B-symmetric chromatic function of signed graphs

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Overview

Stanley's chromatic symmetric function.

R. Stanley, *A symmetric function generalization of the chromatic polynomial of a graph*, Advances in Math. **111**(1) 166–194 (1995).

$$X_G(x_1, x_2, \dots) := \sum_{\substack{\varkappa: V(G)\to\mathbb{N} \\ \text{proper}}} \prod_{v\in V(G)} x_{\varkappa(v)}$$

Power function basis. $\quad p_m := \sum_{i=1}^{\infty} x_i^m.$

Example.

$$\begin{aligned} X_{\bullet\!\!-\!\!\bullet} = \; & \widehat{x_1x_1} + x_1x_2 + x_1x_3 + \dots \\ & x_2x_1 + \widehat{x_2x_2} + x_2x_3 + \dots \\ & x_3x_1 + x_3x_2 + \widehat{x_3x_3} + \dots \\ & \quad \vdots \qquad\quad \vdots \qquad\qquad\quad \ddots \\ = \; & p_1^2 - p_2. \end{aligned}$$

Chromatic symmetric function in power basis.

$$X_G(x_1,x_2,\dots) = \sum_{\substack{\varkappa:V(G)\to\mathbb{N}\\ \text{all}}} \prod_{v\in V(G)} x_{\varkappa(v)} \prod_{e=(v_1,v_2)\in E(G)} (1-\delta_{\varkappa(v_1),\varkappa(v_2)})$$

$$= \sum_{\substack{\varkappa:V(G)\to\mathbb{N}\\ \text{all}}} \prod_{v\in V(G)} x_{\varkappa(v)} \sum_{S\subseteq E_G} (-1)^{|S|} \prod_{e\in S} \delta_{\varkappa(v_1),\varkappa(v_2)}$$

$$\prod_{e\in S} \delta_{\varkappa(v_1),\varkappa(v_2)} = \begin{cases} 1 & \text{all vertices of a connected component of the spanning subgraph with } S \text{ edges are colored by } \varkappa \text{ into the same color} \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{X_G = \sum_{S\subseteq E_G} (-1)^{|S|} p_{\lambda(S)},}$$

where $\lambda(S) \vdash |V(G)|$ is a partition of the number of verticies according to the connected components of the spanning subgraph S, and for $\lambda(S) = (\lambda_1,\dots,\lambda_k)$, $p_{\lambda(S)} := p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_k}$.

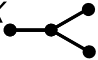
Chromatic symmetric function. Examples.

$$X_G = \sum_{S \subseteq E_G} (-1)^{|S|} p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_k}$$

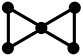
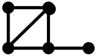
Examples. X  $= p_1^2 - p_2,$

X  $= p_1^3 - 2p_1p_2 + p_3,$ X  $= p_1^3 - 3p_1p_2 + 2p_3.$

X  $= p_1^4 - 3p_1^2p_2 + p_2^2 + 2p_1p_3 - p_4,$

X  $= p_1^4 - 3p_1^2p_2 + 3p_1p_3 - p_4.$

Two graphs with the same chromatic symmetric function:

X  $= X$ 

Chromatic symmetric function. Conjectures.

Tree conjecture.
X_G distingushes trees.

A $(3+1)$ poset is the disjoint union of a 3-element chain and 1-element chain.
A poset P is $(3+1)$-*free* if it contains no induced $(3+1)$ posets.

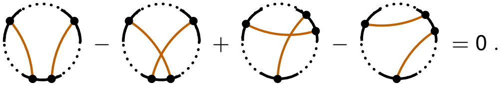
Incomparability graph inc(P) of P: vertices are elements of P; (uv) is an edge if neither $u \leqslant v$ nor $v \leqslant u$.

***e*-positivity conjecture.**
The expansion of $X_{inc(P)}$ in terms of elementary symmetric functions has positive coefficients for $(3+1)$-free posets P.

Vassiliev knot invariants. Chord diagrams.

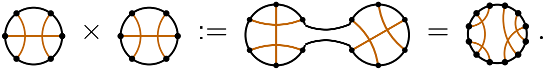
Algebra of chord diagrams.

