weighted graph* is a graph G without loops and multiple edges given together with a *weight* $w : V(G) \to \mathbb{N}$ that assigns a positive integer to each vertex of the graph.

Ordinary simple graphs can be treated as weighted graphs with the weights of all vertices equal to 1.

Let $\mathscr{H}_n$ be a vector space spanned by all weighted graphs of the total weight n modulo the *weighted contraction/deletion relation* $G = (G - e) + (G/e)$, where the graph $G \setminus e$ is obtained from G by removing the edge e and G/e is obtained from G by a contraction of e such that if a multiple edge arises, it is reduced to a single edge and the weight $w(v)$ of the new vertex v is set up to be equal to the sum of the weights of the two ends of the edge e.

$$\mathscr{H} := \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \dots$$

Multiplication: disjoint union of graphs;
Comultiplication: splitting the vertex set into two subsets.
The primitive space $P(\mathscr{H}_n)$ is of dimension 1 and spanned by a single vertex of weight n.
The Hopf algebra $\mathscr{H}$ has a one-dimensional primitive space in each grading.

Milnor–Moore Theorem: $\mathscr{H}_n$ is isomorphic to $\mathbb{C}[q_1, q_2, \dots]$.

Weighted chromatic polynomial.

The image of an ordinary graph G (considered as a weighted graph with weights of all vertices equal to 1) in $\mathscr{H}$ can be represented by a polynomial $W_G(q_1, q_2, \dots)$ in the variables q_n.

S. Noble, D. Welsh, *A weighted graph polynomial from chromatic invariants of knots*, Annales de l'institut Fourier **49**(3) 1057–1087 (1999):

$$(-1)^{|V(G)|} W_G\Big|_{q_j=-p_j} = X_G(p_1, p_2, \dots).$$

Examples. $W_{\bullet\!-\!\bullet} = (\bullet\ \bullet) + \underset{2}{\bullet} = q_1^2 + q_2$

$$W_{\bullet\!-\!\bullet\!-\!\bullet} = (\bullet \quad \bullet\!-\!\bullet) + \underset{2}{\bullet}\!-\!\bullet = (\bullet \quad \bullet \quad \bullet) + 2(\ \bullet\ \underset{2}{\bullet}\ ) + (\ \underset{3}{\bullet}\ )$$

$$= q_1^3 + 2q_1q_2 + q_3$$

Star basis.

S. Cho, S. van Willigenburg, *Chromatic bases for symmetric functions*, The electronic journal of combinatorics **23**(1) (2016) #P1.15.

For every $n \in \mathbb{N}$, pick a connected graph G_n with n vertices.

Theorem. *The symmetric chromatic functions $X_{G_n}(x_1, x_2, \dots)$ generate (multiplicatively) the whole algebra of symmetric functions in $x_1, x_2, \dots$.*

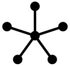
Proof (Corollary of CDL-III'1994).

Consider G_n as an element of the Hopf algebra $\mathscr{H}_n$. Because of connectivity its projection to the one-dimensional primitive space $P(\mathscr{H}_n)$ is non-zero. □

Remark. Instead of graph G_n with n-vertices we can choose any conncted weighted graph $\widetilde{G}_n$ with the total weight n.

Examples. 1) If G_n is a single vertex of weight n then the corresponding basis is the the power functions basis.

2) If $G_n = K_n$ the complete graph with n vertices (of weight 1), then we get the basis of elementary symmetric functions.

3) Let G_n be a star with n vertices. $G_6 =$ 

Then the symmetric chromatic functions $s_n := X_{G_n}$ form a basis for the algebra of all symmetric functions.Its expression in terms of power functions is

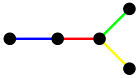
$$s_n = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} p_1^{n-k-1} p_{k+1}.$$

Symmetric chromatic function in star basis.

Theorem. (I.Shah)

$$\boxed{X_G(s_1, s_2, \dots, s_n) = \sum_{\{leaves\}\subseteq E_1 \sqcup E_2 \sqcup E_3 = E(G)} (-s_1)^{|E_2|} s_{\lambda(E_1,E_2)}}\ ,$$

where $\lambda(E_1, E_2) := (\lambda_1, \lambda_2, \dots, \lambda_l) \vdash |E_1|$ is a "partition" of $|E_1|$ defined as follows. Let $G_1, \dots G_l$ be the connected components of the spanning subgraph of G with the set of edges $E_1 \cup E_2$. Then λ_k is the number of E_1-edges of the connected component G_k; $s_{\lambda(E_1,E_2)} := s_{\lambda_1+1} s_{\lambda_2+1} \dots s_{\lambda_l+1}$ is a product of star variables.