$\mathcal{A}_n$ is a $\mathbb{C}$-vector space spanned by chord diagrams modulo four term relations:



The vector space $\mathcal{A} := \bigoplus_{n\geqslant 0} \mathcal{A}_n$ has a natural bialgebra structure.

Multiplication:



Comultiplication: $\delta : \mathcal{A}_n \to \bigoplus_{k+l=n} \mathcal{A}_k \otimes \mathcal{A}_l$ is defined on chord diagrams by the sum of all ways to split the set of chords into two disjoint parts:

$$\delta(D) := \sum_{J\subseteq[D]} D_J \otimes D_{\overline{J}}.$$

Primitive space $\mathcal{P}(\mathcal{A})$ is the space of elements $D \in \mathcal{A}$ with the property

$$\delta(D) = 1 \otimes D + D \otimes 1.$$

$\mathcal{P}(\mathcal{A})$ is also a graded vector space $\mathcal{P}(\mathcal{A}) = \bigoplus_{n\geq 1} \mathcal{P}_n$.

Vassiliev invariants. Bialgebra structure.

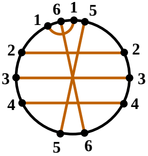
The classical **Milnor—Moore** theorem: *any commutative and cocommutative bialgebra $\mathcal{A}$ is isomorphic to the symmetric tensor algebra of the primitive space, $\mathcal{A} \cong \mathcal{S}(\mathcal{P}(\mathcal{A}))$.*

Let $p_1, p_2, \dots$ be a basis for the primitive space $\mathcal{P}(\mathcal{A})$ then any element of $\mathcal{A}$ can be uniquely represented as a polynomial in commuting variables $p_1, p_2, \dots$.
The dimensions of $\mathcal{P}_n$:

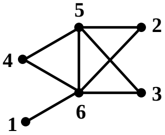
n	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{P}_n$	1	1	1	2	3	5	8	12	18	27	39	55

Vassiliev invariants. Weighted graphs.

S. Chmutov, S. Duzhin, S. Lando, *Vassiliev knot invariants III. Forest algebra and weighted graphs*, Advances in Soviet Mathematics **21** 135–145 (1994).



A chord diagram



The intersection graph

Definition. A *weighted graph* is a graph G without loops and multiple edges given together with a *weight* $w : V(G) \to \mathbb{N}$ that assigns a positive integer to each vertex of the graph.

Ordinary simple graphs can be treated as weighted graphs with the weights of all vertices equal to 1.

Bialgebra of weighted graphs.

Let $\mathcal{H}_n$ be a vector space spanned by all weighted graphs of the total weight n modulo the *weighted contraction/deletion relation* $G = (G \setminus e) + (G/e)$, where the graph $G \setminus e$ is obtained from G by removing the edge e and G/e is obtained from G by a contraction of e such that if a multiple edge arises, it is reduced to a single edge and the weight $w(v)$ of the new vertex v is set up to be equal to the sum of the weights of the two ends of the edge e.

$$\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots$$

Multiplication: disjoint union of graphs;
Comultiplication: splitting the vertex set into two subsets.
The primitive space $P(\mathcal{H}_n)$ is of dimension 1 and spanned by a single vertex of weight n.

The bialgebra $\mathcal{H}$ has a one-dimensional primitive space in each grading and thus is isomorphic to $\mathbb{C}[q_1, q_2, \dots]$.

Weighted chromatic polynomial.

The image of an ordinary graph G (considered as a weighted graph with weights of all vertices equal to 1) in $\mathcal{H}$ can be represented by a polynomial $W_G(q_1, q_2, \dots)$ in the variables q_n.

S. Noble, D. Welsh, *A weighted graph polynomial from chromatic invariants of knots*, Annales de l'institut Fourier **49**(3) 1057–1087 (1999):

$$(-1)^{|V(G)|} W_G\Big|_{q_j=-p_j} = X_G(p_1, p_2, \dots).$$

Examples. $W_{\bullet\!-\!\bullet} = (\bullet\ \bullet) + \underset{2}{\bullet} = q_1^2 + q_2$

$$\begin{aligned} W_{\bullet-\bullet-\bullet} &= (\bullet \quad \bullet\!-\!\bullet) + \underset{2}{\bullet}\!-\!\bullet = (\bullet \quad \bullet \quad \bullet) + 2(\ \bullet\ \underset{2}{\bullet}\ ) + (\ \underset{3}{\bullet}\ ) \\ &= q_1^3 + 2q_1q_2 + q_3 \end{aligned}$$

Plugging in $q_m = -p_m = \sum_{i=1}^{\infty} x_i^m$ into the weighted chromatic polynomial a a graph G with weight function $w : V(G) \to \mathbb{N}$ we get the weighted chromatic function

$$X_{G,w}(x_1, x_2, \dots) := \sum_{\substack{\varkappa : V(G) \to \mathbb{N} \\ \text{proper}}} \prod_{v \in V(G)} x_{\varkappa(v)}^{w(v)}.$$

Generating function of weighted polynomial.

S. Chmutov, M. Kazarian, S. Lando, *Polynomial graph invariants and the KP hierarchy*, `arXiv:1803.09800`. To appear in *Selecta Mathematica*.

$$\mathcal{W}(q_1,q_2,\dots) := \sum_{\substack{G \text{ connected} \\ \text{non-empty}}} \frac{W_G(q_1,q_2,\dots)}{|\mathrm{Aut}(G)|}$$
$$= \frac{1}{1!}q_1 + \frac{1}{2!}\left(q_1^2+q_2\right) + \frac{1}{3!}\left(4q_1^3+9q_1q_2+5q_3\right)$$
$$+\frac{1}{4!}\left(38q_1^4+144q_1^2q_2+45q_2^2+140q_1q_3+79q_4\right)+\dots,$$

Theorem. $F(p_1,p_2,\dots) := \mathcal{W}(\alpha_1p_1,\alpha_2p_2,\alpha_3p_3,\alpha_4p_4,\dots)$ *is a solution of the KP hierarchy of PDEs*,
where $\alpha_n = \frac{2^{n(n-1)/2}(n-1)!}{c_n}$ and $c_1=1$, $c_2=1$, $c_3=5$, $c_4=79$, $c_5=3377$, ... is the [A134531] sequence from Sloane's Encyclopedia of Integer Sequences.

Signed graphs.

T. Zaslavsky, *Signed graph coloring*, Discrete Mathematics **39**(2) 215–228 (1982).

Definition. A *signed graph* is a graph G with a function $\sigma : E(G) \to \{\pm\}$.

A *proper coloring* of a signed graph G is a function $\varkappa : V(G) \to \mathbb{Z} \setminus \{0\}$ such that for all adjacent vertices v, w, $\varkappa(v) \neq \sigma(v, w)\varkappa(w)$.

Definition. A *signed chromatic polynomial*
$\chi_G(2n) := \#$ proper colorings of G by $\{-n, \ldots, -1, 1, \ldots, n\}$.

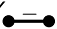
Definition. *B-symmetric chromatic function*

$$\boxed{Y_G(\ldots, x_{-2}, x_{-1}, x_1, x_2, \ldots\ldots) := \sum_{\substack{\varkappa : V(G) \to \mathbb{Z} \setminus \{0\} \\ \text{proper}}} \prod_{v \in V(G)} x_{\varkappa(v)}}$$

The function Y_G is invariant under the action of the group B_∞ on the subscripts of the variables. The group B_∞ of *signed permutations* consists of permutations s of $\mathbb{Z} \setminus \{0\}$ permuting only finitely many integers and such that $s(-i) = -s(i)$ for all $i \in \mathbb{Z} \setminus \{0\}$.

B-symmetric chromatic function. Examples.