Example. $D_5 :=$  The set E_1 has to contain all the leaves b, g, y. So there only two choices for E_1, $r \notin E_1$ and $r \in E_1$.

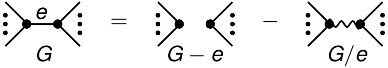
- $E_1 = \{b, g, y\}$, $E_2 = \emptyset \implies s_2 s_3$
  $E_2 = \{r\} \implies -s_1 s_4$
- $E_1 = \{b, g, y, r\}$, $E_2 = \emptyset \implies s_5$

So the result is $X_{D_5} = -s_1 s_4 + s_2 s_3 + s_5$. Compare to

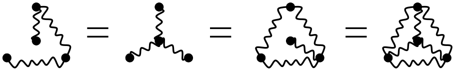
$X_{D_5} = p_1^5 - 4p_1^3 p_2 + 4p_1^2 p_3 + 2p_1 p_2^2 - 3p_1 p_4 - p_2 p_3 + p_5$.

Star basis.

Proof.
The idea is to use the weighted contraction/deletion relation, only postpone the actual contraction replacing the edges by *squiggle* edges.



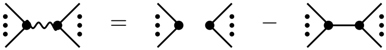
Squiggle calculus. Since all squiggles are going to be contracted we can rearrange squiggles within a connected component as we like.



To prove the theorem we apply the weighted contraction/deletion relation to all edges of our graph G. We will get a combination of terms obtained from G by deleting some edges, which form the part E_3 of the tripartition, and replacing the remaining $E_1 \sqcup E_2$ edges by squiggles. Such a term comes with the coefficient $(-1)^{|E_1|+|E_2|}$. Let $G_1, \dots G_l$ be the connected components of this term with $E_1 \sqcup E_2$ squiggle edges. For every component G_k we rearrange the squiggles to a star.

Star basis.

Proof (continuation).
Then using the weighted contraction/deletion relation in a form



we resolve every squiggle in these stars into straight edge and non-edge.

The straight edges (i.e. the squiggles resolved into the straight edges) form the set E_1. The set E_2 is formed by squiggles resolved to non-edge by deletion. When we delete a squiggle of E_2 from a star, an extra factor s_1 pops up. So we will get a term which is the product of star variables with coefficient $(-1)^{|E_1|+|E_2|}(-1)^{|E_1|} = (-1)^{|E_2|}$. It remains to note that if E_1 does not contain a leaf edge, then we have two choices. One include that leaf from the beginning, that means it will go to E_3. The another one is to include it in E_2, that is delete it on the process of converting a squiggle stars to usual stars. Both choices give the same product of star variables, but they differ by sign because of $(-1)^{|E_2|}$. So they will be canceled out from the final result. □

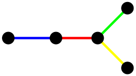
Symmetric chromatic function in paths basis.

The same proof works for the expression in terms of the basis consisting of of the symmetric chromatic function of paths.

$a_2 := X$, $a_3 := X$, $a_4 := X$, $\ldots$

Theorem.
$$\boxed{X_G(a_1, a_2, \ldots, a_n) = \sum_{E_1 \sqcup E_2 \sqcup E_3 = E(G)} (-1)^{|E_2|} a_{\lambda(E_1,E_2)}},$$

where $a_{\lambda(E_1,E_2)}$ is defined as follows. Let $G_1, \ldots G_l$ be the connected components of the spanning subgraph of G with the set of edges $E_1 \sqcup E_2$. For each connected component G_k we construct a path with $|E_1 \sqcup E_2|$ edges and then remove $|E_2|$ edges from this path for all possible choices of E_2. The resulting collection of paths constitutes the product of a-variables $a_{\lambda(E_1,E_2)}$.

Example. $D_5 :=$  , $X_{D_5} = a_1 a_4 - a_2 a_3 + a_5$.

Compare to $X_{D_5} = -s_1 s_4 + s_2 s_3 + s_5$ and
$X_{D_5} = p_1^5 - 4p_1^3 p_2 + 4p_1^2 p_3 + 2p_1 p_2^2 - 3p_1 p_4 - p_2 p_3 + p_5$.

Happy birthday Sergei!!!