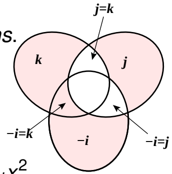
$$Y_G(\ldots, x_{-2}, x_{-1}, x_1, x_2, \ldots\ldots) := \sum_{\substack{\varkappa: V(G)\to\mathbb{Z}\setminus\{0\}\\ \text{proper}}} \prod_{v\in V(G)} x_{\varkappa(v)}.$$

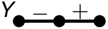
Examples. Y  $=$

$$\begin{array}{l} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_{-2}(\ldots x_{-2} + x_{-1} + x_1 + \widehat{x_2} + \ldots) \\ x_{-1}(\ldots x_{-2} + x_{-1} + \widehat{x_1} + x_2 + \ldots) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

$$= p_{1,0}^2 - p_{1,1},$$

where $\boxed{p_{a,b} := \sum_{i\in\mathbb{Z}\setminus\{0\}} x_i^a x_{-i}^b}$ are the *signed power functions*.

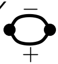


Y  $= \sum_{-i\neq j\neq k} x_i x_j x_k$

$$= \sum_{i,j,k} x_i x_j x_k - \sum_{i,k} x_i x_{-i} x_k - \sum_{i,j} x_i x_j^2 + \sum_j x_{-j} x_j^2$$

$$= p_{1,0}^3 - p_{1,1}p_{1,0} - p_{1,0}p_{2,0} + p_{2,1}.$$

Y  $= Y_\bullet = p_{1,0}.$

Y  $= p_{1,0}^2 - p_{2,0} - p_{1,1}.$

Doubly weighted signed chromatic polynomial.

Definition. A *doubly weighted signed graph* is a signed graph G with a *pair of weights* $w_1 : V(G) \to \mathbb{Z}_{\geqslant 0}$ and $w_2 : V(G) \to \mathbb{Z}_{\geqslant 0}$ for each vertex of the graph. Now we can allow the graphs with negative loops and two parallel edges, one positive and one negative.
Ordinary signed graphs can be considered as doubly weighted signed graphs with the weights of all vertices equal to $(1, 0)$.

Definition. A *weighted signed chromatic function* is given by

$$Y_G(\dots, x_{-2}, x_{-1}, x_1, x_2, \dots\dots) := \sum_{\substack{\varkappa:V(G)\to\mathbb{Z}\setminus\{0\}\\ \text{proper}}} \prod_{v\in V(G)} x_{\varkappa(v)}^{w_1(v)} x_{-\varkappa(v)}^{w_2(v)}.$$

For a graph consisting of a single vertex with weights (a, b) the doubly weighted signed chromatic polynomial is $Y_{\underset{(a,b)}{\bullet}} = p_{a,b} := \sum_{i\in\mathbb{Z}\setminus\{0\}} x_i^a x_{-i}^b$. $\boxed{p_{a,b} = p_{b,a}}$

Switching of a doubly weighted signed graph G at its vertex v of weights (a, b) is the graph G^v obtained by reversing the signs of all edges incident to v and switching the weights of v to (b, a).

Theorem. $Y_G = Y_{G^v}$.

Subsets of edges formula.

$$Y_G = \sum_{\substack{\varkappa: V(G)\to\mathbb{Z}\setminus\{0\}\\ \text{all}}} \prod_{v\in V(G)} x_{\varkappa(v)}^{w_1(v)} x_{-\varkappa(v)}^{w_2(v)} \prod_{e=(v_1,v_2)\in E(G)} \left(1-\delta_{\varkappa(v_1),\sigma(e)\varkappa(v_2)}\right)$$

$$= \sum_{\substack{\varkappa: V(G)\to\mathbb{Z}\setminus\{0\}\\ \text{all}}} \prod_{v\in V(G)} x_{\varkappa(v)}^{w_1(v)} x_{-\varkappa(v)}^{w_2(v)} \sum_{S\subseteq E_G} (-1)^{|S|} \prod_{e\in S} \delta_{\varkappa(v_1),\sigma(e)\varkappa(v_2)}$$

The product $\prod_{e\in S} \delta_{\varkappa(v_1),\sigma(e)\varkappa(v_2)}$ is equal to 1 iff each connected component of the spanning subgraph with S edges is balanced (that is the product of signs of all its edges is 1) and all its vertices are colored into the same or opposite color depending on the parity of the number of negative edges on a path between these vertices.

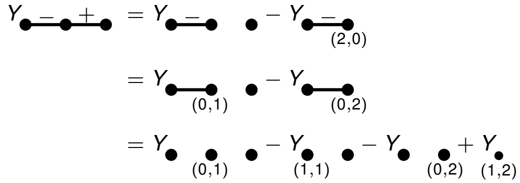
$$\boxed{Y_G = \sum_{\substack{S\subseteq E_G\\ \text{balanced}}} (-1)^{|S|} \prod_{\substack{C\subseteq S\\ \text{connected}}} p_{w_1(C),w_2(C)},}$$

where the sum runs over those spanning subgraphs S each connected component of which is balanced; C is a connected component of S; removing the negative edges splits the vertices of C into two parts connecting by negative edges and all edges withing each part are positive, $w_1(C)$ and $w_2(C)$ are the the total sums of weights of vertices of each part.

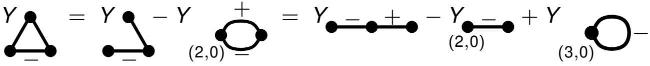
Contraction/Deletion formula.

Theorem. *Let e be a positive edge of a doubly weighted signed graph G. Then, $Y_G = Y_{G\setminus e} - Y_{G/e}$, where both weights of the endvertices of e are summing up under the contraction of edge e.*

Examples.



$$= p_{1,0}^3 - p_{1,1}p_{1,0} - p_{1,0}p_{2,0} + p_{2,1}.$$



$$= (p_{1,0}^3 - p_{1,1}p_{1,0} - p_{1,0}p_{2,0} + p_{2,1}) - (p_{1,0}p_{2,0} - p_{2,1}) + p_{3,0}.$$

$$= p_{1,0}^3 - p_{1,1}p_{1,0} - 2p_{1,0}p_{2,0} + 2p_{2,1} + p_{3,0}.$$

Bialgebra of doubly weighted signed graphs.

Let $\mathcal{SH}_n$ be a vector space spanned by all doubly weighted signed graphs of the total sum of two weights equal n modulo

- the *sined weighted contraction/deletion relation* $G = (G \setminus e) + (G/e)$,
- the *switching relation* $G = G^v$,

where the graph $G \setminus e$ is obtained from G by removing the edge e and G/e is obtained from G by a contraction of e such that the arising multiple edges of the same sing are reduced to a single edge of that sign, negative loops are deleted, and the weights $(w_1(v), w_2(v))$ of the new vertex v is set up to be equal to the sum of the weights of the two ends of the edge e.

$$\mathcal{SH} := \mathcal{SH}_0 \oplus \mathcal{SH}_1 \oplus \mathcal{SH}_2 \oplus \dots$$

Multiplication: disjoint union of graphs;
Comultiplication: splitting the vertex set into two subsets.

Theorem. *The primitive space* $P(\mathcal{SH}_n)$ *has of dimension* $\left\lceil \frac{n}{2} \right\rceil$ *and spanned by a single vertex of weights* $(a, n-a)$.

In particular any signed graphs G with n vertices may be considered as doubly weighted with weights of each vertex $(1,0)$. As such it determines an element of $\mathcal{SH}_n$, and thus it can be uniquely expressed as a polynomial in single vertices of weight (a, b) with $a + b \leqslant n$. This polynomial is exactly Y_G expressed in terms of the signed power functions $p_{a,b}$.

Open problems.

- Is there any relation of the bialgebra $\mathcal{SH}$ of signed graphs with bialgebras of Lando-Krasilnikov of framed graphs? Perhaps not.
- Is there any relations of the *B*-symmetric chromatic function of signed graphs with knot theory? Perhaps with Vassiliev invariants of knots in some special 3-manifolds?
- Are there any analogs of Stanley's conjectures for *B*-symmetric functions?
- Are there any integrable hierarchy of PDEs for which the generating function of the doubly weighted signed chromatic polynomials would provide a solution? This would be a *B*-analog of the KP hierarchy.

THANK YOU!!!